Solving PDEs Associated with Economic Models

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This package EconPDEs.jl introduces a fast and robust way to solve systems of PDEs + algebraic equations (i.e. DAEs) associated with economic models. This note details the underlying algorithm.

Difference with Achdou et al. (2016) Achdou et al. (2016) focus on linear PDEs of the form

$$0 = f_1(V) + f_2(x)\partial_x V + f_3(x)\partial_{xx} V + \partial_t V$$

In contrast, the package solves non-linear PDEs of the form

$$0 = f_1(V) + f_2(x, \partial_x V) + f_3(x, \partial_x V)\partial_{xx} V + \partial_t V$$

Step 1: Write Finite Difference Scheme The system of PDEs is written on a state space grid and derivatives are substituted by finite difference approximations. As in Achdou et al. (2016), first order derivatives are upwinded. This helps to ensure that the boundary counditions are satisfied, as well as making the scheme monotonous.

Step 2: Solve Finite Difference Scheme Denote V_t the solution of the PDE. We can alwyays write the HJB is $0 = F(V_t) + \partial_t V$. I propose to solve for V using a fully implicit Euler method. Updates take the form

$$\forall t \leq T \qquad 0 = F(V_{t+1}) + \frac{1}{\Delta}(V_{t+1} - V_t)$$

Each time step requires to solve a non-linear equation, which is solved using a Newton-Raphson method.

If the Newton-Raphson step is not successful, I decrease Δ (since this non-linear step converges if the guess is sufficiently close to the solution). If it is successful, I increase Δ, to speed up the algorithm.

This method is most similar to a method used in the fluid dynamics literature. In this context, it is called the Pseudo-Transient Continuation method, and is denoted Ψtc. Formal conditions for the convergence of the algorithm are given in Kelley and Keyes (1998).

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References

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll, "Heterogeneous Agent Models in Continuous Time," 2016. Working Paper.

Kelley, Carl Timothy and David E Keyes, "Convergence analysis of pseudo-transient continuation," *SIAM Journal on Numerical Analysis*, 1998, *35* (2), 508–523.