Transmission Line Theory

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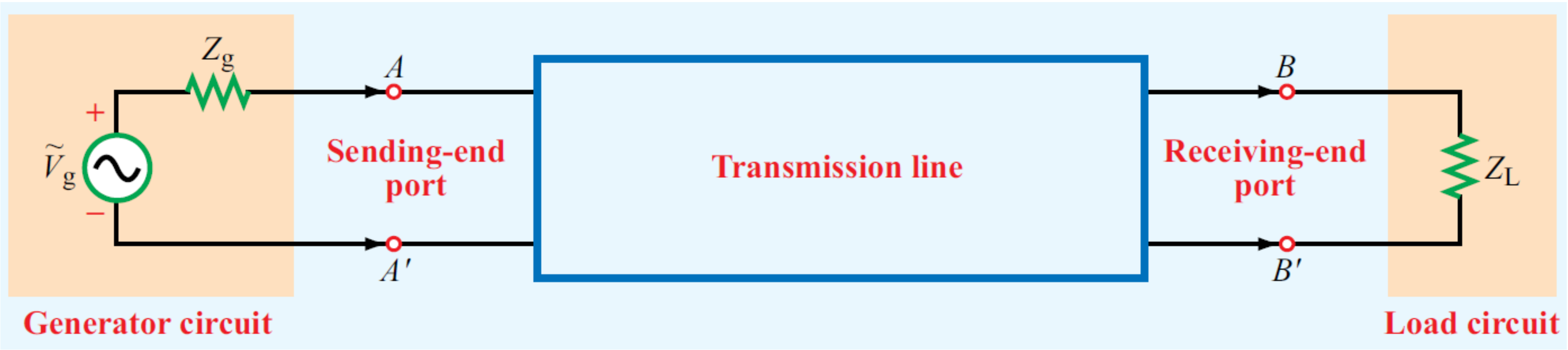
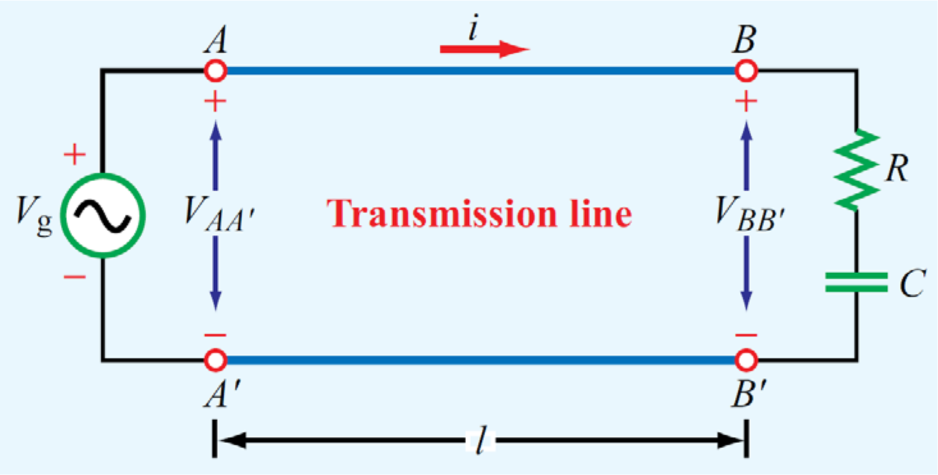
Introduction

- Transmission line: a bridge between circuit theory and electromagnetic theory.
- By modeling transmission lines in the form of equivalent circuits, we can use Kirchhoff's voltage and current laws to develop wave equations whose solutions provide an understanding of wave propagation, standing waves, and power transfer.
- Fundamentally, a transmission line is a two-port network, with each port consisting of two terminals. One of the ports, the line's sending end, is connected to a source (also called the generator). The other port, the line's receiving end, is connected to a load.



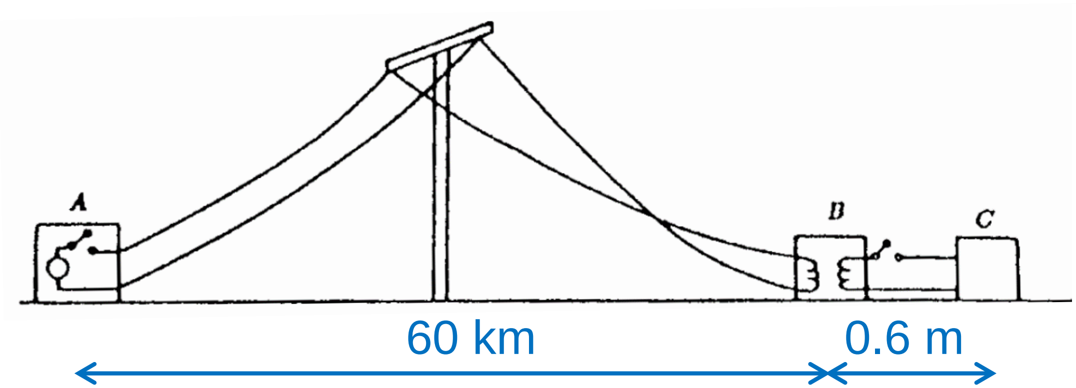
Role of wavelength

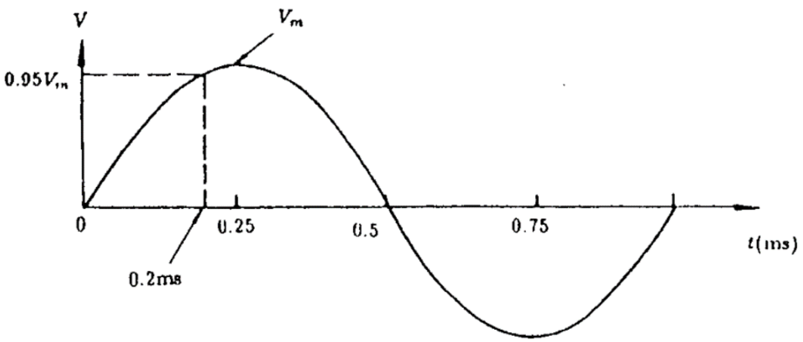
- Is the pair of wires between terminals AA' and and terminals BB' a transmission line? If so, under what set of circumstances should we explicitly treat the pair of wires as a transmission line, as opposed to ignoring their presence?
  - The factors that determine whether or not we should treat the wires as a transmission line are governed by the length of the line l and the frequency f of the signal provided by the generator.
  - When l/λ is very small, transmission-line effects may be ignored, but when $l/\lambda \gtrsim 0.01$, it may be necessary to account not only for the phase shift due to the time delay, but also for the presence of reflected signals that may have been bounced back by the load toward the generator.



Example

- Assume that $V = V_m \sin 2\pi f t$ with $f = 1\ \mathrm{kHz}$ when A is closed. The corresponding wavelength is 300 km and is comparable to the distance between A and B. This voltage signal will not appear instantaneously at point B, as a certain time is required for the signal to travel from points A to B. → transmission line



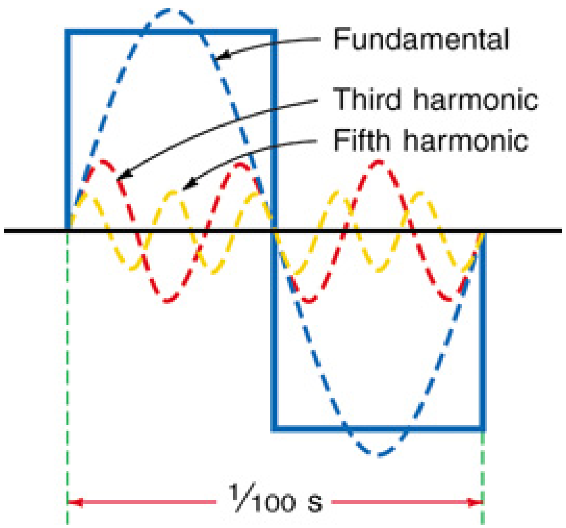


- How about B-C? Because the distance between B and C is small, the voltage at point B will appear almost instantaneously at point C. → electric circuit

- What if the frequency is increased to 100 MHz?

Transmission line vs. electric circuit

- Generally, when the wavelength of the applied voltage is comparable to or shorter than the length of the conductors, the conductors must be treated as a transmission line.
- A single pulse is composed of an infinite number of Fourier components with the amplitudes of the spectrum decreasing for higher frequencies.



- Empirical formula to estimate the upper-frequency f of the pulse:

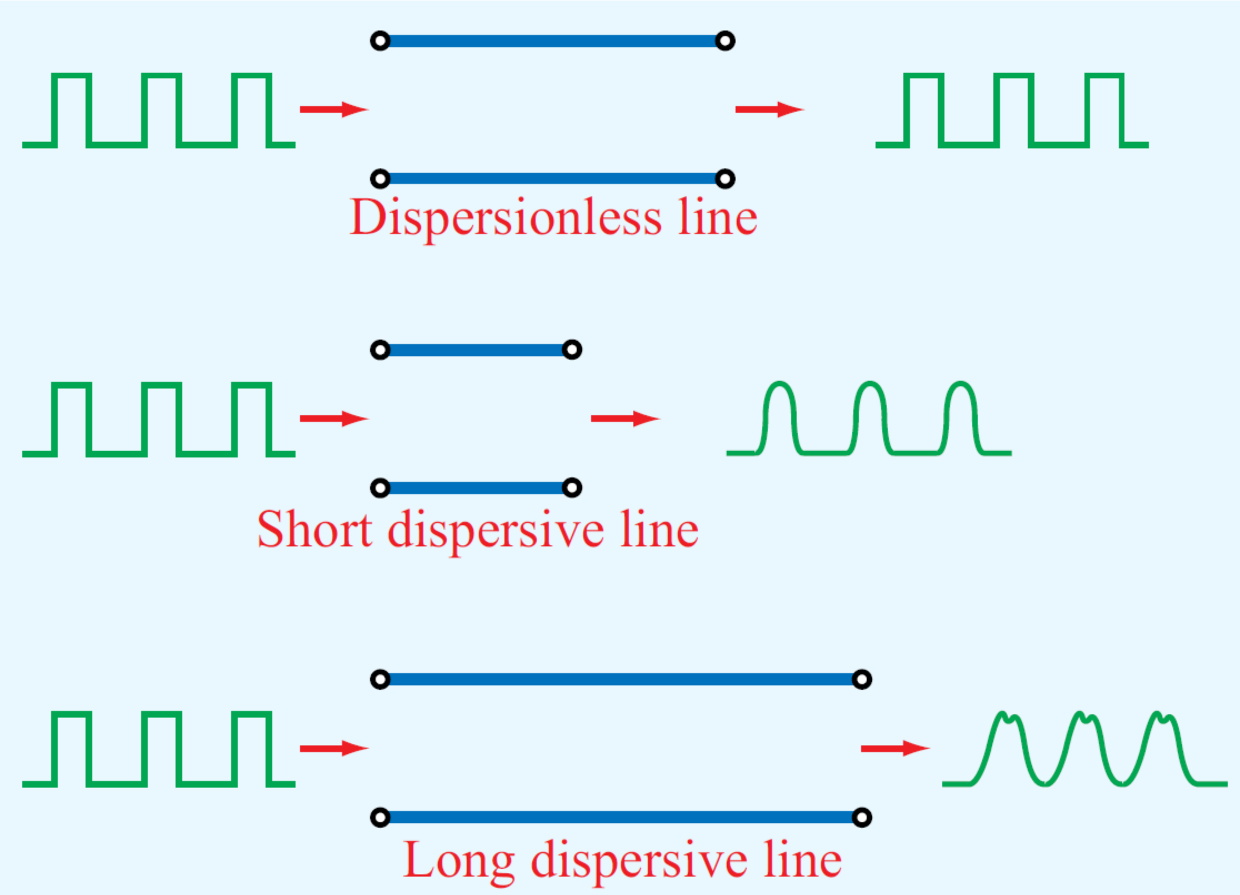
$$f = \frac{K}{t_r}$$

where, t_r is the rise time of the pulse and K is a constant between 0.35 and 0.45 depending on the shape of the pulse.

