Rejection Sampling - Beta Distribution

October 22, 2021

1 Generating Random Numbers from the Beta Distribution Using Rejection Sampling

Arman Rezaei - 9723034

- The *target* distribution $f(x)$ is the Beta distribution with parameters $\alpha = 3$ and $\beta = 4$
- The initial *trial* distribution $g(x)$ is a Normal distribution

1.1 Comparing the Normal & Beta Distributions

[ ]:

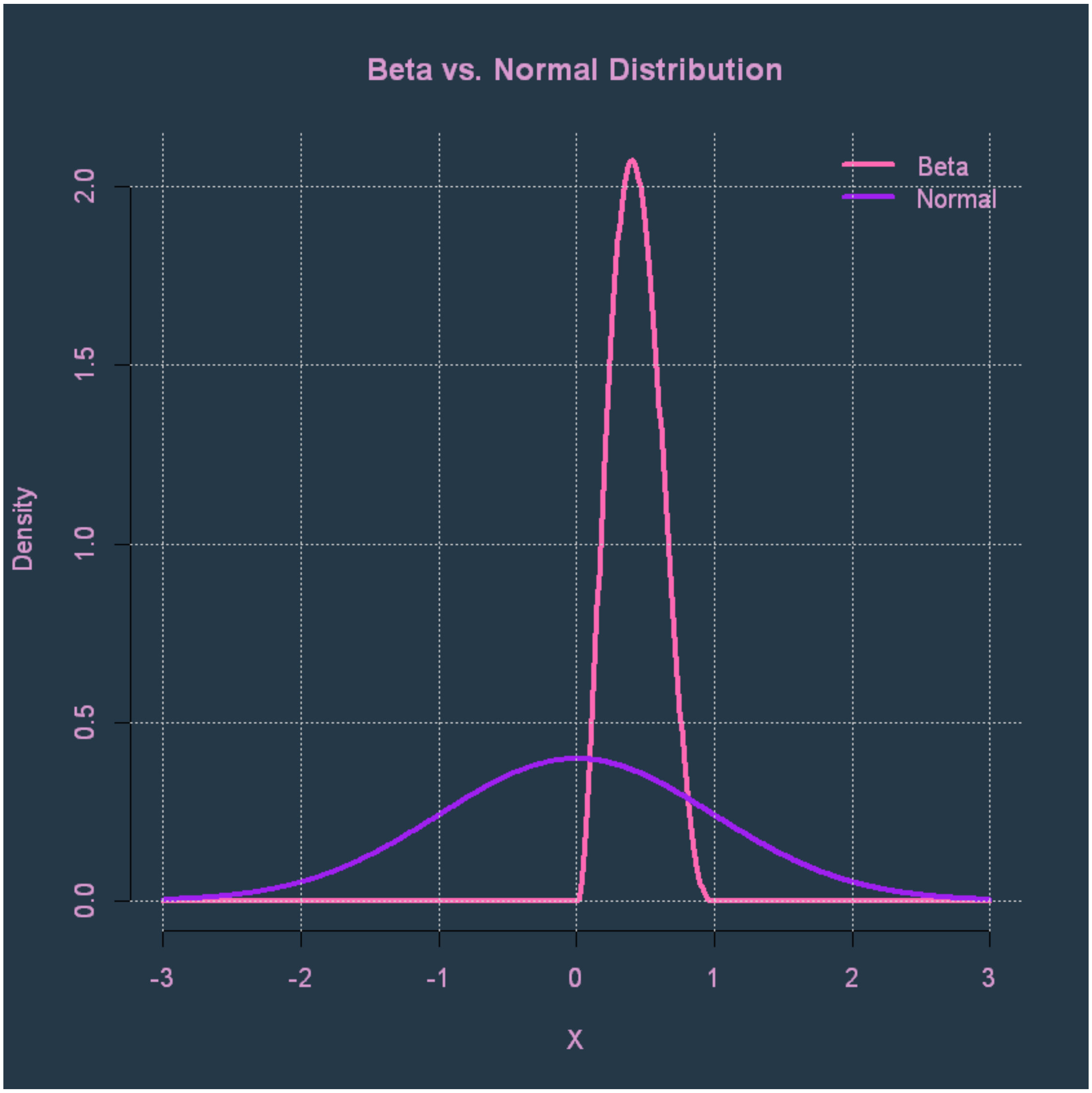
```
# range
x <- seq(-3, 3, .01)

# beta dist params
alpha = 3
beta = 4

# plotting params
theme.col = "#d99ad0"
par(bty="n", bg="#253847", col.axis=theme.col, col.lab=theme.col, col.
 ↪main=theme.col)

# actual plots
plot(x, dbeta(x, alpha, beta), type="l", col="hotpink", lwd=3.5, xlab="X",␣
 ↪ylab="Density")
lines(x, dnorm(x), col="purple", lwd=3.5)

# misc
title(main="Beta vs. Normal Distribution")
legend("topright", legend=c("Beta", "Normal"), lty=1, lwd=3.5, col=c("hotpink",␣
 ↪"purple"), bty="n", text.col=theme.col)
grid()
```



