Analysis proofs and exercises

Facundo Quiroga

February 11, 2016

Contents

1 Limits

1.1 Rational and nth-roots

1. $\lim\limits_{x\to 1} \frac{x^2+x-2}{x^2-3x+2} = -3$
2. $\lim\limits_{x\to 1^+} \frac{|x-1|}{x-1} = 1$
3. $\lim\limits_{x\to a^+} \frac{|x-a|}{x-a} = 1$
4. $\lim\limits_{x\to 1} \frac{|x-1|}{x-1} = NE$
5. $\lim\limits_{x\to 1} \frac{x-1}{\sqrt{x}-1} = 2$
6. $\lim\limits_{x\to -a} \frac{x-b}{x+a} = NE$ si $a \neq b$
7. $\lim\limits_{x\to -a^-} \frac{x-b}{x+a} = s\infty$ s=-sign(-a-b)
8. $\lim\limits_{x\to a} \frac{x^2-a^2}{x-a} = 2a$
9. $\lim\limits_{x\to a} \frac{x^2-a^2}{|x-a|} = NE$
10. $\lim\limits_{x\to a} \frac{x^3-a^3}{x-a} =$
11. $\lim\limits_{x\to a} \frac{x^3-a^3}{x^2-a^2} = \pm\infty$
12. $\lim\limits_{x\to 0} \frac{x-|x|}{2}$
13. $\lim\limits_{x\to 0} \frac{\sqrt{x+1}-1}{x}$
14. $\lim\limits_{x\to +\infty} \frac{\sqrt{x+1}-1}{x}$

15. $\lim_{x\to 0} \frac{\sqrt{x^2+1}}{x}$

16. $\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x}$

17. $\lim_{x\to 0} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

18. $\lim_{x\to 0} \frac{x^{2/3}-x^{3/7}}{x^{2/5}+x^{3/4}}$

19. $\lim_{x\to +\infty} \frac{a_n x^n+\cdots+a_0}{b_m x^m+\cdots+b_0}$ para 1) $n<m$, 2) $n=m$ y 3) $n>m$

1.2 Trigonometric

1. $\lim_{x\to 0} \frac{\sin(x)}{x}$

2. $\lim_{x\to 0} \frac{\sin(ax)}{x}$

3. $\lim_{x\to 0} \frac{\sin(|x|)}{x}$

4. $\lim_{x\to 0} \frac{\sin(\pi-x)}{x}$

5. $\lim_{x\to 0} \frac{\sin(x)}{x^2}$

6. $\lim_{x\to 0} \frac{\tan(x)}{x}$

7. $\lim_{x\to 0} \frac{\cos(x)-1}{x}$

8. $\lim_{x\to 0} \frac{\cos(x)-1}{x^2}$

9. $\lim_{x\to 0} \frac{\sin(x^2)}{x}$

10. $\lim_{x\to 0} \frac{x\sin(x)}{\sin(x^2)}$

11. $\lim_{x\to +\infty} \frac{\sin(x)}{x}$

1.3 Exponential

1. $\lim_{x\to +\infty} \frac{2^x}{x}$

2. $\lim_{x\to +\infty} \frac{2^x}{\ln(x)}$

3. $\lim_{x\to +\infty} \frac{a^x}{x}$ para $0<a<1$ y $1<a$

4. $\lim_{x\to +\infty} \frac{x^x}{e^x}$

5. $\lim_{x\to +\infty} \frac{x^x}{e^{\sqrt{x}}}$

6. $\lim_{x\to 0^+} xln(x)$

7. $\lim_{x\to +\infty} \frac{ln(x)}{x}$

8. $\lim_{x\to 0^+} \frac{ln(x+1)}{x}$

9. $\lim_{x \to 0^+} \frac{x}{ln(x+1)}$

10. $\lim_{x \to +\infty} (1 + \frac{b}{x})^x$

11. $\lim_{x \to +\infty} (a + \frac{b}{x})^x$

12. $\lim_{x \to +\infty} (a + \frac{b}{x})^{cx}$

13. $\lim_{x \to +\infty} (a + \frac{b}{x})^{cx+d}$

14. $\lim_{x \to +\infty} (a + \frac{b}{x})^{cx+d}$

15. $\lim_{x \to +\infty} (1 + \frac{1}{x^2})^x$

1.4 Mix

1. $\lim_{x \to +\infty} \frac{\sin(2x)-\cos(x^4)}{x}$

2. $\lim_{x \to +\infty} \frac{\sin(2x)-\cos(x^4)}{\ln(x)}$

3. $\lim_{x \to +\infty} x(a^{1/x} - 1)$ for $a > 0$

4. $\lim_{x \to -\infty} \frac{1}{xe^x} = -\infty$

5. $\lim_{x \to +\infty} \ln(1 + e^x) - x$

6. $\lim_{x \to +\infty} \frac{x^2-\ln(x)}{e^x-x}$

7. $\lim_{x \to -1} \frac{x-2}{e^{x+1}-1}$

2 Derivatives properties

Note that the limits in these proofs are very different from those you evaluate when calculating, say, the derivative of an actual function like x^2 or e^x by definition.

While in both cases you have a $0/0$ limit to solve, when working with an actual function the difficulty lies in trying to rewrite the numerator to factor out an h or do something equivalent.

However, in the following cases you need to try to rewrite the numerator $f(x+h) - f(x)$ as a combination of other sorts of derivatives (depending on how f is defined), and when you do the h usually takes care of itself because it's absorbed by the definition of the other derivatives.

2.1 Linearity: multiply by a constant

Property. *Linearity: multiply by a constant*

Preconditions:

- $f(x) = cg(x)$
- $g'(x)$ exists

Claim: $f'(x) = cg'(x)$

Proof: The main idea of this proof is to simply write down the limit definition of the derivative of f and factor out the constant c from the limit.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{cg(x+h) - cg(x)}{h} = \lim_{h \to 0} \frac{c[g(x+h) - g(x)]}{h} \\ &= c \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = cg'(x) \end{aligned}$$

□

2.2 Linearity: sum of functions

Property. *Linearity: sum of functions*

Preconditions:

- $f(x) = g(x) + z(x)$
- $g'(x)$ and $z'(x)$ exist

Claim: $f'(x) = g'(x) + z'(x)$

Proof: The main idea of this proof is to simply write down the limit definition of the derivative of f in terms of g and z, and split the limit in two terms, one for the derivative of g and the other for z.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[g(x+h) + z(x+h)] - [g(x) + z(x)]}{h} \\ &= \lim_{h \to 0} \frac{[g(x+h) - g(x)] + [z(x+h) - z(x)]}{h} \\ &= \lim_{h \to 0} \frac{g(x+h) - g(x))}{h} + \lim_{h \to 0} \frac{z(x+h) - z(x)}{h} \\ &= g'(x) + z'(x) \end{aligned}$$

□

2.3 Product Rule

Property. *Product Rule*

Preconditions:

- $f(x) = g(x)z(x)$
- $g'(x)$ and $z'(x)$ exist

Claim: $f'(x) = g'(x)z(x) + z(x)'g(x)$

Proof: The main idea of this proof is to add $-g(x+h)z(x) + g(x+h)z(x)$ to the numerator of the limit definition of the derivative. This will again allow us to split the limit into two terms, whose limits can be evaluated individually, and give the final result shown above. After splitting into the two terms and factoring, the terms will be slightly asymmetric, but that wont be a problem for the proof if you remember that for functions that are continuous at x, by definition the following holds: $\lim_{h \to 0} f(x+h) = f(x)$. So when you see a $g(x+h)$ on one term and a $z(x)$ on the other that is asymmetric, don't worry, because if you take the $\lim_{h \to 0}$ of the first, you actually get $g(x)$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{[g(x+h)z(x+h)]-[g(x)z(x)]}{h} \\
&= \lim_{h\to 0} \frac{[g(x+h)z(x+h)]-[g(x)z(x)]-g(x+h)z(x)+g(x+h)z(x)}{h} \\
&= \lim_{h\to 0} \frac{[g(x+h)z(x+h)-g(x+h)z(x)]+[-g(x)z(x)+g(x+h)z(x)]}{h} \\
&= \lim_{h\to 0} \frac{[g(x+h)z(x+h)-g(x+h)z(x)]+[g(x+h)z(x)-g(x)z(x)]}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)[z(x+h)-z(x)]+[g(x+h)-g(x)]z(x)]}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)[z(x+h)-z(x)]+[g(x+h)-g(x)]z(x)]}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)[z(x+h)-z(x)]}{h} + \lim_{h\to 0} \frac{[g(x+h)-g(x)]z(x)}{h} \\
&= [\lim_{h\to 0} g(x+h)][\lim_{h\to 0} \frac{z(x+h)-z(x)}{h}] + z(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= g(x)z'(x) + z(x)g'(x)
\end{aligned}
$$

Note: This property will also allow us to prove a rule for the derivative of $\frac{g(x)}{z(x)}$. □

2.4 Quotient rule: special case for $1/g(x)$

Property. *Quotient rule: special case for* $1/g(x)$

Preconditions:

- $f(x) = \frac{1}{g(x)}$
- $g(x) \neq 0$
- $g'(x)$ exists

Claim: $f'(x) = \frac{-g'(x)}{g(x)^2}$

Proof: Note: This derivative will allow us to easily prove a rule for the derivative of $\frac{g(x)}{z(x)}$.

The main idea of this proof is to unify the denominators of the terms in $\frac{1}{g(x+h)} - \frac{1}{g(x)}$, and then from the result (1) identify the definition of $g'(x)$ and (2) find a $g(x)^2$ that can be factored out (actually $g(x+h)g(x)$, which is the same as $g(x)^2$ if $h \to 0$).

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{g(x)-g(x+h)}{g(x+h)g(x)}}{h} \\
&= \lim_{h\to 0} \frac{g(x)-g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{g(x)-g(x+h)}{h} \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \\
&= -\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \\
&= -g'(x) \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \\
&= -g'(x) \frac{1}{g(x)g(x)} \\
&= \frac{-g'(x)}{g(x)^2}
\end{aligned}
$$

□

2.5 Quotient rule (full)

Property. *Quotient rule (full)*

Preconditions:

- $f(x) = \frac{g(x)}{z(x)}$
- $z(x) \neq 0$
- $g'(x)$ and $z'(x)$ exists

Claim: $f'(x) = \frac{g'(x)z(x)-z'(x)g(x)}{z(x)^2}$

Proof: The main idea of this proof is that $\frac{a}{b} = a\frac{1}{b}$, so we can solve the derivative of a quotient with the product rule we derived before.

$$
\begin{aligned}
f'(x) = (\frac{g(x)}{z(x)})' &= (g(x)\frac{1}{z(x)})' \\
&= g'(x)\frac{1}{z(x)} + \frac{1}{z(x)}' g(x) \\
&= g'(x)\frac{1}{z(x)} + \frac{-z'(x)}{z(x)^2} g(x) \\
&= g'(x)\frac{1}{z(x)}\frac{z(x)}{z(x)} + \frac{-z'(x)g(x)}{z(x)^2} \\
&= g'(x)\frac{1}{z(x)}\frac{z(x)}{z(x)} + \frac{-z'(x)g(x)}{z(x)^2} \\
&= \frac{g'(x)z(x) - z'(x)g(x)}{z(x)^2}
\end{aligned}
$$

□

2.6 Chain rule

Property. *Quotient rule (full)*

Preconditions:

- $f(x) = z(g(x))$
- $z'(g(x))$ and $g'(x)$ exists

Claim: $f'(x) = z'(g(x))g'(x)$

Proof: First, a proof that's flawed because we can't be sure that $\Delta_h = g(x+h) - g(x) \neq 0$:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{z(g(x+h))-z(g(x))}{h} \\
&= \lim_{h\to 0} \frac{z(g(x+h))-z(g(x))}{h} \frac{\Delta_h}{\Delta_h} \quad \text{where } \Delta_h = g(x+h)-g(x) \text{ (wrong)} \\
&= \lim_{h\to 0} \frac{z(g(x)+\Delta_h)-z(g(x))}{\Delta_h} \lim_{h\to 0} \frac{\Delta_h}{h} \\
&= \lim_{h\to 0} \frac{z(g(x)+\Delta_h)-z(g(x))}{\Delta_h} \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{z(g(x)+\Delta_h)-z(g(x))}{\Delta_h} g'(x) \\
&= z'(g(x))g'(x) \quad \text{(wrong)}
\end{aligned}$$

The general idea of the proof is correct, but it doesn't work because $\lim\limits_{h\to 0} \frac{z(g(x)+\Delta_h)-z(g(x))}{\Delta_h}$ is undefined when $\Delta_h = 0$, and because in the first term we are taking the limit of $h \to 0$, not $\Delta_h \to 0$.

We'll fix the first problem by defining a function $\phi(h)$ that is the same as that expression except when $\Delta_h = 0$, and the second by doing a rigorous proof that in this case it's the same.

$$\phi(h) = \begin{cases} \frac{z(g(x)+\Delta_h)-z(g(x))}{\Delta_h} & \text{if } \Delta_h \neq 0 \\ z'(g(x)) & \text{if } \Delta_h = 0 \end{cases}$$

Note that now $z(g(x)+\Delta_h) - z(g(x)) = \phi(h)\Delta_h$, since it is true by definition when $\Delta_h \neq 0$, and when $\Delta_h = 0$ both sides are 0.