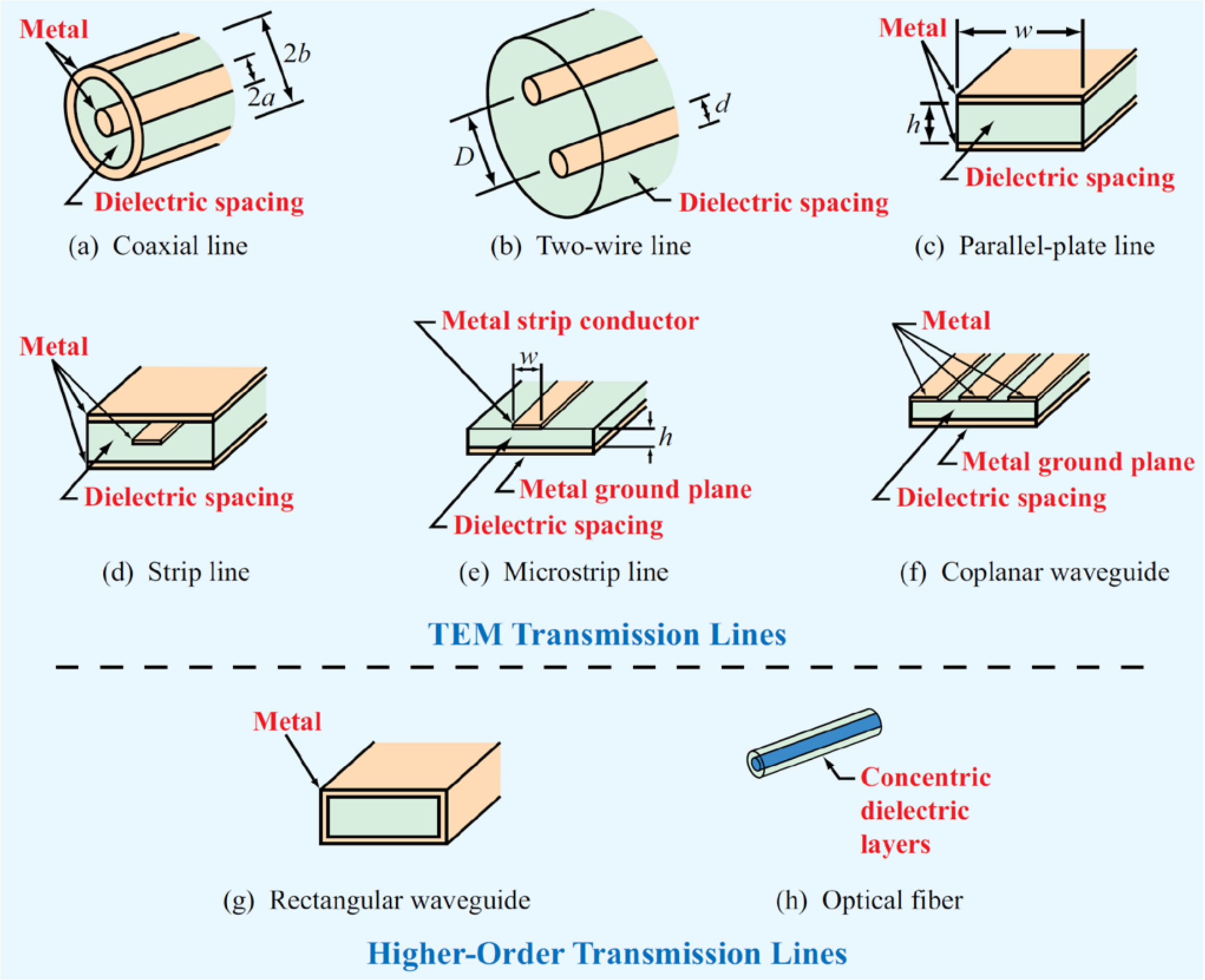
- In pulsed power, the conductors must be treated as transmission line, because the equivalent wavelength of the pulse in most cases is comparable to the length of the conductors.

Dispersive effects

- A dispersive transmission line is one on which the wave velocity is not constant as a function of the frequency f.

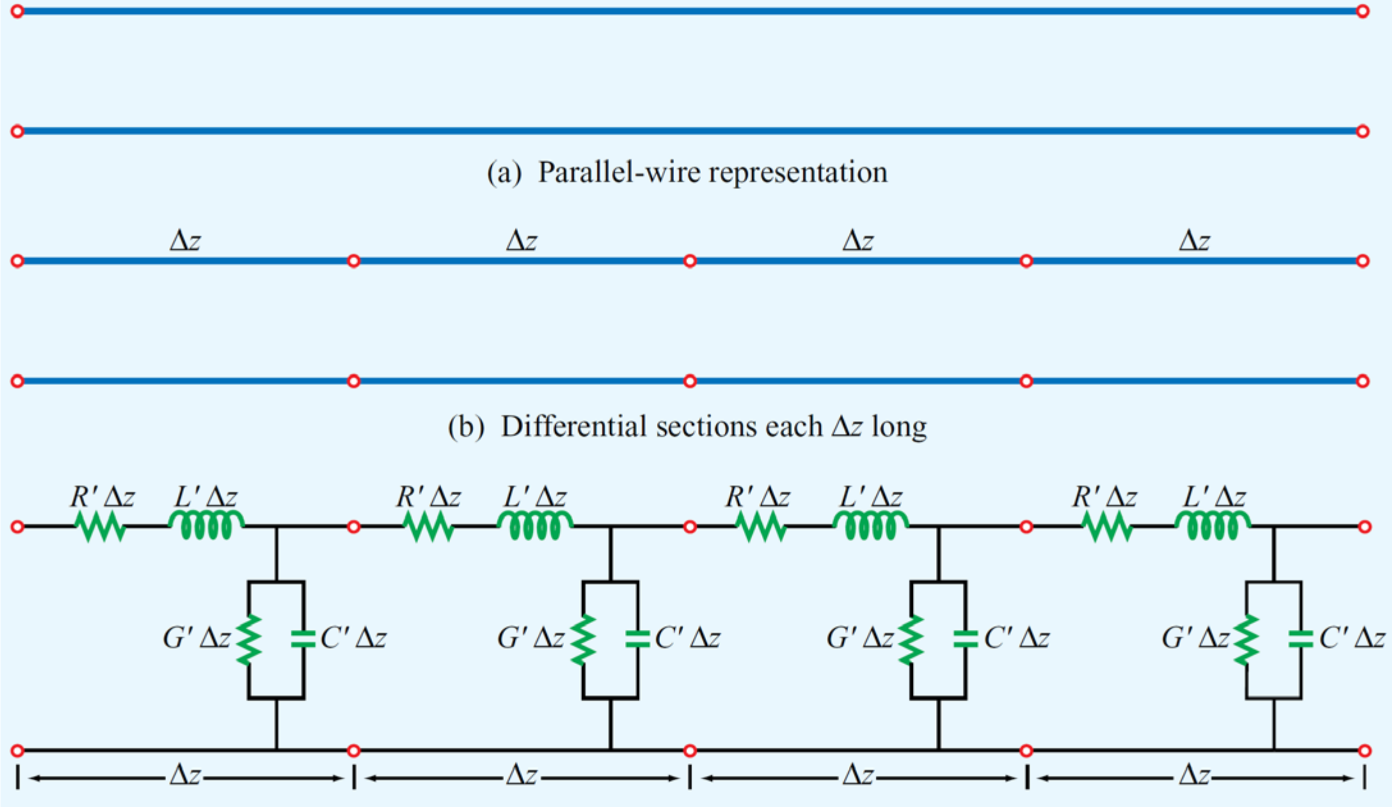


Transverse electromagnetic (TEM) and higher-order transmission lines



Lumped-element model

- A transmission line can be represented by a parallel wire configuration, regardless of its specific shape or constitutive parameters.
- To obtain equations relating voltages and currents, the line is subdivided into small differential sections.



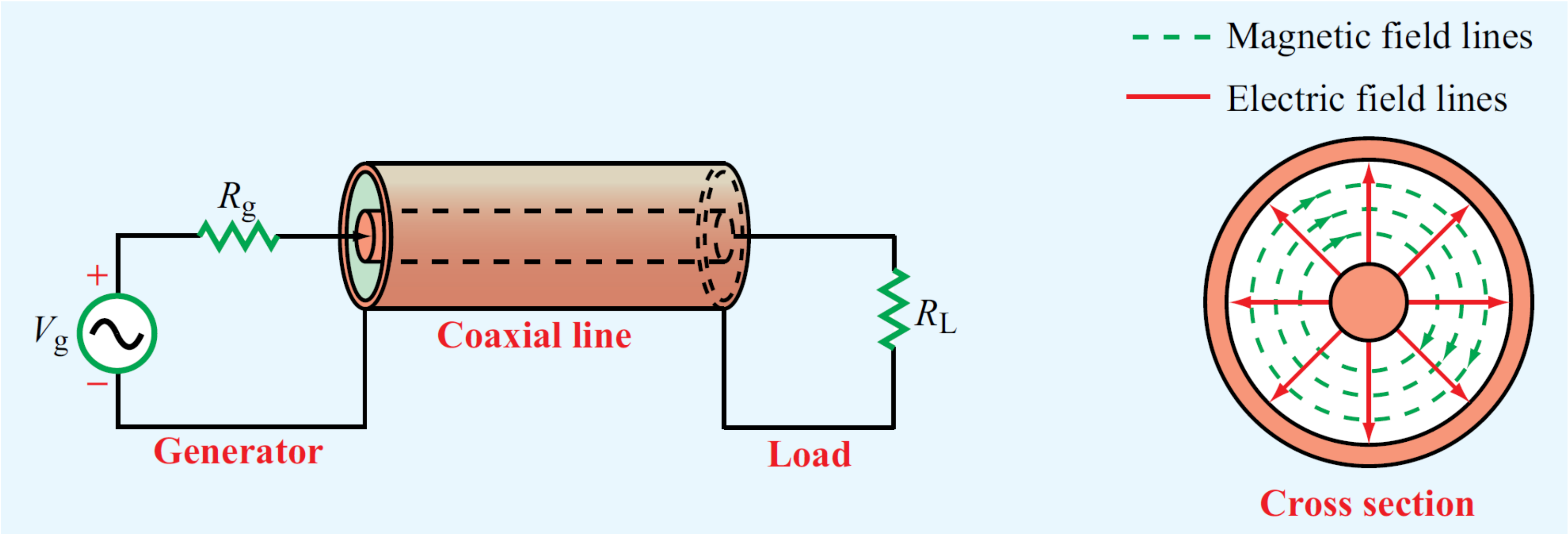
(c) Each section is represented by an equivalent circuit.

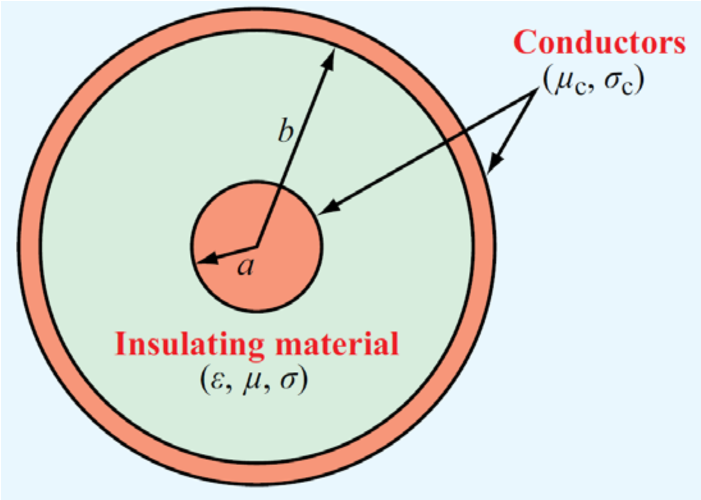
Transmission parameters

Parameter	Coaxial	Two-Wire	Parallel-Plate	Unit
R'	$\dfrac{R_s}{2\pi}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$	$\dfrac{2R_s}{\pi d}$	$\dfrac{2R_s}{w}$	Ω/m
L'	$\dfrac{\mu}{2\pi}\ln(b/a)$	$\dfrac{\mu}{\pi}\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]$	$\dfrac{\mu h}{w}$	H/m
G'	$\dfrac{2\pi\sigma}{\ln(b/a)}$	$\dfrac{\pi\sigma}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\dfrac{\sigma w}{h}$	S/m
C'	$\dfrac{2\pi\epsilon}{\ln(b/a)}$	$\dfrac{\pi\epsilon}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\dfrac{\epsilon w}{h}$	F/m

Notes: (1) Refer to **Fig. 2-4** for definitions of dimensions. (2) μ, ϵ, and σ pertain to the insulating material between the conductors. (3) $R_s = \sqrt{\pi f \mu_c/\sigma_c}$. (4) μ_c and σ_c pertain to the conductors. (5) If $(D/d)^2 \gg 1$, then $\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right] \approx \ln(2D/d)$.