1.2 Calculating M

There is, however, one problem. Lets have a look at our PDF functions:

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha - 1}(1 - x)^{\beta - 1}$$

Which for $\alpha = 3$ and $\beta = 4$ we have

$$f(x; 3, 4) = 60 \cdot x^2 \cdot (1 - x)^3$$

And for the standard normal distribution:

$$g(x;0,1) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

In order to calculate M:

$$\frac{f(x)}{g(x)} = 60\sqrt{2\pi}\frac{x^2(1-x)^3}{e^{\frac{-x^2}{2}}} \leq M$$

Calculating the derivative of this function in order to find M can be a nightmare. Therefore, we will use a much simpler (and more inefficient) $g(x)$ such as the Standard Uniform distribution's PDF:

$$g(x) = 1$$

And now calculating M is only a matter of calculating the derivative of f and founding its global maximum.

$$f'(x) = -180x^2(1-x)^2 + 120x(1-x)^3 = 0 \Rightarrow x = \frac{2}{5} \Rightarrow f(x) \leq f(\frac{2}{5}) \simeq 2.1 = M$$

1.3 Generating Beta Samples

[ ]:
```
# sample size
n <- 10000
M <- 2.1

# function to determine acception/rejection
accept <- function(x) {
    u <- runif(1)
    if ( 1/M * 60 * x^2 * (1-x)^3 > u ) {
        return(TRUE)
    }
    return(FALSE)
}


# trials
X <- rep(0, n)
niter <- 0
sample <- NULL

for (i in 1:n) {
    # keep trying until acception
    flag <- FALSE
    while (flag == FALSE) {
        # increment counter
        niter <- niter + 1
```

```
        # generate candidate
        sample <- runif(1)
        flag <- accept(sample)
    }

    X[i] <- sample
}
```

1.4 Measuring Closeness to Reality

[ ]:
```
niter
```

21047

Since the probability of acceptance is $1/M$, the results should be consistent with it:

$$\text{Theoretical number of iterations required} = M \cdot n = 2.1 \times 10000 = 21000$$