Then:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{z(g(x+h))-z(g(x))}{h} \\
&= \lim_{h\to 0} \frac{z(g(x)+\Delta_h)-z(g(x))}{h} \\
&= \lim_{h\to 0} \frac{\phi(h)\Delta_h}{h} \\
&= \lim_{h\to 0} \phi(h) \lim_{h\to 0} \frac{\Delta_h)}{h} \\
&= \lim_{h\to 0} \phi(h) g'(x)
\end{aligned}$$

Now all we need to prove is that $\lim\limits_{h\to 0} \phi(h) = z'(g(x))$. Since the quantity in the left hand side is a limit, we'll proof this equality using the limit definition of $z'(g(x))$ (we can only do this because we put as a precondition that z is differentiable at $g(x)$), which is:

$$\begin{aligned}
&\forall \epsilon' > 0, \exists \delta' > 0, \forall h,\ 0 < |h - g(x)| < \delta' \\
&\to |\frac{z(g(x)+h)-z(g(x))}{h} - z'(g(x))| < \epsilon'
\end{aligned}$$

Now, we need a similar proof for $\lim_{h\to 0}\phi(h)$. Given $\epsilon > 0$, we need a $\delta > 0$ such that:

$$\begin{aligned}&\forall h\ 0 < |h - g(x)| < \delta\\&\to |\phi(h) - z'(g(x))| < \epsilon\end{aligned}$$

When $\Delta_h = 0$ this is trivially true, because we defined $\phi(h) = z'(g(x))$ for that case, basically any δ will do. When $\Delta_h \neq 0$ we need a $\delta > 0$ such that:

$$\begin{aligned}&\forall h\ 0 < |h - g(x)| < \delta\\&\to |\frac{z(g(x)+\Delta_h) - z(g(x))}{\Delta_h} - z'(g(x))| < \epsilon\end{aligned}$$

Note that this is almost what we got before, only that now we have Δ_h instead of h, so it would seem the δ' from the limit definition of $z'(g(x))$ won't work directly.

We can get this to work anyway using the continuity of g at x; choosing $\epsilon'' = \delta'$, we have:

$$\exists\delta'', \forall h\ 0 < |h| < \delta'' \to |g(x+h) - g(x)| < \epsilon''$$

Therefore, we have the following chain of implications:

$$\begin{aligned}&\exists\delta'', 0 < |h| < \delta'' \to |g(x+h) - g(x)| < e'' \quad (\text{or } |\Delta_h| < \delta')\\&\to 0 < |\Delta_h| < \delta' \quad \text{because we are dealing with the case } \Delta_h \neq 0\\&\to |\frac{z(g(x)+\Delta_h) - z(g(x))}{\Delta_h} - z'(g(x))| < \epsilon\\&\to |\phi(h) - z'(g(x))| < \epsilon \quad \text{because we are dealing with the case } \Delta_h \neq 0\end{aligned}$$

So the δ we were looking for is just δ''. With that, we have proven $\lim_{h\to 0}\phi(h) = z'(g(x))$

And so finally we can say that:

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\phi(h)g'(x)\\&= z'(g(x))g'(x)\end{aligned}$$

□

2.7 Inverse function rule

Property. *Inverse function rule (take 1: by definition)*

Preconditions:

- $f(y) = g^{-1}(y)$
- $g'(x)$ exists
- $g'(x) \neq 0$
- $y = g(x)$

Claim: $f'(y) = \frac{1}{g'(f(y))}$

Proof:

$$\begin{aligned}
f'(y) &= \lim_{h\to 0} \frac{f(y+h)-f(y)}{h} \\
&= \lim_{h\to 0} \frac{g^{-1}(y+h)-g^{-1}(y)}{h} \\
&= \lim_{y'\to y} \frac{g^{-1}(y')-g^{-1}(y)}{y'-y} \quad \text{Substituting h by y'-y} \\
&= \lim_{x'\to x} \frac{x'-x}{g(x')-g(x)} \quad \text{a bit of handwaving using the fact that y=g(x)} \\
&= \frac{1}{\lim\limits_{x'\to x} \frac{g(x')-g(x)}{x'-x}} \quad \text{because we know the limit in the denominator exists and it's not 0 (precondition)} \\
&= \frac{1}{g'(x)} \quad \text{by definition} \\
&= \frac{1}{g'(f(y))}
\end{aligned}$$

□

Property. *Inverse function rule. Take 2: by implicit differentiation using the chain rule*

Preconditions:

- $f(x) = g^{-1}(x)$
- $g'(x)$ exists
- $g'(x) \neq 0$

Claim: $f'(x) = \frac{1}{g'(f(x))}$

Proof:

$$\begin{aligned}
g(f(x)) = x g(g^{-1}(x)) &= x \\
(g(g^{-1}(x)))' &= x' \quad \text{Deriving both sides} \\
g'(g^{-1}(x)) {g^{-1}}'(x) &= 1 \\
{g^{-1}}'(x) &= \frac{1}{g'(g^{-1}(x))} \quad \text{By the chain rule} \\
f(x) &= \frac{1}{g'(f(x))}
\end{aligned}$$

□

3 Derivatives of common functions

3.1 Constant function $f(x) = c \to f'(x) = 0$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0
\end{aligned}$$

3.2 Polinomial: $(\sum_{i=0} n a_i x^i)' = \sum_{i=1} n a_i i x^{i-1}$

Easy by induction

3.3 Exponential: $(e^x)' = e^x$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x(e^h-1)}{h} \\
&= e^x \lim_{h\to 0} \frac{e^h-1}{h} \\
&= e^x
\end{aligned}
$$

3.4 Logarithm: $ln'(x) = \frac{1}{x} \quad (x > 0)$

3.4.1 Take 1, by definition

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{ln(x+h)-ln(x)}{h} \\
&= \lim_{h\to 0} \frac{ln(\frac{x+h}{x})}{h} \\
&= \lim_{h\to 0} \frac{ln(1+\frac{h}{x})}{h} \\
&= \lim_{h\to 0} \frac{ln(1+\frac{h}{x})}{h} \\
&= \lim_{h\to 0} \frac{ln(1+\frac{h}{x})x}{xh} \quad \text{because } x > 0 \\
&= \frac{1}{x} \lim_{h\to 0} ln(1+\frac{h}{x})\frac{x}{h} \\
&= \lim_{u\to\infty} \frac{ln(1+\frac{1}{u})u}{x} \quad \text{substitution } u = \frac{x}{h},\ h\to 0 \Rightarrow u\to\infty \\
&= \frac{1}{x} \lim_{u\to\infty} ln(1+\frac{1}{u})u \\
&= \frac{1}{x} \lim_{u\to\infty} ln((1+\frac{1}{u})^u) \\
&= \frac{1}{x} ln(\lim_{u\to\infty}(1+\frac{1}{u})^u) \\
&= \frac{1}{x} ln(e) \quad \text{by definition of e} \\
&= \frac{1}{x}
\end{aligned}
$$

3.4.2 Take 2, with implicit derivation

Defining $y = exp(x)$:

$$
\begin{aligned}
ln(y) &= x \\
ln(y)' &= x' && \text{deriving both sides} \\
ln(y)' &= 1 && \text{if } f(x) = x, f'(x) = 1 \\
ln'(y)y' &= 1 && \text{chain rule} \\
ln'(y)y &= 1 && \text{since } y = exp(x) \to y' = exp'(x) = exp(x) = y \\
ln'(y) &= \frac{1}{y} && y \neq 0 \text{ because } exp(x) > 0
\end{aligned}
$$

(all derivatives taken with respect to x)