Coaxial transmission line





$$R' = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) \quad \text{where } R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}}$$

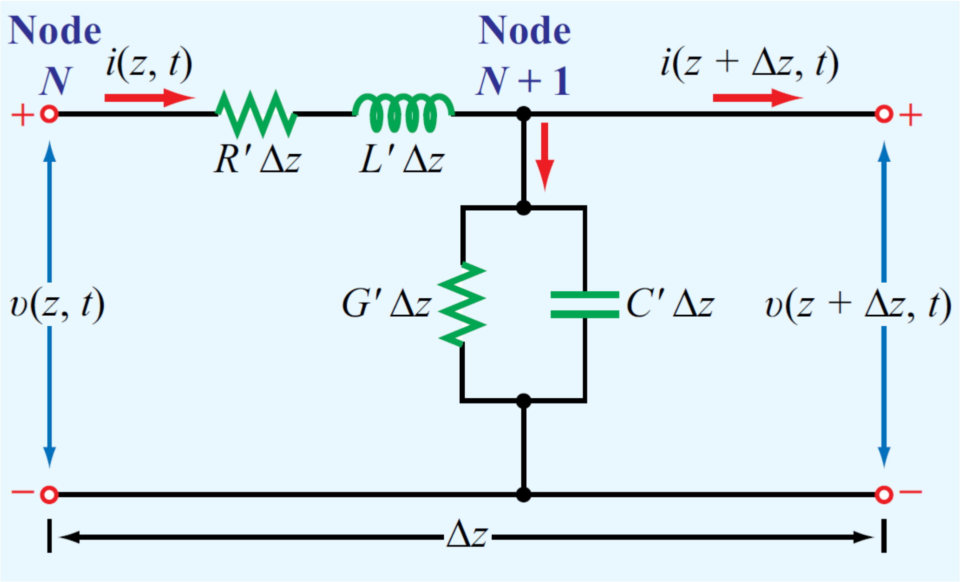
Surface resistance

- For all TEM lines,

$$L'C' = \mu\epsilon \qquad \frac{G'}{C'} = \frac{\sigma}{\epsilon}$$

Transmission line equation

- Equivalent circuit of a two-conductor transmission line of differential length Δz.



- Kirchhoff's voltage law

$$v(z,t) - R'\Delta z\, i(z,t) - L'\Delta z\,\frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0$$

- Kirchhoff's current law

$$i(z,t) - G'\Delta z\, v(z+\Delta z,t) - C'\Delta z\,\frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0$$

Transmission line equation

- By rearranging KVL and KCL equations, we obtain two first-order differential equations known as telegrapher's equations.

$$-\frac{\partial v(z,t)}{\partial z} = R'\, i(z,t) + L'\, \frac{\partial i(z,t)}{\partial t}$$

$$-\frac{\partial i(z,t)}{\partial z} = G'\, v(z,t) + C'\, \frac{\partial v(z,t)}{\partial t}$$

- Phasor notation for sinusoidal signals

$$v(z,t) = Re[\tilde{V}(z)e^{j\omega t}] \qquad i(z,t) = Re[\tilde{I}(z)e^{j\omega t}]$$

- Telegrapher's equation in phasor form

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\tilde{I}(z)$$

$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\tilde{V}(z)$$

Wave propagation on a transmission line

- The two first-order coupled equations can be combined to give two second-order uncoupled wave equations, one for $\tilde{V}(z)$ and another for $\tilde{I}(z)$.

$$\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0$$

$$\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0$$

$$\text{where } \gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

Propagation constant

- Complex propagation constant γ consists of a real part α, called the attenuation constant of the line with units of Np/m, and an imaginary part β, called the phase constant of the line with units of rad/m.

$$\gamma = \alpha + j\beta$$

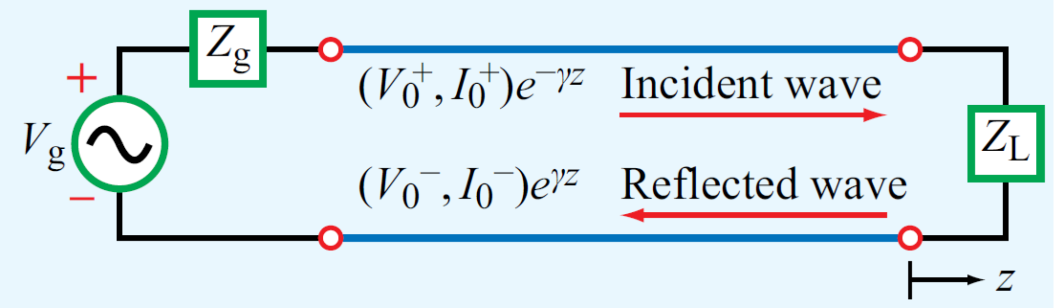
Neper unit: $L_{Np} = \ln(x_1/x_2)$

1 Np = $20\log_{10}e$ dB ~ 8.686 dB

- The wave equations have traveling wave solutions of the following form:

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$



Wave propagation on a transmission line

- The voltage-current relationship

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = -\frac{1}{(R' + j\omega L')}\frac{d\tilde{V}(z)}{dz} = \frac{\gamma}{(R' + j\omega L')}[V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}]$$

$$= \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

$$\text{where} \quad \frac{V_0^+}{I_0^+} = Z_0 = -\frac{V_0^-}{I_0^-}$$

Characteristic impedance

- Characteristic impedance of the line

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

Unit: Ohm

- It should be noted that Z_0 is equal to the ratio of the voltage amplitude to the current amplitude for each of the traveling waves individually (with an additional minus sign in the case of the $-z$ propagating wave), but it is not equal to the ratio of the total voltage $\tilde{V}(z)$ to the total current $\tilde{I}(z)$, unless one of the two waves is absent.

Lossless transmission line

- In many practical situations, the transmission line can be designed to exhibit low ohmic losses by selecting conductors with very high conductivities ($R' \approx 0$) and dielectric materials with negligible conductivities ($G' \approx 0$).
- Propagation constant for lossless line

$$\gamma = \alpha + j\beta \approx \sqrt{(j\omega L')(j\omega C')} = j\omega\sqrt{L'C'} \qquad \alpha = 0, \qquad \beta = \omega\sqrt{L'C'}$$

- Characteristic impedance of lossless line

$$Z_0 \approx \sqrt{\frac{j\omega L'}{j\omega C'}} = \sqrt{\frac{L'}{C'}}$$

- Phase velocity

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\mu\epsilon}} \approx \frac{1}{\sqrt{\mu_0\epsilon_r\epsilon_0}} = \frac{1}{\sqrt{\mu_0\epsilon_0}}\frac{1}{\sqrt{\epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}}$$

Not depend on frequency (nondispersive)

- Guide wavelength

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{L'C'}} = \frac{2\pi}{\omega\sqrt{\mu\epsilon}} \approx \frac{1}{f\sqrt{\mu_0\epsilon_r\epsilon_0}} = \frac{c}{f}\frac{1}{\sqrt{\epsilon_r}} = \frac{\lambda_0}{\sqrt{\epsilon_r}}$$

Characteristic parameters of transmission lines

	Propagation Constant $\gamma = \alpha + j\beta$	Phase Velocity u_p	Characteristic Impedance Z_0
General case	$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$	$u_p = \omega/\beta$	$Z_0 = \sqrt{\dfrac{(R' + j\omega L')}{(G' + j\omega C')}}$
Lossless $(R' = G' = 0)$	$\alpha = 0,\ \beta = \omega\sqrt{\epsilon_r}/c$	$u_p = c/\sqrt{\epsilon_r}$	$Z_0 = \sqrt{L'/C'}$
Lossless coaxial	$\alpha = 0,\ \beta = \omega\sqrt{\epsilon_r}/c$	$u_p = c/\sqrt{\epsilon_r}$	$Z_0 = \left(60/\sqrt{\epsilon_r}\right)\ln(b/a)$
Lossless two-wire	$\alpha = 0,\ \beta = \omega\sqrt{\epsilon_r}/c$	$u_p = c/\sqrt{\epsilon_r}$	$Z_0 = \left(120/\sqrt{\epsilon_r}\right) \cdot \ln\left[(D/d) + \sqrt{(D/d)^2 - 1}\right]$ $Z_0 \approx \left(120/\sqrt{\epsilon_r}\right)\ln(2D/d)$, if $D \gg d$
Lossless parallel-plate	$\alpha = 0,\ \beta = \omega\sqrt{\epsilon_r}/c$	$u_p = c/\sqrt{\epsilon_r}$	$Z_0 = \left(120\pi/\sqrt{\epsilon_r}\right)(h/w)$

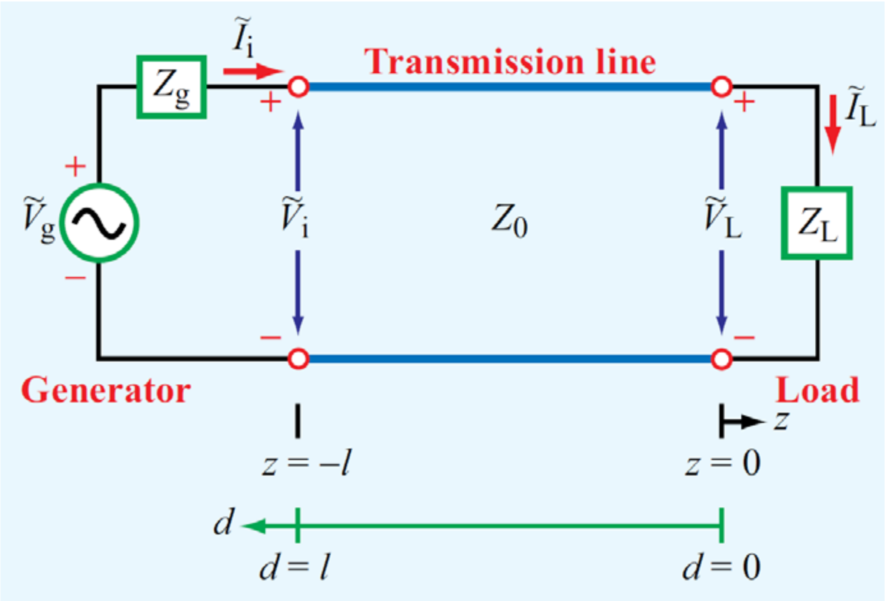
Notes: (1) $\mu = \mu_0$, $\epsilon = \epsilon_r\epsilon_0$, $c = 1/\sqrt{\mu_0\epsilon_0}$, and $\sqrt{\mu_0/\epsilon_0} \approx (120\pi)\ \Omega$, where ϵ_r is the relative permittivity of insulating material. (2) For coaxial line, a and b are radii of inner and outer conductors. (3) For two-wire line, d = wire diameter and D = separation between wire centers. (4) For parallel-plate line, w = width of plate and h = separation between the plates.