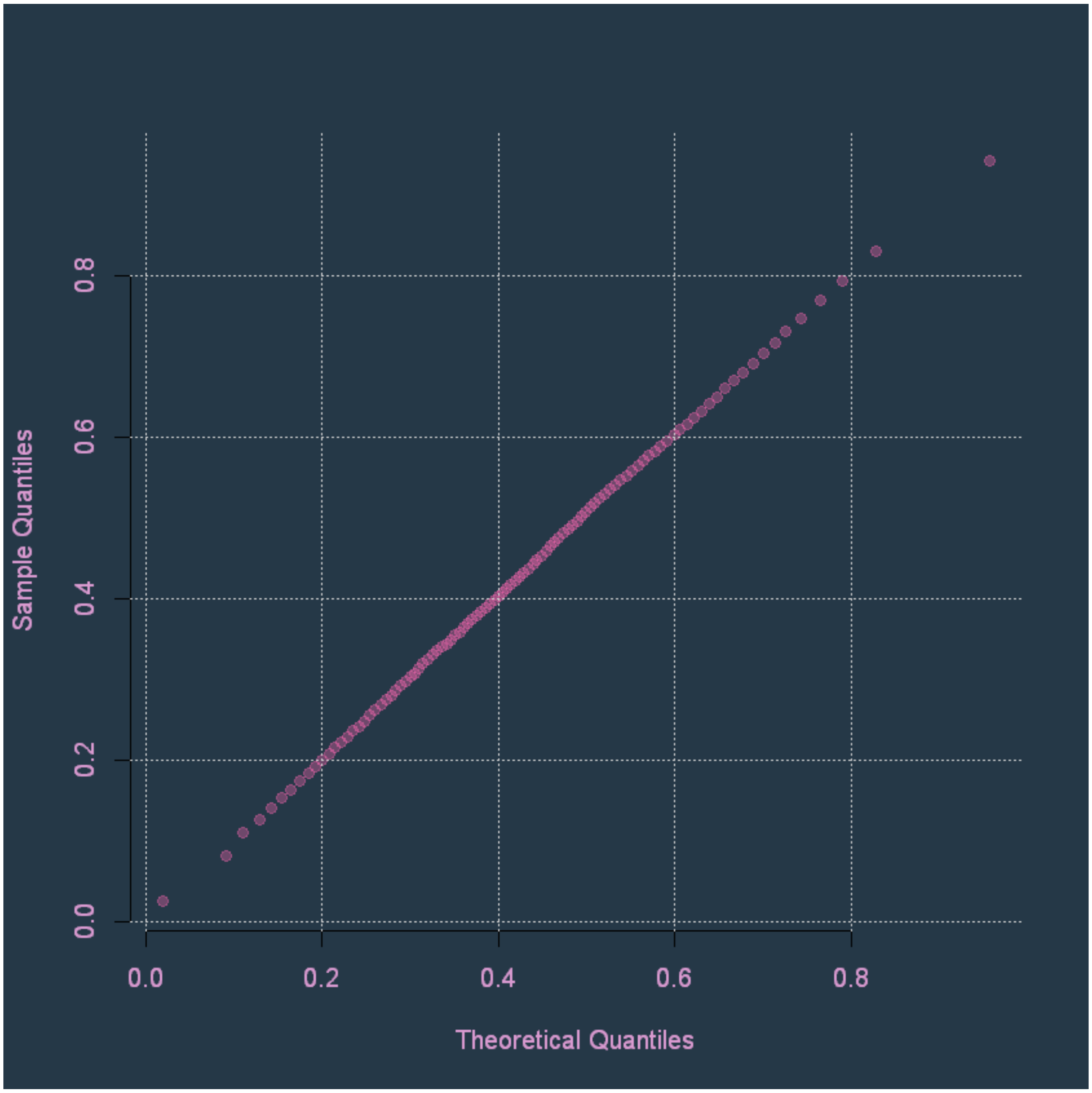
$$P_{theoretical}(\text{accept}) = \frac{1}{M} \simeq 0.48$$

$$P_{actual}(\text{accept}) = \frac{n}{n_{iter}} = \frac{10000}{21200} \simeq 0.47$$

The results seem consistent enough. Lets use a quantile-quantile plot to visually see whether the random numbers generated are close to the actual distribution.

[ ]:
```
probs <- seq(0, 1, 0.01)
x <- quantile(rbeta(n, alpha, beta), probs)
y <- quantile(X, probs)

par(bty="n", bg="#253847", col.axis=theme.col, col.lab=theme.col, col.
 ↪main=theme.col)
plot(x, y, xlab="Theoretical Quantiles", ylab="Sample Quantiles",␣
 ↪col="#FF69B455", pch=19)
# lines(c(x[10], x[90]), c(y[10], y[90]), col="purple", lwd=2)
grid()
```



We can see that there is a linear relation between the quantiles, therefore proving the theory that the generated results have a Beta distribution.