3.4.3 Take 3, using the inverse function derivative property

$$
\begin{aligned}
ln(x)' &= \frac{1}{exp'(ln(x))} && \text{ln is the inverse of } exp\text{, ie, } exp(ln(x)) = x \\
&= \frac{1}{exp(ln(x))} && \text{because } exp'(y) = exp(y) \\
&= \frac{1}{x}
\end{aligned}
$$

3.5 Inverse trigonometric functions

3.5.1 asin(x)

$$
\begin{aligned}
asin(y)' &= \frac{1}{sin'(asin(y))} \\
&= \frac{1}{cos(asin(y))} \\
&= \frac{1}{\sqrt{1-(sin(asin(y)))^2}} && \text{because } cos(x) = \sqrt{1-sin(x)^2} \\
&= \frac{1}{\sqrt{1-y^2}}
\end{aligned}
$$

3.5.2 acos(x)

$$
\begin{aligned}
acos(y)' &= \frac{1}{cos'(acos(y))} \\
&= \frac{1}{-sin(acos(y))} \\
&= \frac{1}{-\sqrt{1-(cos(acos(y)))^2}} && \text{because } sin(x) = \sqrt{1-cos(x)^2} \\
&= \frac{1}{-\sqrt{1-y^2}} \\
&= -\frac{1}{\sqrt{1-y^2}}
\end{aligned}
$$

3.5.3 atan(x)

$$\begin{aligned} atan(y)' &= \frac{1}{tan'(atan(y))} \\ &= \frac{1}{1+tan(atan(y))^2} \qquad \text{because } tan'(x) = \frac{1}{cos(x)^2} = 1 + tan(x)^2 \\ &= \frac{1}{1+y^2} \end{aligned}$$

3.6 Hiperbolic functions

3.6.1 senh(x)

$$\begin{aligned} sinh(x)' &= (\frac{e^x - e^{-x}}{2})' \\ &= \frac{e^x + e^{-x}}{2} \\ &= cosh(x) \end{aligned}$$

3.6.2 cosh(x)

$$\begin{aligned} cosh(x)' &= (\frac{e^x + e^{-x}}{2})' \\ &= \frac{e^x - e^{-x}}{2} \\ &= sinh(x) \end{aligned}$$

3.6.3 tanh(x)

Note first:

$$\begin{aligned} sinh(x)^2 &= (\frac{e^x - e^{-x}}{2})^2 \\ &= \frac{(e^x - e^{-x})(e^x - e^{-x})}{2^2} \\ &= \frac{e^{2x} - 1 - 1 + e^{2x}}{2^2} \\ &= \frac{2e^{2x} - 2}{2^2} \\ &= \frac{e^{2x} - 1}{2} \end{aligned}$$

Similarly:

$$\begin{aligned} sinh(x)^2 &= (\frac{e^x + e^{-x}}{2})^2 \\ &= \frac{e^{2x} + 1}{2} \end{aligned}$$

$$
\begin{aligned}
tanh(x)' &= \frac{sinh(x)'}{cosh(x)} \\
&= \frac{cosh(x)^2 - sinh(x)^2}{sinh(x)^2} \\
&= \frac{cosh(x)^2}{sinh(x)^2} - 1 \\
&= \frac{1}{tanh(x)^2} - 1 \\
&= cotanh(x)^2 - 1
\end{aligned}
$$

4 Derivative theorems

In the following subsections, I stands for (a, b) or $[a, b]$

4.1 A continuous function in $I = [a, b]$ is bounded (Extreme Value Theorem)

Property. *A continuous function in $I = [a, b]$ is bounded*

Preconditions:

- $f(x)$ is continuous on I

Claim: $Thereexistsf$

Proof:

$$f'(y)$$

□

4.2 Differentiability → continuity

Property. *Differentiability → continuity*

Preconditions:

- f is differentiable in $[a, b]$

Claim: f is continuous in (a, b)

Proof: Let f be differentiable at x_0. Then:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

So:

$$
\begin{aligned}
\lim_{x \to x_0} f(x) - f(x_0) &= \lim_{x \to x_0} \frac{(x - x_0)(f(x) - f(x_0))}{x - x_0} \\
&= (\lim_{x \to x_0} x - x_0) \lim_{x \to x_0} \frac{(f(x) - f(x_0))}{x - x_0} \\
&= (\lim_{x \to x_0} x - x_0) f'(x_0) \\
&= 0 f'(x_0) \\
&= 0
\end{aligned}
$$

Therefore:

□

$$\lim_{x \to x_0} f(x) - f(x_0) = 0$$

$$\lim_{x \to x_0} f(x) = f(x_0) \quad \textit{Limit of sum of functions, limit of a constant}$$

4.3 $f'(x) = 0$ at local maximums or minimums

Property. *If x_0 is a local maximum then $f'(x_0) = 0$*

Preconditions:

- $I = (a, b)$
- $f'(x_0) exists$
- x_0 is a local maximum in I

Claim: $f'(x_0) = 0$

Proof: Since x_0 is a local maximum in I, then $f(x) < f(x_0),\ \forall x \in I.$ Therefore, taking the limit by considering only those $x \in I$:

$$L^+ = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \quad \text{since } f(x) - f(x_0) \leq 0 \text{ and } x - x_0 > 0$$

But also:

$$L^- = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \quad \text{since } f(x) - f(x_0) \leq 0 \text{ and } x - x_0 < 0$$

But, since $f'(x_0) = L$ exists, then $L = L^+ = L^-$. This means that $L \leq 0$ and $0 \leq L$, which can only be true if $L = 0$, that is if $f'(x_0) = 0$

Note: if $x_0 = a$ or $x_0 = b$ then □

The proof for minimums proceeds analogously, or can be done applying the result for maximums to the function $g(x) = -f(x)$, since all minimums in f transform into maximums in g.

4.4 If a function is convex (concave) at a critical point x_0, then x_0 is a local maximum (minimum)

Property. *If $f'(x_0) = 0$ and $f''(x_0) < 0$ then x_0 is a local maximum*

Preconditions:

- $f'(x_0) = 0$
- $f''(x_0) > 0$

Claim: x_0 is a local maximum

Proof: Since $f''(x_0) < 0$, then $f''(x) < 0$ in an interval $I = (x_0^a, x_0^b)$ around x_0. Therefore, the function f' is *increasing* in that interval. Given that $f'(x_0) = 0$, and f' is increasing in I, then $f'(x) < 0$ if $x \in (x_0^a, x_0)$ and analogously $f'(x) > 0$ if (x_0, x_0^b).

But this means f is increasing in (x_0^a, x_0) and decreasing in (x_0, x_0^b). Therefore, x_0 is a local maximum.

□

The proof is analogous if the function is concave for minimums, or can be proved directly applying this result to the function $g(x) = -f(x)$.

4.5 Rolle's theorem

Property. *Rolle's theorem*

Preconditions:

- $I = [a, b] \neq \emptyset$
- $f(x)$ differentiable in I
- $f(a) = f(b)$

Claim: $\exists x_0 \in I$ such that $f'(x_0) = 0$

Proof: We'll divide this in two cases:

- $f(x) = d$, where d is a constant
  In this case, $f'(x) = 0 \; \forall x \in I$, so since $I \neq \emptyset$ we can pick any $x_0 \in I$ (say, $c = (a + b)/2$)
- $f(x) \neq d$
  Since f is not a constant, it must have at least one local maximum or minimum in I. Assume $x_0 \in I$ is a maximum or minimum. By the previous theorem, $f'(x_0) = 0$, so the proof is done.