Voltage reflection coefficient

- With $\gamma = j\beta$ for the lossless line, the total voltage and current become

$$\tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$



- At the load ($z = 0$)

$$\tilde{V}_L = \tilde{V}(z=0) = V_0^+ + V_0^-$$

$$\tilde{I}_L = \tilde{I}(z=0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

- The load impedance

$$Z_L = \frac{\tilde{V}_L}{\tilde{I}_L} = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right) Z_0 \quad \Rightarrow \quad V_0^- = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) V_0^+$$

- Voltage reflection coefficient

$$\boxed{\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L + 1}} \qquad \boxed{\Gamma = -\frac{I_0^-}{I_0^+}}$$

$$z_L = Z_L / Z_0$$

Normalized load impedance

Voltage reflection coefficient

- Z_0 of a lossless line is a real number, but Z_L is in general a complex quantity. Hence, in general Γ also is complex and given by

$$\Gamma = |\Gamma| e^{-j\theta_r}$$

$$\boxed{\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L + 1}}$$

- Matched load ($Z_L = Z_0$) → $\Gamma = 0$ and $V_0^- = 0$
- Open load ($Z_L = \infty$) → $\Gamma = 1$ and $V_0^- = V_0^+$
- Short load ($Z_L = 0$) → $\Gamma = -1$ and $V_0^- = -V_0^+$

Load	$\lvert\Gamma\rvert$	θ_r
Z_0 $Z_L = (r + jx)Z_0$	$\left[\frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2}$	$\tan^{-1}\left(\frac{x}{r-1}\right) - \tan^{-1}\left(\frac{x}{r+1}\right)$
Z_0 Z_0	0 (no reflection)	irrelevant
Z_0 (short)	1	$\pm 180°$ (phase opposition)
Z_0 (open)	1	0 (in-phase)
Z_0 $jX = j\omega L$	1	$\pm 180° - 2\tan^{-1} x$
Z_0 $jX = \frac{-j}{\omega C}$	1	$\pm 180° + 2\tan^{-1} x$

Standing waves

- Using $V_0^- = \Gamma V_0^+$, the total voltage and current become

$$\tilde{V}(z) = V_0^+(e^{-j\beta z} + \Gamma e^{+j\beta z})$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{+j\beta z})$$

Only one unknown V_0^+

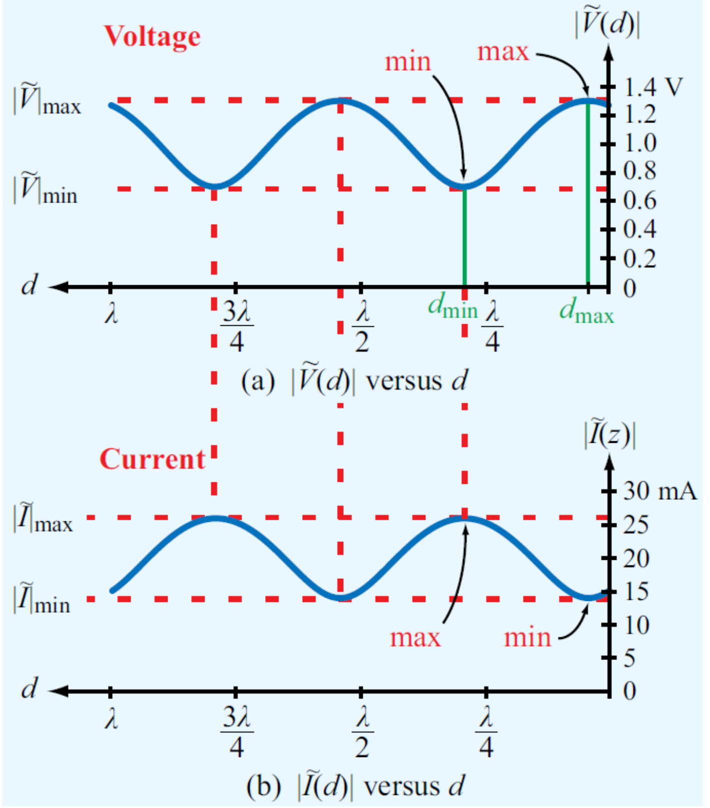
- Replacing z with $-d$, Magnitude of $\tilde{V}(d)$ and $\tilde{I}(d)$

$$\left|\tilde{V}(d)\right| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

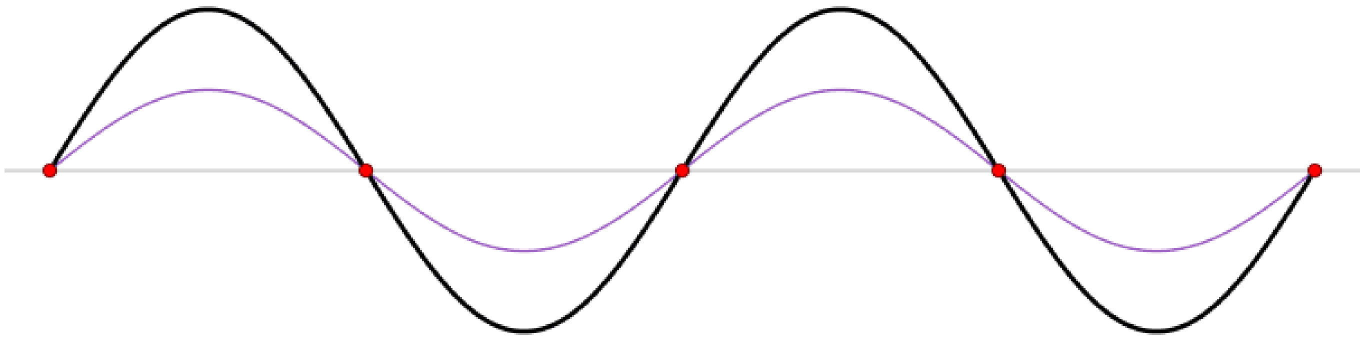
$$\left|\tilde{I}(d)\right| = \frac{|V_0^+|}{Z_0}[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

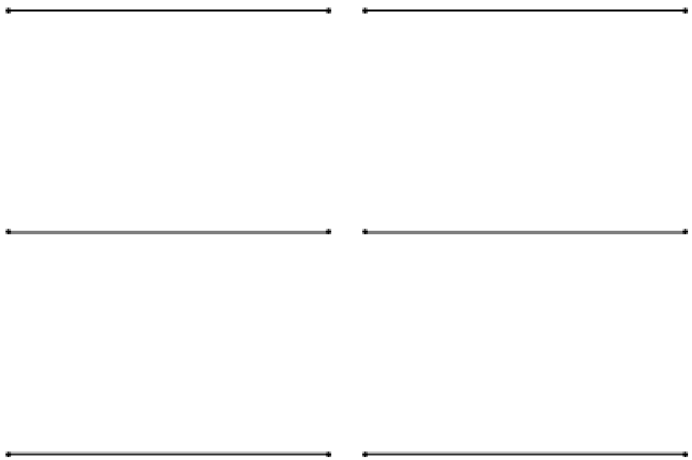
- The sinusoidal patterns are called standing waves and are caused by the interference of the two traveling waves.
  - ➢ Maximum values when $2\beta d - \theta_r = 2n\pi$
  - ➢ Minimum values when $2\beta d - \theta_r = (2n+1)\pi$
- With no reflected wave present, there are no interference and no standing waves.

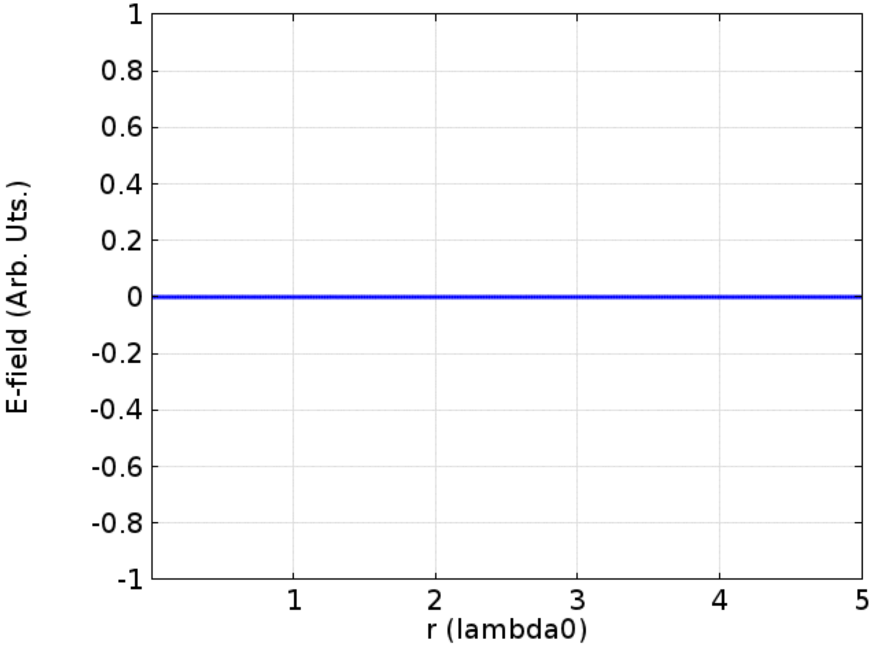
Figure 2-14 Standing-wave pattern for (a) $|\tilde{V}(d)|$ and (b) $|\tilde{I}(d)|$ for a lossless transmission line of characteristic impedance $Z_0 = 50\ \Omega$, terminated in a load with a reflection coefficient $\Gamma = 0.3e^{j30°}$. The magnitude of the incident wave $|V_0^+| = 1$ V. The standing-wave ratio is $S = |\tilde{V}|_{max}/|\tilde{V}|_{min} = 1.3/0.7 = 1.86$.



Standing waves



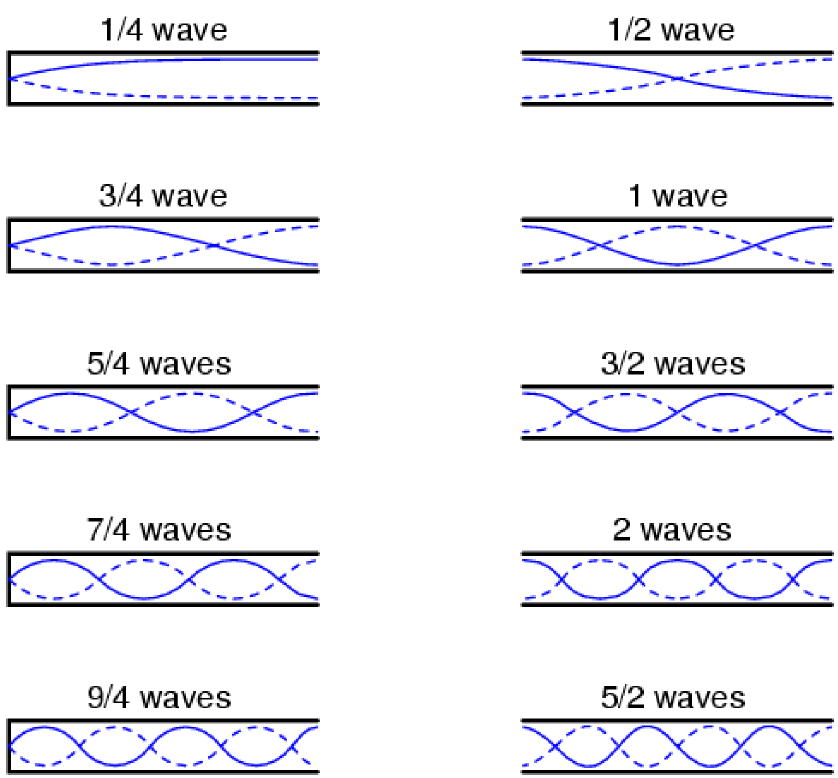


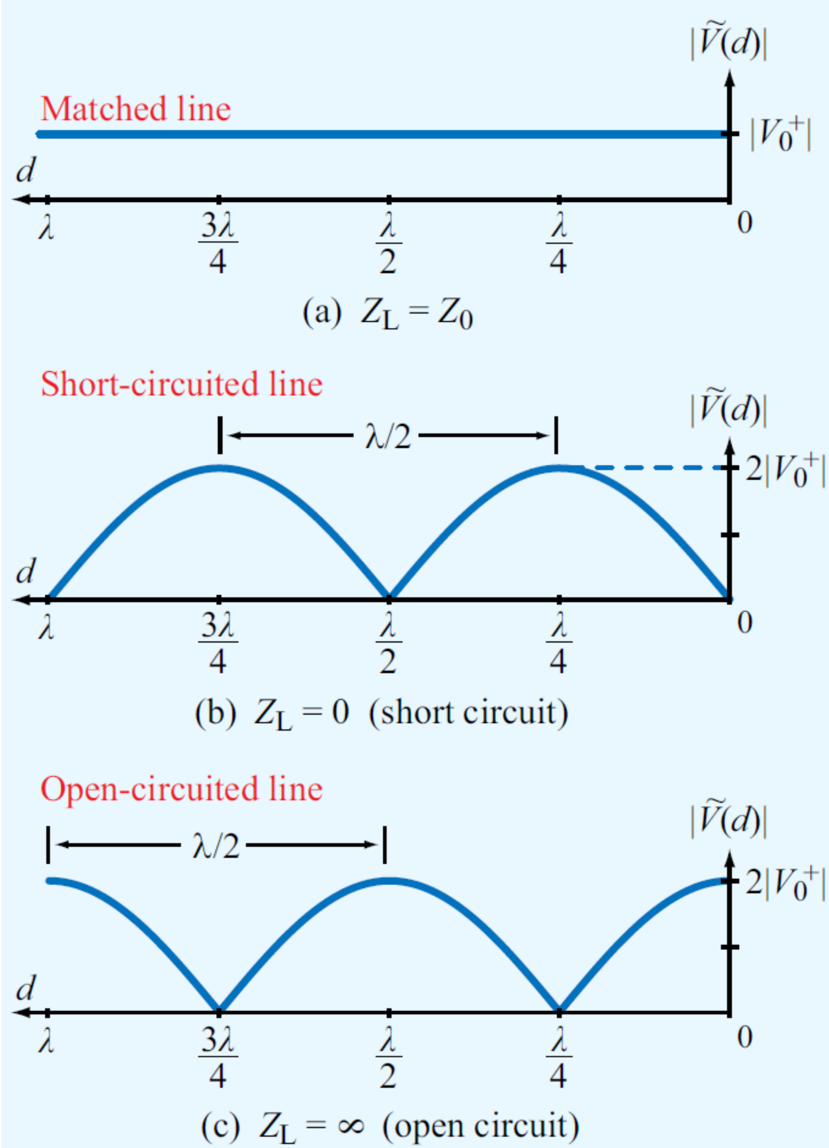


Voltage standing-wave patterns

- Matched load ($Z_L = Z_0$) → $\Gamma = 0$
- Open load ($Z_L = \infty$) → $\Gamma = 1$
- Short load ($Z_L = 0$) → $\Gamma = -1$

Standing sound waves in open-ended tubes





Voltage standing wave ratio (VSWR)

- Maximum and minimum values of $|\tilde{V}|$
  - $|\tilde{V}|_{max} = |\tilde{V}(d_{max})| = |V_0^+|[1 + |\Gamma|]$ at $2\beta d_{max} - \theta_r = 2n\pi$
  - $|\tilde{V}|_{min} = |\tilde{V}(d_{min})| = |V_0^+|[1 - |\Gamma|]$ at $2\beta d_{min} - \theta_r = (2n+1)\pi$

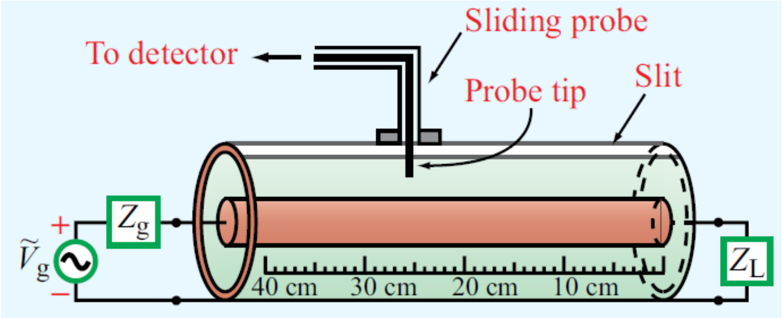
- Voltage standing wave ratio (VSWR)

$$\boxed{S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}}$$

  - A measure of the mismatch between the load and the transmission line
  - For a matched load with $\Gamma = 0$, $S = 1$, and for a line with $|\Gamma| = 1$, $S = \infty$.

- Measuring Z_L using slotted-line probe

$$\underset{\uparrow \atop \text{Measure}}{S} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \qquad \Gamma = \frac{Z_L - Z_0}{Z_L + \underset{\uparrow \atop \text{Known}}{Z_0}}$$



Wave impedance of lossless lines

- Since the voltage and current magnitudes are oscillatory with position along the line and in phase opposition with each other, the wave impedance $Z(d)$ must vary with position also.

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^+(e^{+j\beta d} + \Gamma e^{-j\beta d})}{V_0^+(e^{+j\beta d} - \Gamma e^{-j\beta d})} Z_0 = Z_0 \left[\frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}}\right] = Z_0 \left[\frac{1 + \Gamma_d}{1 - \Gamma_d}\right]$$

- Phase-shifted voltage reflection coefficient

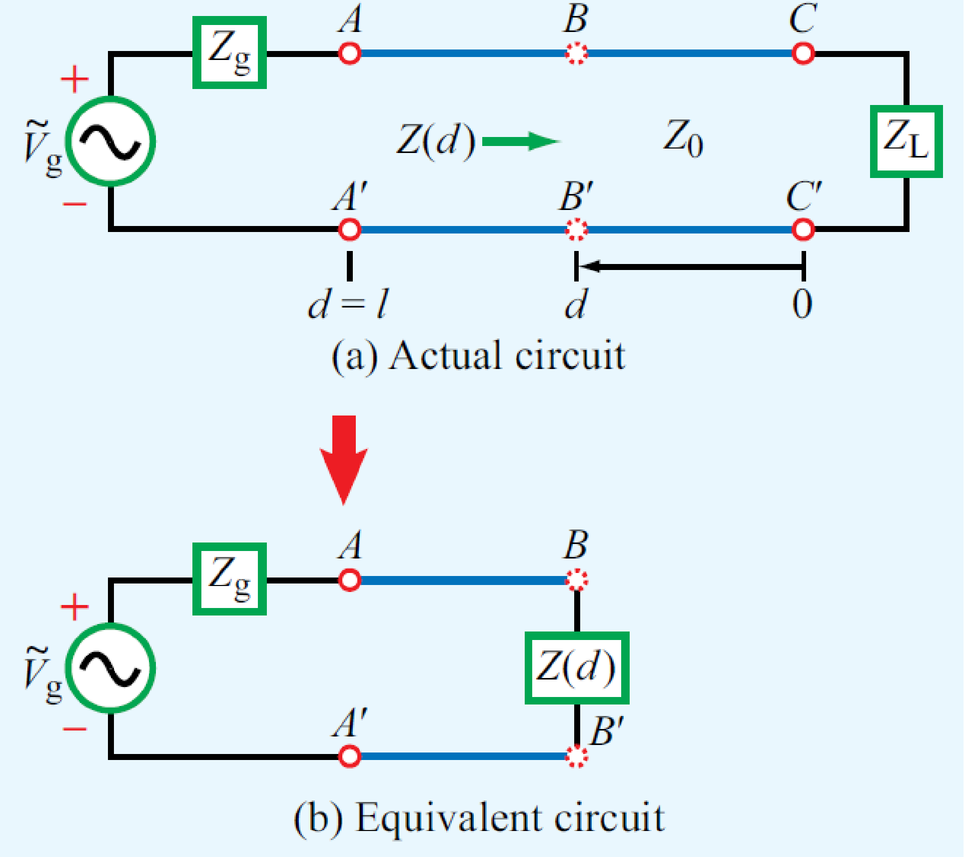
$$\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j\theta_r} e^{-j2\beta d} = |\Gamma| e^{j(\theta_r - 2\beta d)}$$

  - Γ_d has the same magnitude as Γ, but its phase is shifted by $2\beta d$ relative to that of Γ.

- $Z(d)$ is the ratio of the total voltage (incident and reflected-wave voltages) to the total current at any point d on the line, in contrast with the characteristic impedance of the line Z_0, which relates the voltage and current of each of the two waves individually ($Z_0 = V_0^+/I_0^+ = -V_0^-/I_0^-$).

Input impedance

- In the actual circuit (a), at terminals BB' at an arbitrary location d on the line, $Z(d)$ is the wave impedance of the line when "looking" to the right (i.e., towards the load). Application of the equivalence principle allows us to replace the segment to the right of terminals BB' with a lumped impedance of value $Z(d)$.



(a) Actual circuit

(b) Equivalent circuit

Input impedance

- Of particular interest in many transmission-line problems is the input impedance at the source end of the line, at $d = l$, which is given by

$$Z_{in} = Z(l) = Z_0 \left[\frac{1 + \Gamma_l}{1 - \Gamma_l} \right]$$

Where, $\Gamma_l = \Gamma e^{-j2\beta l} = |\Gamma| e^{j(\theta_r - 2\beta d)}$

- Using $\quad \Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

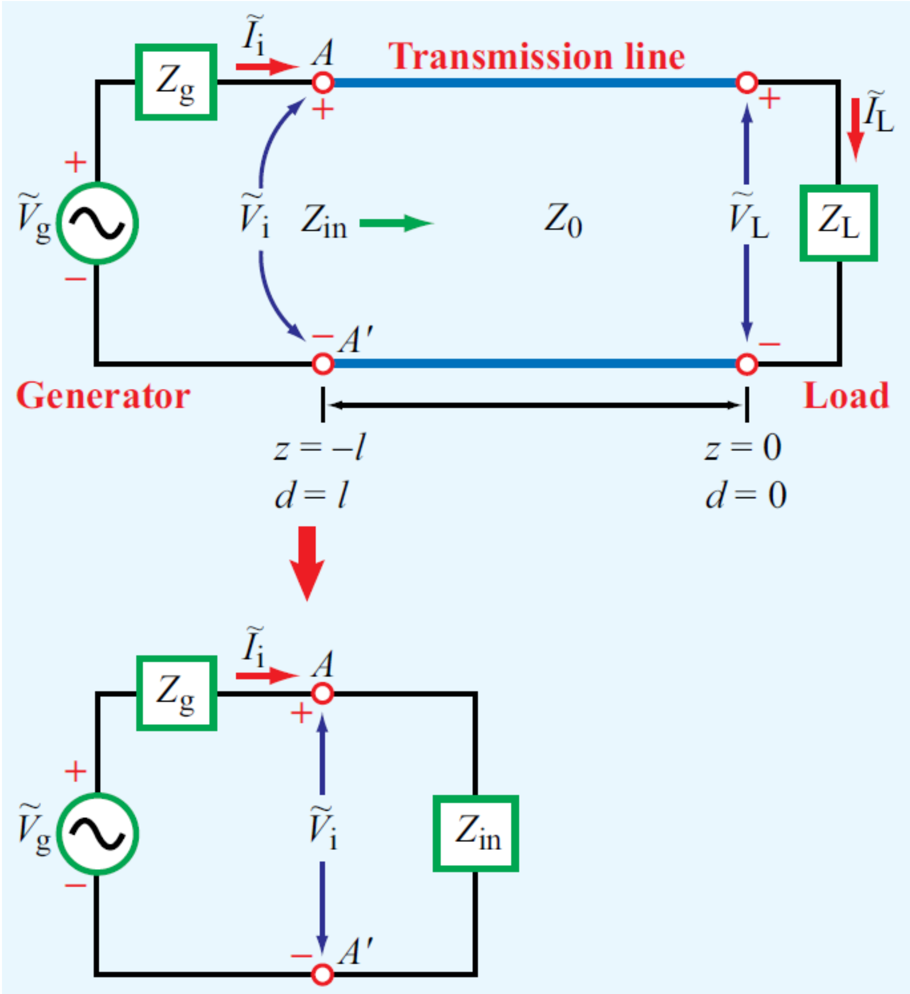
- We obtain the input impedance

$$\boxed{Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right]}$$

- Final solution is

$$\tilde{V}_i = \tilde{I}_i Z_{in} = \frac{Z_{in}}{Z_g + Z_{in}} \tilde{V}_g \qquad = \tilde{V}(-l)$$

$$\boxed{V_0^+ = \left(\frac{Z_{in}}{Z_g + Z_{in}} \tilde{V}_g \right) \left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}} \right)}$$



Short-circuited line

- Since $Z_L = 0$, the input impedance is

$$Z_{in}^{sc} = Z_0 \left[\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right] = jZ_0 \tan \beta l$$

Purely reactive

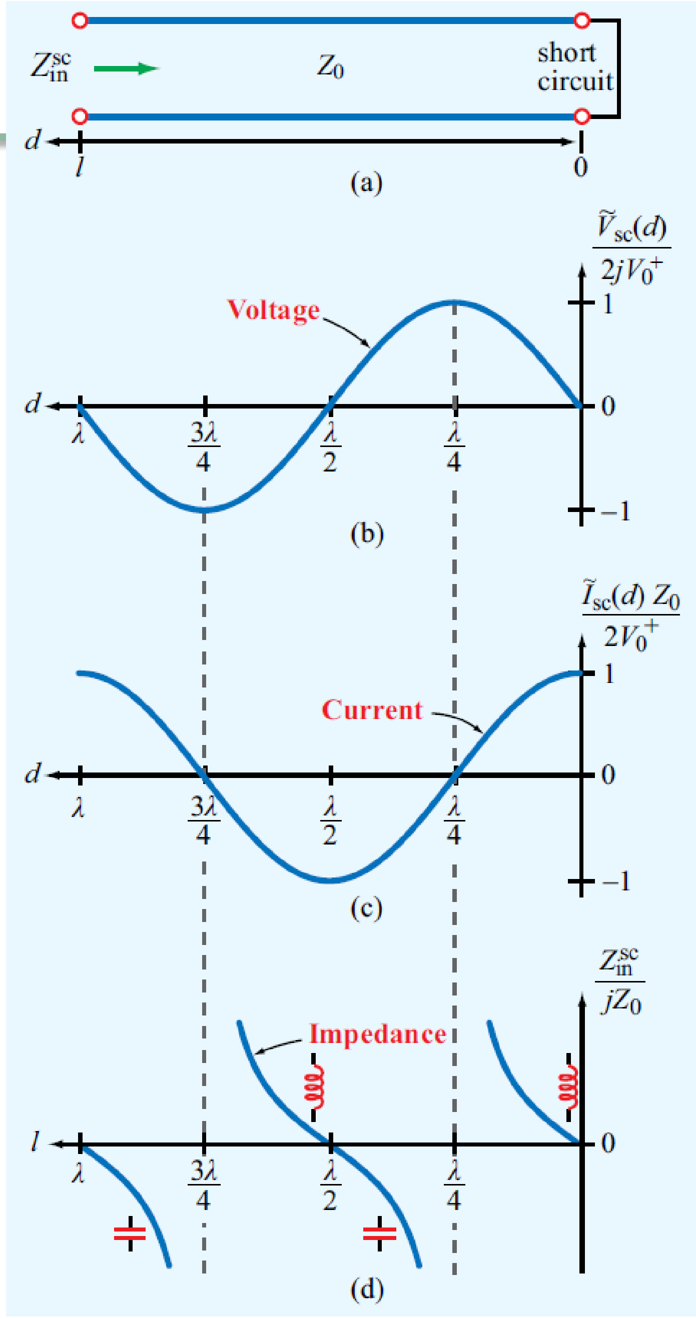
- If $\tan \beta l \geq 0$, the line appears inductive to the source, acting like an equivalent inductor L_{eq}

$$L_{eq} = \frac{Z_0 \tan \beta l}{\omega}$$

- If $\tan \beta l \leq 0$, the input impedance is capacitive, in which case the line acts like an equivalent capacitor with capacitance C_{eq}

$$C_{eq} = -\frac{1}{\omega Z_0 \tan \beta l}$$

➢ Through proper choice of the length of a short-circuited line, we can make them into equivalent capacitors and inductors of any desired reactance.