□

4.6 Mean value theorem

Property. *Mean value theorem*

Preconditions:

- $I = [a, b] \neq \emptyset$
- $f(x)$ differentiable in I

Claim: $\exists x_0 \in I$ such that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$

Proof: This is similar to Rolle's theorem, but now we don't know if $f(a) = f(b)$, so we can't do exactly the same. Still, if we can twist the function f a bit into a function g for which $g(a) = g(b)$ and apply Rolle's theorem to g, maybe that'll help things along. The simplest way to do this is to substract a line from f; which line? the one that goes from $(a, f(a))$ to $(b, f(b))$:

$$\begin{aligned} g(x) &= f(x) - \text{"line" from } (a, f(a)) \text{ to } (b, f(b)) \\ &= f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] \end{aligned}$$

Note that:

$$\begin{aligned} g'(x) &= f'(x) - \frac{f(b) - f(a)}{b - a} \\ g(a) &= f(a) - f(a) + 0 = 0 \\ g(b) &= f(b) - [f(a) + f(b) - f(a)] = 0 \end{aligned}$$

Let's apply Rolle's theorem to g at x_0; then

$$\begin{aligned} &\exists x_0 \in I \text{ such that } g'(x_0) = 0 \\ &\exists x_0 \in I \text{ such that } f'(x_0) - \frac{f(b) - f(a)}{b - a} = 0 \\ &\exists x_0 \in I \text{ such that } f'(x_0) = \frac{f(b) - f(a)}{b - a} \end{aligned}$$

□

4.7 Cauchy's mean value theorem

Property. *Cauchy's mean value theorem*

Preconditions:

- $I = [a, b] \neq \emptyset$
- $f(x)$ differentiable in I
- $g(x)$ differentiable in I
- $g(b) \neq g(a)$

Claim: $\exists x_0 \in I$ such that $\frac{f'(x_0)}{g'(x_0)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Proof: This is a generalization of the mean value theorem, that simplifies to the former when $g(x) = x$, so that $g'(x) = 1$, $g(a) = a$ and $g(b) = b$.

Again we'll design a new function h for which $h(a) = h(b)$ and use Rolle's theorem to do the proof. Define:

$$h(x) = (g(b) - g(a))(f(x) - f(a)) - (f(b) - f(a))(g(x) - g(a))$$

Then:

$$\begin{aligned}
h'(x) &= (g(b) - g(a))f'(x) - (f(b) - f(a))g'(x) \\
h(a) &= (g(b) - g(a))0 - (f(b) - f(a))0 \\
&= 0 \\
h(b) &= (g(b) - g(a))(f(b) - f(a)) - (f(b) - f(a))(g(b) - g(a)) \\
&= 0
\end{aligned}$$

Therefore $h(a) = h(b)$ and h is differentiable in I, so we can apply Rolle's theorem to h to get:

$$\begin{aligned}
&\exists x_0 \in I \text{ such that} & h'(x_0) &= 0 \\
&\exists x_0 \in I \text{ such that} & (g(b) - g(a))f'(x_0) - (f(b) - f(a))g'(x_0) &= 0 \\
&\exists x_0 \in I \text{ such that} & (g(b) - g(a))f'(x_0) &= +(f(b) - f(a))g'(x_0) \\
&\exists x_0 \in I \text{ such that} & \frac{f'(x_0)}{g'(x_0)} &= \frac{f(b) - f(a)}{g(b) - g(a)}
\end{aligned}$$

□

4.8 Lagrange's remainder formula for Taylor's polynomial

Property. *Lagrange's remainder formula for Taylor's polynomial: Take 1, with Cauchy's MVT*

Preconditions:

- $f(x)$ n times differentiable everywhere
- $I_{a,x} = [min(a, x), max(a, x)]$
- $p_a(x) = f(a) + \sum_{i=1}^{n-1} f^i(a)(x - a)^i \frac{1}{i!}$ is defined.
- $\forall c \in I_{a,x} r_a(x) = f^n(c)(x - a)^n \frac{1}{n!}$ is defined.

Claim: $\forall x,\ \exists c \in I_{a,x}$ such that $f(x) = p_a(x) + r_a(x) = f(a) + \sum_{i=1}^{n-1} f^i(a)(x - a)^i \frac{1}{i!} + f^n(c)(x - a)^n \frac{1}{n!}$

Proof: Let $g(t) = p_t(x)$. That is, g is like p, but it is a function of t, the point where Taylor's polynomial is centered. Then $r_t(x) = f(x) - p_t(x)$ is the remainder or error of Taylor's polynomial when approximating $f(x)$ with the approximation centered at t.

Now let's differentiate g wrt t (note that here g is not a function of x!):

$$
\begin{aligned}
g'(t) &= (f(t) + \sum_{i=1}^{n-1} f^i(t)(x-t)^i \frac{1}{i!})' \\
&= f'(t) + \sum_{i=1}^{n-1} f^i(t)(x-t)^{i-1} i(-1)\frac{1}{i!} + f^{i+1}(t)(x-t)^i \frac{1}{i!} \\
&= f'(t) + \sum_{i=1}^{n-1} f^i(t)(x-t)^{i-1}(-1)\frac{1}{i-1!} + f^{i+1}(t)(x-t)^i \frac{1}{i!} \\
&= f'(t) + \sum_{i=1}^{n-1} f^i(t)(x-t)^{i-1}(-1)\frac{1}{i-1!} + \sum_{i=1}^{n-1} f^{i+1}(t)(x-t)^i \frac{1}{i!} \\
&= f'(t) - \sum_{i=1}^{n-1} f^i(t)(x-t)^{i-1}\frac{1}{i-1!} + \sum_{i=1}^{n-1} f^{i+1}(t)(x-t)^i \frac{1}{i!} \\
&= f'(t) - \sum_{i=1}^{n-1} f^i(t)(x-t)^{i-1}\frac{1}{i-1!} + \sum_{i=2}^{n} f^{i}(t)(x-t)^{i-1} \frac{1}{i-1!} \\
&= f'(t) - f^1(t)(x-t)^{1-1} - \sum_{i=2}^{n-1} f^i(t)(x-t)^{i-1}\frac{1}{(i-1)!} + \sum_{i=2}^{n-1} f^{i}(t)(x-t)^{i-1} \frac{1}{(i-1)!} + f^n(t)(x-t)^{n-1}\frac{1}{(n-1)!} \\
&= f'(t) - f'(t) - \sum_{i=2}^{n-1} f^i(t)(x-t)^{i-1}\frac{1}{i-1!} + \sum_{i=2}^{n-1} f^{i}(t)(x-t)^{i-1} \frac{1}{(i-1)!} + f^n(t)(x-t)^{n-1}\frac{1}{(n-1)!} \\
&= f^n(t)(x-t)^{n-1}\frac{1}{(n-1)!}
\end{aligned}
$$