Open-circuited line

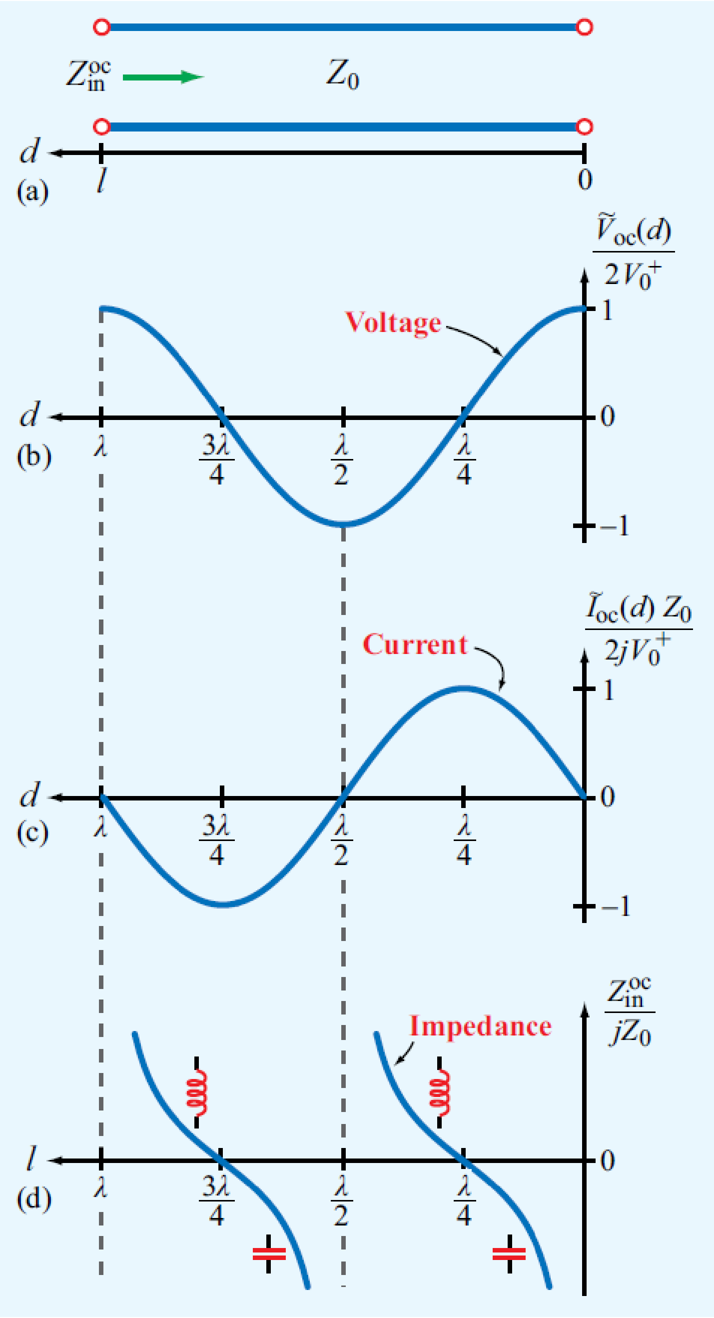
- Since $Z_L = \infty$, the input impedance of an open-circuited line is

$$Z_{in}^{oc} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right] = -jZ_0 \cot\beta l$$

- A network analyzer is a radio-frequency (RF) instrument capable of measuring the impedance of any load connected to its input terminal.
- The combination of the two measurements (Z_{in}^{sc} and Z_{in}^{oc}) can be used to determine the characteristic impedance of the line Z_0 and its phase constant β.

$$Z_0 = \sqrt{Z_{in}^{sc} Z_{in}^{oc}}$$

$$\tan\beta l = \sqrt{-\frac{Z_{in}^{sc}}{Z_{in}^{oc}}}$$



Quarter-wavelength ($4/\lambda$) transformer

- If $l = n\lambda/2$, where n is an integer, $\tan\beta l = 0$. Then,

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right] = Z_L$$

  - A half-wavelength line does not modify the load impedance.

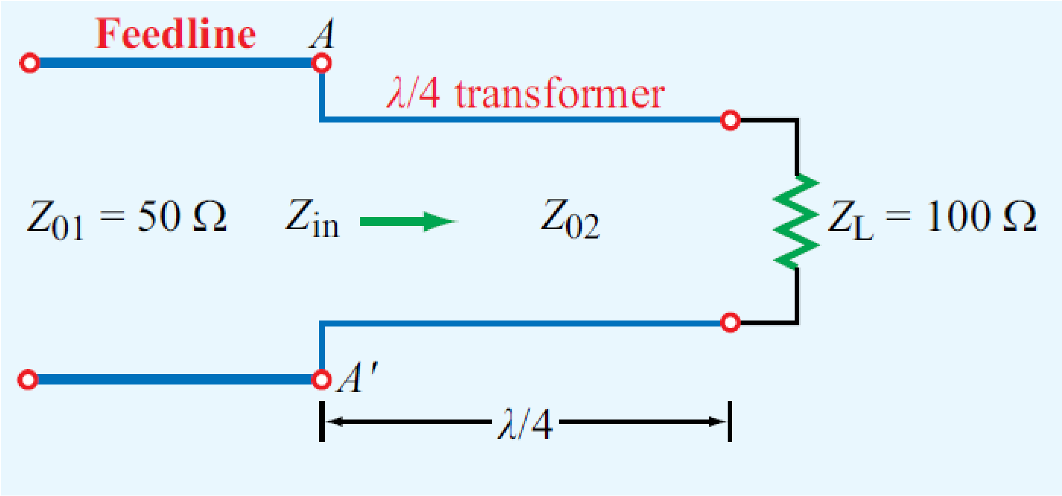
- If $l = n\lambda/2 + \lambda/4$, where n is 0 or a positive integer, $\tan\beta l = \infty$. Then,

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right] = \frac{Z_0^2}{Z_L}$$

- $\lambda/4$ transformer (example)

$$Z_{in} = \frac{Z_{02}^2}{Z_L} = Z_{01}$$

$$Z_{02} = \sqrt{Z_{01} Z_L} = 70.7\ \Omega$$



Properties of standing waves on a lossless transmission line

Voltage maximum Voltage minimum	$\lvert\tilde{V}\rvert_{\max} = \lvert V_0^+\rvert[1+\lvert\Gamma\rvert]$ $\lvert\tilde{V}\rvert_{\min} = \lvert V_0^+\rvert[1-\lvert\Gamma\rvert]$
Positions of voltage maxima (also positions of current minima)	$d_{\max} = \frac{\theta_r\lambda}{4\pi} + \frac{n\lambda}{2}, \quad n = 0, 1, 2, \ldots$
Position of first maximum (also position of first current minimum)	$d_{\max} = \begin{cases} \frac{\theta_r\lambda}{4\pi}, & \text{if } 0 \le \theta_r \le \pi \\ \frac{\theta_r\lambda}{4\pi} + \frac{\lambda}{2}, & \text{if } -\pi \le \theta_r \le 0 \end{cases}$
Positions of voltage minima (also positions of current maxima)	$d_{\min} = \frac{\theta_r\lambda}{4\pi} + \frac{(2n+1)\lambda}{4}, \quad n = 0, 1, 2, \ldots$
Position of first minimum (also position of first current maximum)	$d_{\min} = \frac{\lambda}{4}\left(1 + \frac{\theta_r}{\pi}\right)$
Input impedance	$Z_{in} = Z_0\left(\frac{z_L + j\tan\beta l}{1 + jz_L\tan\beta l}\right) = Z_0\left(\frac{1+\Gamma_l}{1-\Gamma_l}\right)$
Positions at which Z_{in} is real	at voltage maxima and minima
Z_{in} at voltage maxima	$Z_{in} = Z_0\left(\frac{1+\lvert\Gamma\rvert}{1-\lvert\Gamma\rvert}\right)$
Z_{in} at voltage minima	$Z_{in} = Z_0\left(\frac{1-\lvert\Gamma\rvert}{1+\lvert\Gamma\rvert}\right)$
Z_{in} of short-circuited line	$Z_{in}^{sc} = jZ_0\tan\beta l$
Z_{in} of open-circuited line	$Z_{in}^{oc} = -jZ_0\cot\beta l$
Z_{in} of line of length $l = n\lambda/2$	$Z_{in} = Z_L, \quad n = 0, 1, 2, \ldots$
Z_{in} of line of length $l = \lambda/4 + n\lambda/2$ Z_{in} of matched line	$Z_{in} = Z_0^2/Z_L, \quad n = 0, 1, 2, \ldots$ $Z_{in} = Z_0$
$\lvert V_0^+\rvert$ = amplitude of incident wave; $\Gamma = \lvert\Gamma\rvert e^{j\theta_r}$ with $-\pi < \theta_r < \pi$; θ_r in radians; $\Gamma_l = \Gamma e^{-j2\beta l}$.	

Smith chart

- The Smith chart, developed by P. H. Smith in 1939, is a widely used graphical tool for analyzing and designing transmission line circuits.

- Reflection coefficient and normalized load impedance

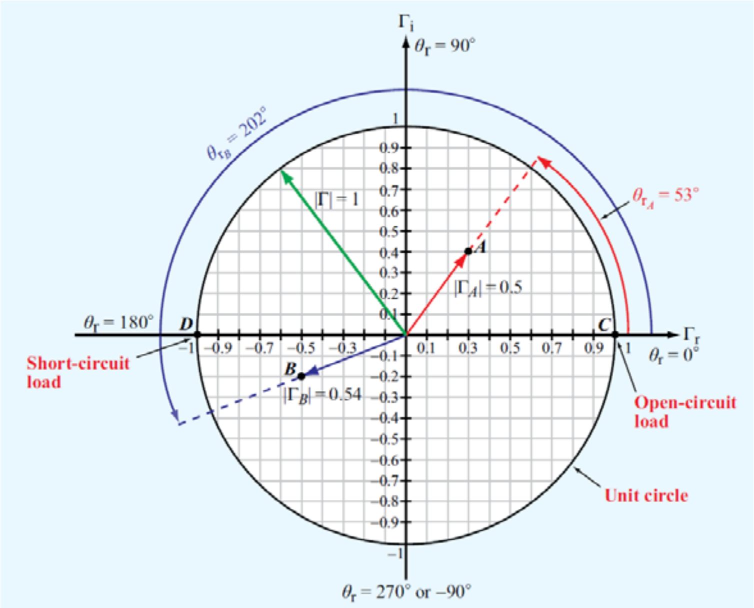
$$\Gamma = |\Gamma| e^{-j\theta_r} = \Gamma_r + j\Gamma_i$$

$$\Gamma = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{z_L - 1}{z_L + 1}$$

$$z_L = \frac{1+\Gamma}{1-\Gamma} = r_L + jx_L = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i}$$

Normalized load resistance

Normalized load reactance



$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$x_L = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

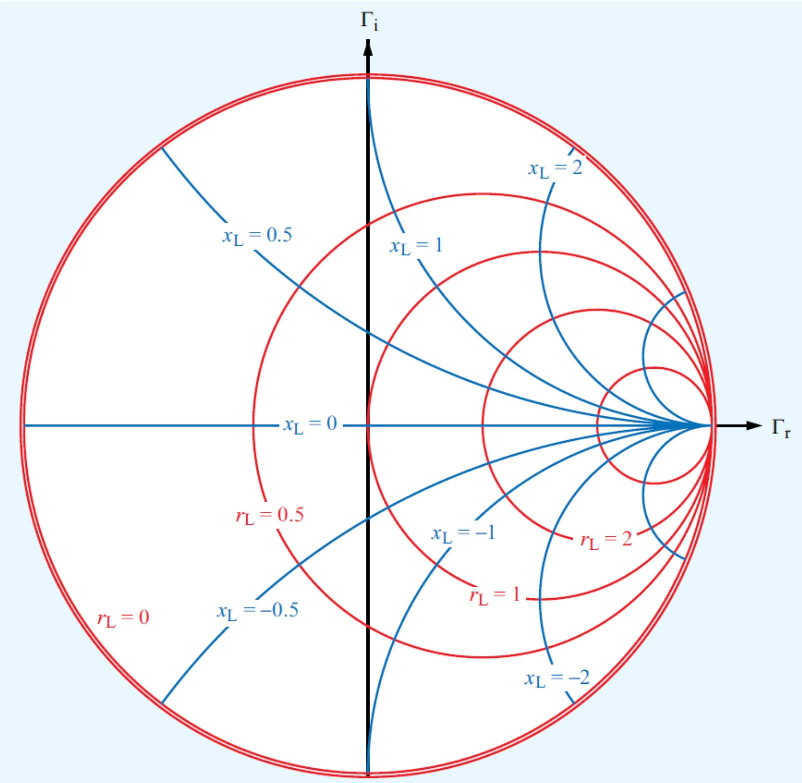
There are many combinations of (Γ_r, Γ_i) that yield the same value for the normalized load resistance.

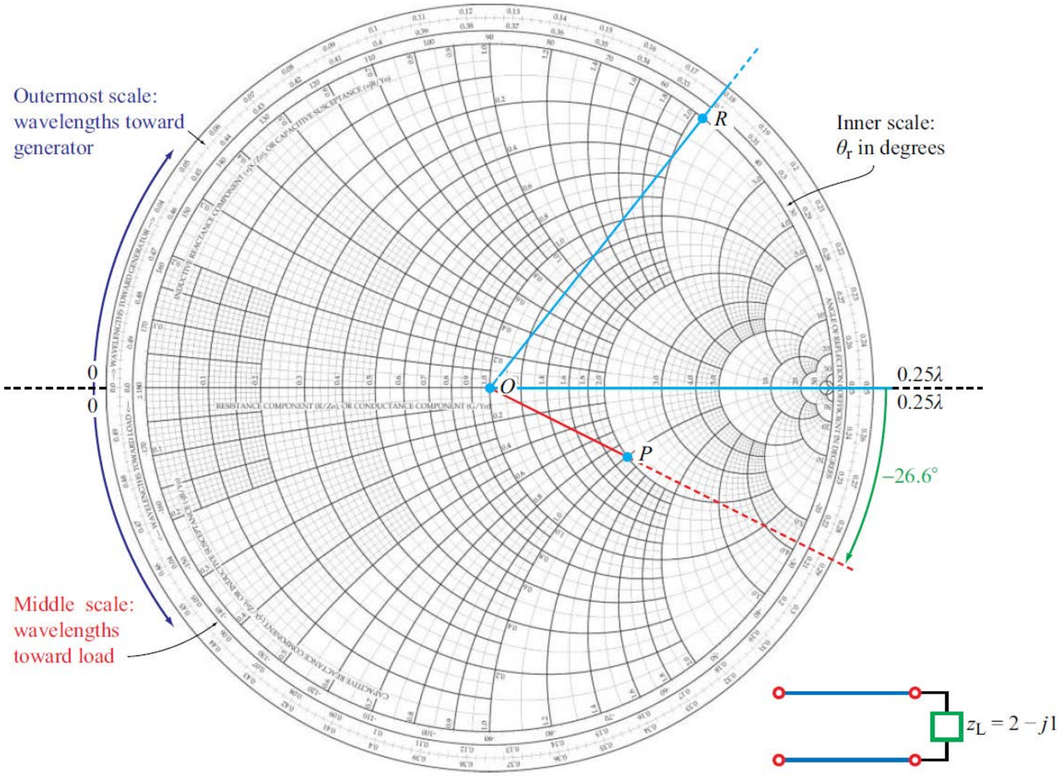
Smith chart: graphical evaluation of (r_L, x_L) in the $\Gamma_r - \Gamma_i$ plane

Smith chart

- Parametric equation for the circle in the $\Gamma_r - \Gamma_i$ plane corresponding to given values of r_L and x_L.

$$\left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r_L}\right)^2 \qquad (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2$$





Wave impedance

- The normalized wave impedance looking toward the load at a distance d from the load is

$$z(d) = \frac{Z(d)}{Z_0} = \frac{1+\Gamma_d}{1-\Gamma_d}$$

$$\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j\theta_r} e^{-j2\beta d} = |\Gamma| e^{j(\theta_r - 2\beta d)}$$

(Phase-shifted voltage reflection coefficient)

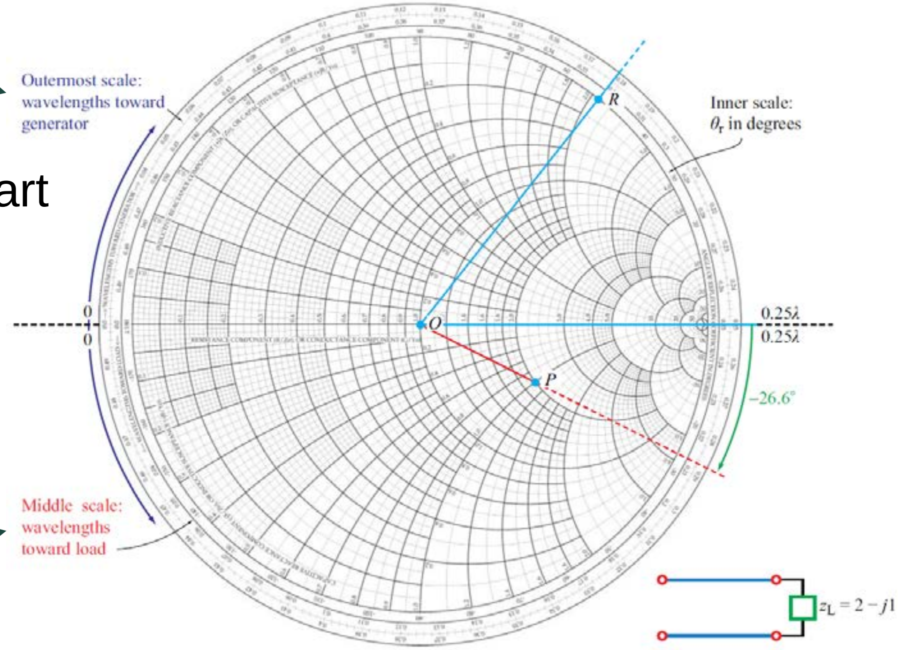
- The transformation from Γ to Γ_d is achieved by maintaining $|\Gamma|$ constant and decreasing its phase θ_r by $2\beta d$, which corresponds to a clockwise rotation (on the Smith chart) over an angle of $2\beta d$ radians.

WTG scale

- A complete rotation around the Smith chart

$$2\beta d = 2\pi \quad \Rightarrow \quad d = \frac{\lambda}{2}$$

WTL scale

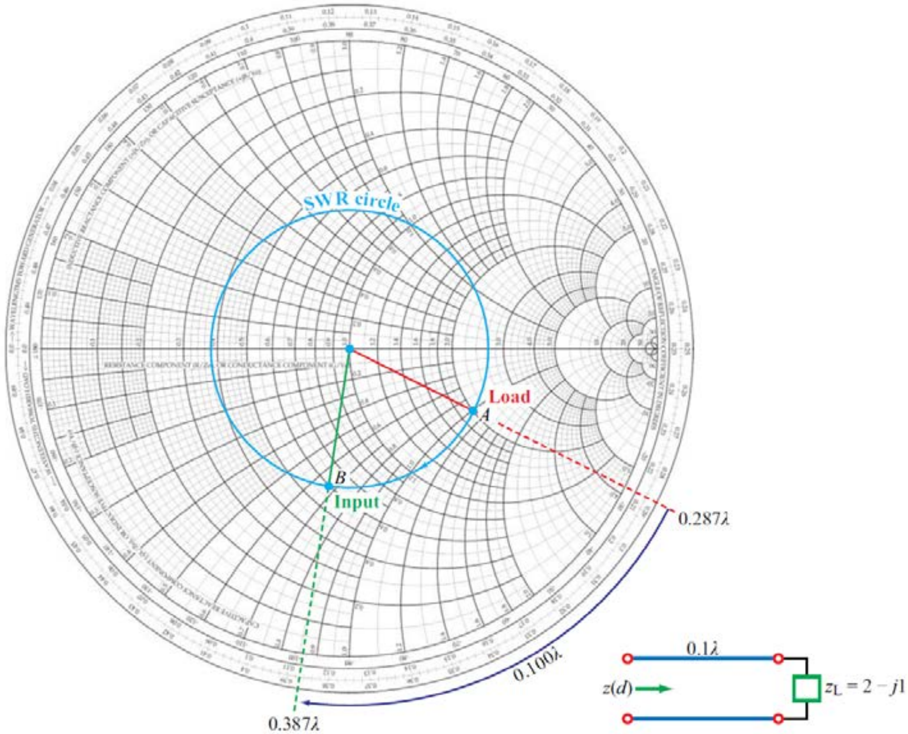


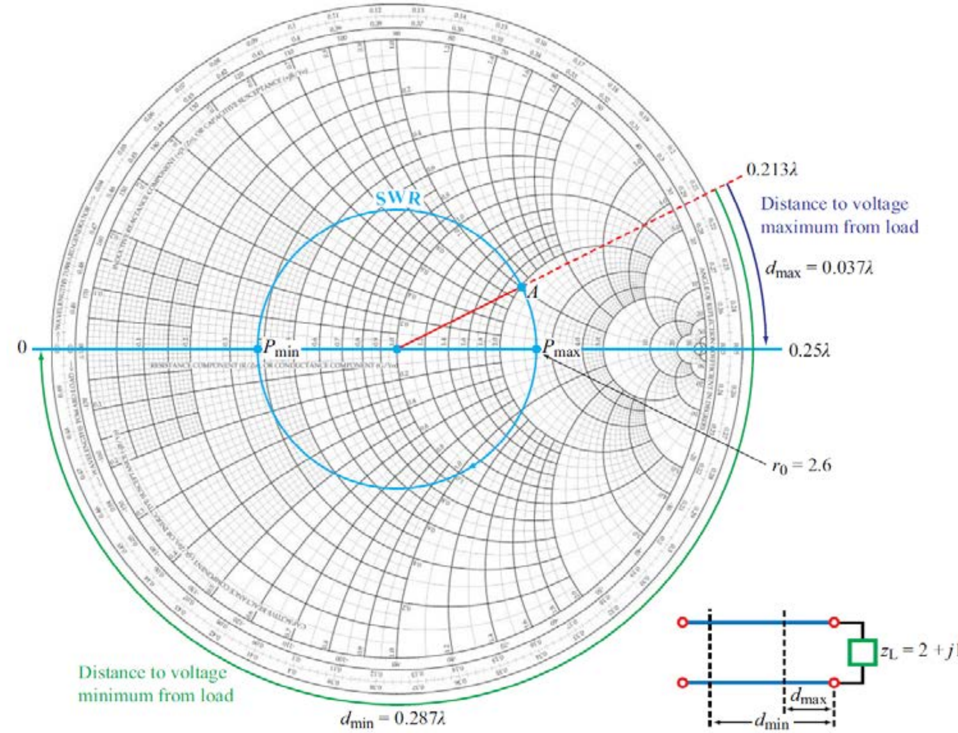
Wave impedance

- Constant-|Γ| circle or constant-SWR circle

$$S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

- If a line is of length l, its input impedance is $Z_{in} = Z_0 z(l)$, with $z(l)$ determined by rotating a distance l from the load along the WTG scale.