This is quite nice! All the derivatives canceled each other, and we are left with just a single term. Now we'll use a simple trick to get the result we want. Note that our claim is similar to a mean value theorem in the sense that it is true for some $c \in I_{a,x}$. So maybe we can figure out another function h to apply Cauchy's MVT to g and h, and that will get us the result. That function is:

$$h(t) = (x-t)^n \quad (x \text{ is a constant here!})$$

Now, note that:

$$
\begin{aligned}
h(x) &= (x-x)^n = 0 & g(x) &= f_x(x) = f(x) \\
h(a) &= (x-a)^n & g(a) &= f_a(x) \\
h'(t) &= -n(x-t)^{n-1} & g'(t) &= f^n(t)(x-t)^{n-1}\frac{1}{(n-1)!}
\end{aligned}
$$

So if we apply Cauchy's MVT to h and g as functions of t in the interval $I_{a,x}$, we get (for a fixed x) that $\exists c \in I_{a,x}$ such that:

$$
\begin{aligned}
g'(c)(h(x)-h(a)) &= h'(c)(g(x)-g(a)) \\
g'(c)(-h(a)) &= h'(c)(f(x)-p_a(x)) \\
g'(c)(-(x-a)^n) &= h'(c)r_a(x) \\
g'(c)(-(x-a)^n) &= h'(c)r_a(x) \\
f^n(c)(x-c)^{n-1}\frac{1}{(n-1)!}(-(x-a)^n) &= -n(x-c)^{n-1}r_a(x) \\
f^n(c)\frac{1}{(n-1)!}(-(x-a)^n) &= -nr_a(x) \\
f^n(c)\frac{1}{n!}(-(x-a)^n) &= -r_a(x) \\
f^n(c)(x-a)^n\frac{1}{n!} &= r_a(x) \\
f^n(c)(x-a)^n\frac{1}{n!} &= f(x)-p_a(x) \\
f^n(c)(x-a)^n\frac{1}{n!}+p_a(x) &= f(x)
\end{aligned}
$$

□

5 Integrals

5.1 Definite integral definition

Preconditions:

- $f(x)$ is bounded on $[a, b]$
- $f(x) \geq 0$ $[a, b]$

To arrive to the definition of the definite integral, we'll make a few other definitions in the way that prepare us to speak about subintervals of $[a, b]$ and the maximum and minimum value of f in those subintervals.

Let P_n be any complete partition of $[a, b]$.

$P_n = \{ I_1, \ldots, I_n \}$ is composed of n non empty subintervals $I_k = x_k, x_{k+1}$, $k = 1, \ldots, n$, such that

$$\bigcup_{k=1}^{n} I_k = [a, b]$$

Also, note that $a = x_1$ and $b = x_n$, and $x_k > x_{k+1}$.

Define the length of a subinterval I_k as $|I_k| = x_{k+1} - x_k$.

Now define, for each interval I_k, f_k and F_k as the smallest and greatest values of f in that interval, which surely exists given the boundedness of f in $[a, b]$. Then define the sums s_n and S_n:

$$
\begin{aligned}
s_n &= \sum_{k=1}^{n} |I_k| f_k \\
S_n &= \sum_{k=1}^{n} |I_k| F_k
\end{aligned}
$$

Since $f_k \leq f(x) \leq F_k$, $\forall x \in$, then $s_n < S_n$. Now, if the following limits both exist:

$$s = \lim_{n \to +\infty} s_n S = \lim_{n \to +\infty} S_n$$

Then we know $s \leq S$. If these limits are the same, ie $s = S$, then $\int_a^b f(x) = s = S$.

5.2 Linearity of integrals

Property. *p*

Preconditions:

- $f(x)$ is integrable on $[a, b]$.

Claim: f

Proof:

$$f'(y)$$

□

5.3 Fundamental theorem of calculus (1 and 2)

Property. *Fundamental theorem of calculus 1*

Preconditions:

- $f(x)$ continuous in $I = [a, b]$
- $x \in I$

Claim: $G'(x) = (\int_a^x f(t)dt)' = f(x)$

Proof: By definition of the derivative, we have:

$$G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} \qquad \text{(here, } a < x + h < b\text{, which can be ensured because we are dealing with a limit)}$$

Note that:

$$\begin{aligned} G(x+h) - G(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt + \int_a^x f(t)dt - \int_a^x f(t)dt \quad \text{by linearity of the integral} \\ &= \int_x^{x+h} f(t)dt \end{aligned}$$

So actually we can simplify the derivative's limit to:

$$G'(x) = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)dt}{h}$$

Now, define M_h as the maximum value f takes in the interval $(x, x+h)$, and m_h the minimum. Note that both these values depend on h. Also note that we know these exist because of the continuity of f. Then:

$$\begin{aligned} \int_x^{x+h} m_h dt \leq \int_x^{x+h} f(t)dt &\leq \int_x^{x+h} M_h dt \qquad (m_h \leq f(t) \leq M_h) \\ m_h \int_x^{x+h} dt \leq \int_x^{x+h} f(t)dt &\leq M_h \int_x^{x+h} dt \qquad M_h \text{ and } m_h \text{ are constants wrt } t \\ m_h h \leq \int_x^{x+h} f(t)dt &\leq M_h h \end{aligned}$$

If $h > 0$, then:

$$
\begin{aligned}
m_h h \leq \int_x^{x+h} f(t)dt \leq M_h h \\
m_h \leq \frac{\int_x^{x+h} f(t)dt}{h} \leq M_h \quad \text{(dividing by h)}
\end{aligned}
$$

Taking the limit as $h \to 0^+$:

$$
\begin{aligned}
&\lim_{h\to 0^+} m_h \leq \lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)dt}{h} \leq \lim_{h\to 0^+} M_h \\
&\lim_{h\to 0^+} m_h \leq G'(x) \leq \lim_{h\to 0^+} M_h \qquad \text{by definition of G'(x) and previous derivation}
\end{aligned}
$$

Now, as $h \to 0^+$, $x' \in (x, x+h)$ tends to x, therefore $\lim_{h\to 0^+} M_h = \lim_{h\to 0^+} m_h = f(x)$ This gives

$$
f(x) \leq G'(x) \leq f(x)
$$

Therefore, this implies the derivative G' exists and furthermore that $G'(x) = f(x)$ (when $h > 0$).
If $h < 0$, the same idea applies, except that when dividing by h now the inequalities get inverted:

$$
\begin{aligned}
m_h h \leq \int_x^{x+h} f(t)dt \leq M_h h \\
m_h \geq \frac{\int_x^{x+h} f(t)dt}{h} \geq M_h \quad \text{(dividing by h)}
\end{aligned}
$$

Taking the limit as $h \to 0^-$:

$$
\begin{aligned}
\lim_{h\to 0^-} m_h \geq \lim_{h\to 0^-} \frac{\int_x^{x+h} f(t)dt}{h} \geq \lim_{h\to 0^-} M_h \\
\lim_{h\to 0^-} m_h \geq G'(x) \geq \lim_{h\to 0^-} M_h \\
f(x) \geq G'(x) \geq f(x)
\end{aligned}
$$

Since both limits agree, we have $G'(x) = f(x)$.

□

Property. *Fundamental theorem of calculus, part 2*

Preconditions:

- $f(x)$ continuous in $I = [a, b]$
- $G(x) = \int_a^x f(t)dt$
- F is a primitive of f, ie, $F'(x) = f(x)$ (which exists because f is continuous)

Claim: $\int_a^b f(t)dt = F(b) - F(a)$

Proof: Since $G'(x) = f(x)$ (by part 1 of the theorem) and $F'(x) = f(x)$ (by definition), F and G can only differ in a constant, so:

$$
G(x) = F(x) + C
$$

Since $G(a) = \int_a^a f(t)dt = 0$, then $G(a) = F(a) + C = 0$, so $C = -F(a)$. Replacing C in the previous equation, we get:

$$G(x) = F(x) - F(a)$$

That equation works for all $x \in I$. In particular it works for $x = b$, which gives us:

$$G(b) = F(b) - F(a)$$
$$\int_a^b f(t)dt = F(b) - F(a)$$

□

5.4 Mean value theorem for Integrals

Property. *Mean value theorem for integrals (version 1, assuming the FTC)*

Preconditions:

- $f(x)$ is continuous on $I = [a, b]$

Claim: $\exists c \in (a, b)$ such that $\int_a^b f(x)dx = f(c)(b - a)$

Proof: Let G be defined as before. Applying the MVT to G in the interval I, we have $\exists c \in (a, b)$ such that:

$$G'(c) = \frac{G(b) - G(a)}{(b - a)}$$
$$f(c) = \frac{G(b) - G(a)}{(b - a)}$$
$$f(c) = \frac{G(b)}{(b - a)}$$
$$f(c)(b - a) = G(b)$$
$$f(c)(b - a) = \int_a^b f(x)dx$$

□

Property. *Mean value theorem for integrals (version 2, without assuming the FTC)*

Preconditions:

- $f(x)$ is continuous on $I = [a, b]$

Claim: $\exists c \in (a, b)$ such that $\int_a^b f(x)dx = f(c)(b - a)$

Proof:

prove me

□

5.4.1 Common integral substitutions

$\int \frac{\sqrt{a^2x^2-b^2}}{x}$

$\int \frac{\sqrt{a^2x^2+b^2}}{x}$

$\int \frac{\sqrt{-a^2x^2+b^2}}{x}$

6 Sequences

6.1 Convergence definition

A sequence converges to L if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N, |a_n - L| < \epsilon$.

6.2 Monotonous and bounded sequences converge

Property. *If a sequence a_n is bounded above and is increasing, then it converges*

Preconditions:

- $\exists L, \forall n \; a_n \leq L$
- $\forall n \; a_n \leq a_{n+1}$

Claim: $\lim_{n \to \infty} a_n$ converges to the supremum of $A = \{ a_n \mid n \in \mathbb{N} \}$.

Proof: Since a_n is bounded, then we have that $\alpha = \sup A$ exists by properties of the real numbers.

By definition of supremum, $\forall \epsilon > 0, \exists N$ such that:

$$\alpha - \epsilon < a_N < \alpha$$

Since a_n is increasing:

$$\forall n, \; N \leq n, \; a_N \leq a_n$$

Also by definition:

$$\forall n, \; a_n \leq \alpha$$

Combining these two results, we get $\forall n \geq N, \; \alpha - \epsilon < a_n < \alpha$. This implies:

$$\forall n \geq N, \; \alpha - a_n < \epsilon$$

Again, since $a_n \leq \alpha$, we have:

$$\forall n, \alpha - a_n = |\alpha - a_n|$$

Therefore, given $\epsilon > 0$, we have $\forall n \geq N, \; |\alpha - a_n| < \epsilon$, which by definition means:

$$\lim_{n \to \infty} a_n = \alpha$$

□

7 Series

Note: Most of the following tests/theorems assume that series/sequences that start at $n = 1$, but the proofs can be easily applied to n starting at other values by considering a sequence such as $a_n = a_{m+k}$.

Also, they assume that certain properties of a sequence hold from $n = 1$ onwards; for example, $\forall n > 1, a_n > 0$. This is to simplify the proofs.

In practice, a property may only hold for n greater than some N. This poses little trouble since a series can be split at N, so that:

$$= \sum_{i=1}^{N} a_i + \sum_{N}^{\infty} a_i$$

The first part of the series up to N is finite, so there it is trivially convergent and we have no need to apply tests to it. The second part is the interesting one, and if we define $S_n = \sum_{N}^{n+N} a_i$, we get a sequence starting at $n = 1$ that hold the partial sums of the interesting part of the series, so we can apply our theorem/test to that part only.

7.1 Divergence theorem

Property. *If* $\sum_{i=1}^{n} a_i$ *converges, then* $\lim\limits_{n\to\infty} a_n = 0$

Preconditions:

- $\sum_{i=1}^{n} a_i = L$ (ie, converges)

Claim: $\lim\limits_{n\to\infty} a_n = 0$

Proof: Let $S_n = \sum_{i=1}^{n} a_i$. Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} S_{n+1} = L$

Now, notice that:

$$\begin{aligned}\lim_{n\to\infty} S_{n+1} - S_n &= \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n \\ &= L - L \\ &= 0\end{aligned}$$

But $S_{n+1} - S_n = a_{n+1}$. So:

$$\begin{aligned}\lim_{n\to\infty} S_{n+1} - S_n &= \lim_{n\to\infty} a_{n+1} \\ &= \lim_{n\to\infty} a_n \\ &= 0\end{aligned}$$

□

Corollary (Divergence Theorem):

Property. *If* $\lim\limits_{n\to\infty} a_n \neq 0$ *then* $\lim\limits_{n\to\infty} \sum_{i=1}^{n} a_i$ *diverges*

Preconditions:

- $\lim\limits_{n\to\infty} a_n \neq 0$

Claim: $\lim\limits_{n\to\infty} \sum_{i=1}^{n} a_i$ diverges.

Proof: Suppose both $\lim\limits_{n\to\infty} a_n \neq 0$ and $\lim\limits_{n\to\infty} \sum_{i=1}^{n} a_i$ converges. Then by the previous property, $\lim\limits_{n\to\infty} a_n = 0$. So by contradiction, $\lim\limits_{n\to\infty} \sum_{i=1}^{n} a_i$ must diverge. □

7.2 Comparison test

Property. *If a series* $\sum_{n=1}^{\infty} a_n$ *is bounded by* $\sum_{n=1}^{\infty} b_n$ *and the latter converges, so does the former*

Preconditions:

- $0 \leq a_n \leq b_n$
- $\sum_{n=1}^{\infty} b_n = L$ (ie, converges)

Claim: $\sum_{n=1}^{\infty} a_n$ converges

Proof: Define:

$$A_n = \sum_{i=1}^{n} a_i \qquad B_n = \sum_{i=1}^{n} b_i \qquad B = \lim_{n\to\infty} B_n$$

Since $0 \leq a_n \leq b_n$, we must have $A_n \leq B_n$. Also, since $0 \leq b_n$, we have $B_n \leq B$. Therefore $A_n \leq B$, so A_n is bounded above. Since $0 \leq a_n$, A_n must form an increasing sequence. Since a bounded above, increasing sequence converges, so does A_n, so $A = \lim\limits_{n\to\infty} A_n = \sum_{n=1}^{\infty} a_n$ exists (ie, the series converges).

□

Corollary:

Property. *If a series $\sum_{n=1}^{\infty} a_n$ is bounded by $\sum_{n=1}^{\infty} b_n$ and the former diverges, so does the latter*

Preconditions:

- $0 \leq a_n \leq b_n$
- $\sum_{n=1}^{\infty} a_n$ diverges

Claim: $\sum_{n=1}^{\infty} b_n$ diverges

Proof: Suppose $\sum_{n=1}^{\infty} b_n$ converges. By the previous theorem, $\sum_{n=1}^{\infty} a_n$ converges. But this contradicts our hypothesis. Therefore, $\sum_{n=1}^{\infty} b_n$ diverges $\square$

7.3 Limit comparison test

Property. *If the limit ratio between the terms of two series exists and is not 0, then either both converge or both diverge*

Preconditions:

- $0 \leq a_n$
- $0 < b_n$
- $0 < \lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$ (ie, converges to a number not zero)

Claim: $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges

Proof: Since $0 < L$, we can find $m, M \in \mathbb{R}$ such that:

$$m < L < M$$

Now, since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$, we can find a number N such that $\frac{a_n}{b_n}$ is really close to L, so much that:

$$\forall n > N,\ m < \frac{a_n}{b_n} < M$$

This can be ensured by taking $\epsilon = \min\{\,|L-m|, |M-L|\,\}$ in the definition of the limit of a sequence.
Multiplying by b_n:

$$\forall n > N,\ mb_n < a_n < Mb_n$$

Since multiplying by a constant does not change the convergence properties of a series, this means that for $n \geq N$, b_n is bounded above by a_n, and viceversa.
Since we can split both series as:

$$= \sum_{i=1}^{N} a_i + \sum_{N}^{\infty} a_i \qquad\qquad = \sum_{i=1}^{N} b_i + \sum_{N}^{\infty} b_i$$

And both $\sum_{i=1}^{N} a_i$ and $\sum_{i=1}^{N} b_i$ are finite,
Therefore, $mb_n < a_n$ and the comparison test, imply that if a_n converges so does b_n. Since $a_n < Mb_n$, again by the comparison test if b_n converges so does a_n.
Finally:

$$a_n \text{ converges} \Leftrightarrow b_n \text{ converges}$$

$\square$

7.4 Absolute converges implies (normal) convergence

Property. *Absolute converges implies (normal) convergence*

Preconditions:

- $\sum_{n=1}^{\infty} |a_n| = L$

Claim: $\sum_{n=1}^{\infty} a_n = M$

Proof: Given that

$$\begin{aligned} a_n &\leq |a_n| \\ a_n + |a_n| &\leq |a_n| + |a_n| \\ a_n + |a_n| &\leq 2|a_n| \\ a_n + |a_n| &\leq 2|a_n| \end{aligned}$$

Define $b_n = a_n + |a_n|$. Notice $0 \leq b_n$. Since $|a_n|$ converges, then $2|a_n|$ converges as well. Since $0 \leq b_n \leq 2|a_n|$, by the comparison test b_n converges as well.

Now, since $b_n = a_n + |a_n|$ converges, and $|a_n|$ converges, then we must have a_n converge as well. □

7.5 Integral test

Property. *$\int_1^{\infty} f(x)$ converges iff $\sum_{n=1}^{\infty} a_n$ converges*

Preconditions:

- $a_n = f(n)$ if $1 \leq n$
- $f(x) > 0$ if $1 \leq x$
- $f(x)$ decreasing if $1 \leq x$

Claim: $I = \int_1^{\infty} f(x)$ converges iff $S = \sum_{n=1}^{\infty} a_n$ converges

Proof: Since f is increasing, we have that if $x \in [i, i+1]$ then $a_{i+1} \leq f(x) \leq a_i$. So if we define $I_n = \int_1^n f(x)$, $S_n = \sum_{i=1}^{n-1} a_i$ and $S'_n = \sum_{i=2}^{n} a_i$ then $S'_n \leq I_n \leq S_n$ because $\int_1^n f(x) = \sum_{i=1}^{n-1} \int_i^{i+1} f(x)$

Suppose I converges. Since $S'_n \leq I_n$, and $I = \lim_{n\to\infty} I_n$, then $\lim_{n\to\infty} S'_n$ converges. Now, since $S_n = a_1 + S'_n$, then $\lim_{n\to\infty} S_n = S$ converges as well.

Conversely, suppose S converges. Since $I_n \leq S_n$, and $S = \lim_{n\to\infty} S_n$, then $\lim_{n\to\infty} I_n = I$ converges.

□

7.6 Ratio test

Property. *If $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists, then $L < 1 \to \sum_{n=1}^{\infty} a_n$ converges, and $L > 1 \to \sum_{n=1}^{\infty} a_n$ diverges*

Preconditions:

- $0 \leq a_n$
- $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists

Claim:

- $L < 1 \to \sum_{n=1}^{\infty} a_n$ converges
- $L > 1 \to \sum_{n=1}^{\infty} a_n$ diverges

Proof: Case $L < 1$:

The idea of this proof is that as $\frac{a_{n+1}}{a_n}$ gets close to L so that all the $\frac{a_{n+1}}{a_n}$ are smaller than some $r < 1$, then since $\frac{a_{n+1}}{a_n} < r$, we have $a_{n+1} < ra_n$, $a_{n+2} < r^2 a_n$, and so $a_{n+k} < r^k a_n$, so we can actually compare the series to a geometric series with $a < 1$, which converges.

So, since $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, by density of the reals:

$$\exists r, L < r < 1$$

When $\frac{a_{n+1}}{a_n}$ gets really close to L, we will have $\frac{a_{n+1}}{a_n} < r$. Formally

$$\exists N,\ \forall n > N,$$

¡r

This implies:

$$\exists N,\ \forall n > N,$$

a$_{n+1} < ra_n$

□

7.7 Root test

Property. *$\int_k^\infty f(x)$ converges iff $\sum_{n=1}^\infty a_n$ converges*

Preconditions:

- $a_n = f(n)$ if $k \leq n$
- $f(x) > 0$ if $k \leq x$
- $f(x)$ decreasing if $k \leq x$

Claim: $\int_k^\infty f(x)$ converges iff $\sum_{n=1}^\infty a_n$ converges

Proof: prove me

□

7.8 Alternating series test

Property. *Alternating series test*

Preconditions:

- $a_n = f(n)$ if $k \leq n$
- $f(x) > 0$ if $k \leq x$
- $f(x)$ decreasing if $k \leq x$

Claim: $\int_k^\infty f(x)$ converges iff $\sum_{n=1}^\infty a_n$ converges

Proof: prove me

□

8 Differential Equations