Impedance to admittance transformation

- In solving certain types of transmission-line problems, it is often more convenient to work with admittances than with impedances.
- Impedance

$$Z = \underbrace{R}_{\text{Resistance}} + j\underbrace{X}_{\text{Reactance}}$$

- Admittance

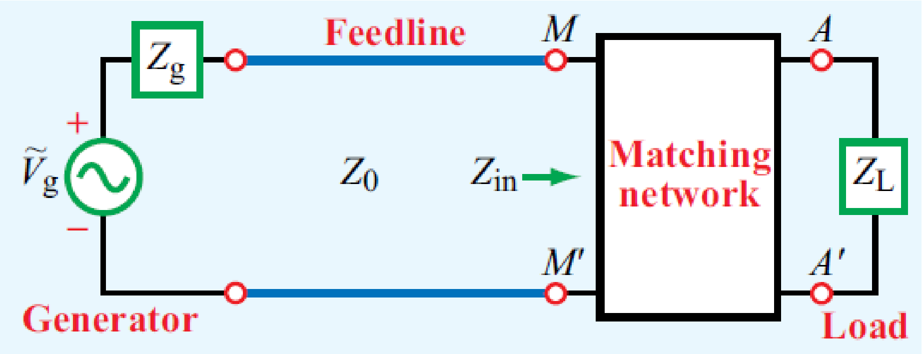
$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R}{R^2 + X^2} + j\frac{-X}{R^2 + X^2} = \underbrace{G}_{\text{Conductance}} + j\underbrace{B}_{\text{Susceptance}}$$

- Normalized admittance

$$y = \frac{Y}{Y_0} = \frac{G}{Y_0} + j\frac{B}{Y_0} = g + jb$$

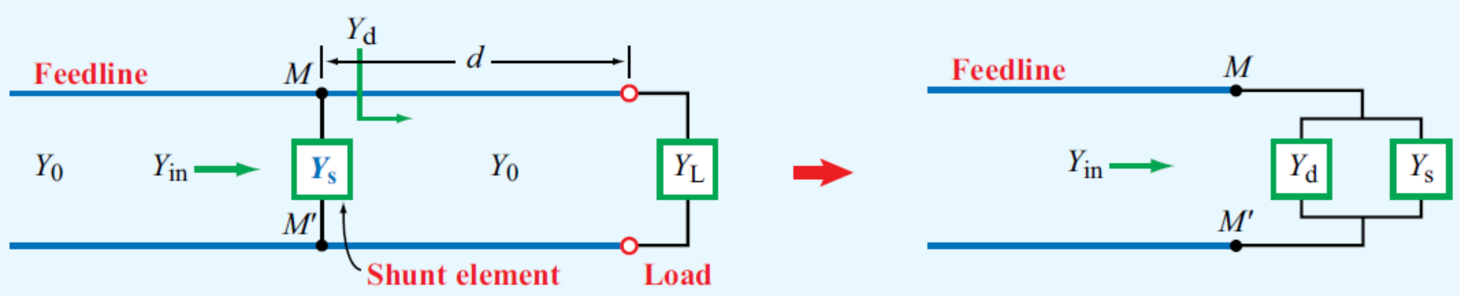
Impedance matching

- Impedance-matching network: It eliminates reflections at terminals MM' for waves incident from the source. Even though multiple reflections may occur between AA' and MM', only a forward traveling wave exists on the feedline.



- Lumped-element matching

$$Y_{in} = Y_d + Y_s = (G_d + jB_d) + jB_s = G_d + j(B_d + B_s) = Y_0 \quad \Rightarrow \quad G_d = Y_0 \quad B_s = -B_d$$



(a) Transmission-line circuit

(b) Equivalent circuit

Matching network in RF inductive discharges

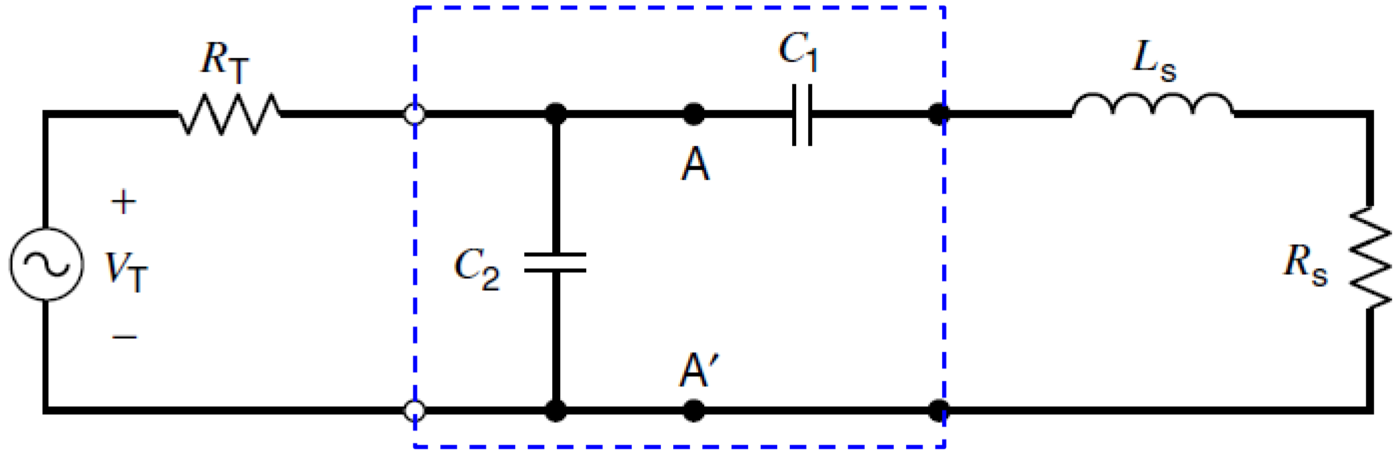
- The admittance looking to the right at the terminals $A – A'$ is

$$Y_A = G_A + jB_A = \frac{1}{R_s + j(X_1 + X_s)} \quad \Rightarrow \quad G_A = \frac{R_s}{R_s^2 + (X_1 + X_s)^2} \quad \text{conductance}$$

$$B_A = -\frac{X_1 + X_s}{R_s^2 + (X_1 + X_s)^2} \quad \text{susceptance}$$

- Determine the values of C_1 and C_2

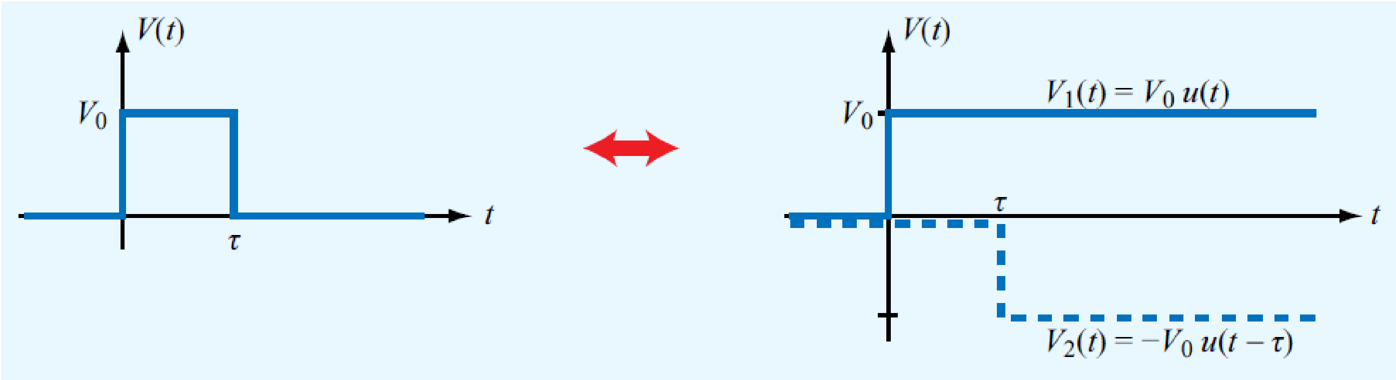
$$G_A = \frac{1}{R_T} \qquad B_2 = \omega C_2 = -B_A$$



Transients on transmission lines

- The transient response of a voltage pulse on a transmission line is a time record of its back and forth travel between the sending and receiving ends of the line, taking into account all the multiple reflections (echoes) at both ends.
- A single rectangular pulse can be described mathematically as the sum of two unit step functions:

$$V(t) = V_1(t) + V_2(t) = V_0 u(t) - V_0 u(t-\tau)$$



(a) Pulse of duration τ (b) $V(t) = V_1(t) + V_2(t)$

- Hence, if we can develop basic tools for describing the transient behavior of a single step function, we can apply the same tools for each of the two components of the pulse and then add the results to obtain the response to $V(t)$.

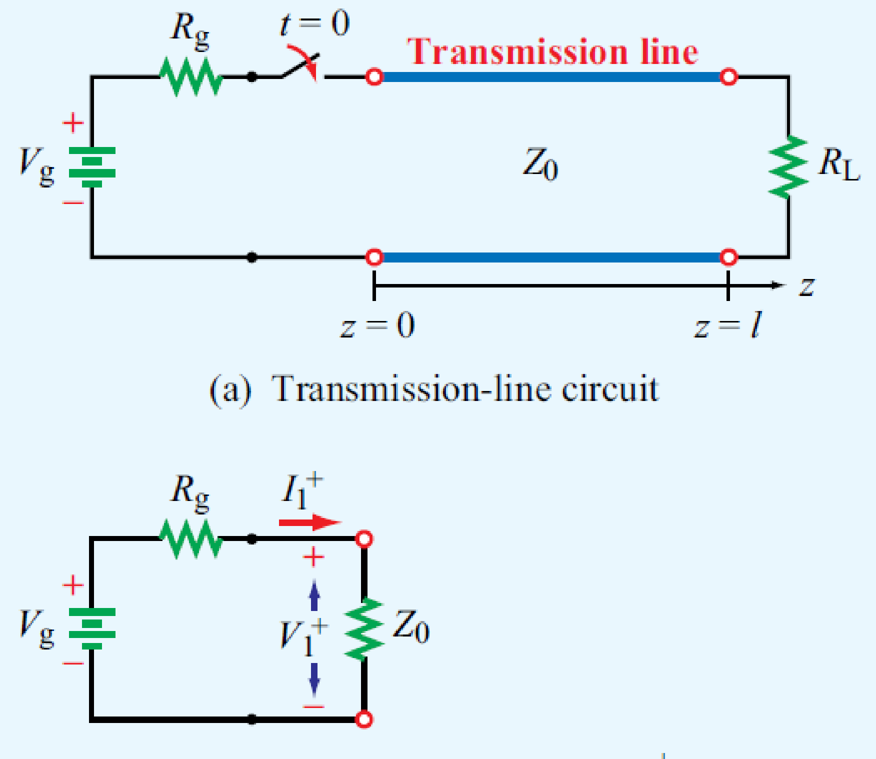
Transient response to a step function

- The switch between the generator circuit and the transmission line is closed at $t = 0$. The instant the switch is closed, the transmission line appears to the generator circuit as a load with impedance Z_0. This is because, in the absence of a signal on the line, the input impedance of the line is unaffected by the load impedance R_L.

- The initial condition:

$$I_1^+ = \frac{V_g}{R_g + Z_0}$$

$$V_1^+ = I_1^+ Z_0 = \frac{Z_0 V_g}{R_g + Z_0}$$



(a) Transmission-line circuit

(b) Equivalent circuit at $t = 0^+$

➢ The combination of V_1^+ and I_1^+ constitutes a wave that travels along the line with velocity $u_p = 1/\sqrt{\epsilon\mu}$, immediately after the switch is closed.

Reflections at both ends

- At $t = T$, the wave reaches the load at $z = l$, and because $R_L \neq Z_0$, the mismatch generates a reflected wave with amplitude

$$V_1^- = \Gamma_L V_1^+ \qquad \text{where } \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

- After this first reflection, the voltage on the line consists of the sum of two waves: the initial wave V_1^+ and the reflected wave V_1^-.
- At $t = 2T$, the reflected wave V_1^- arrives at the sending end of the line. If $R_g \neq Z_0$, the mismatch at the sending end generates a reflection at $z = 0$ in the form of a wave with voltage amplitude V_2^+ given by

$$V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+ \qquad \text{where } \Gamma_g = \frac{R_g - Z_0}{R_g + Z_0}$$

- The multiple-reflection process continues indefinitely, and the ultimate value that $V(z,t)$ reaches as t approaches $+\infty$ is the same at all locations on the transmission line (steady-state voltage).

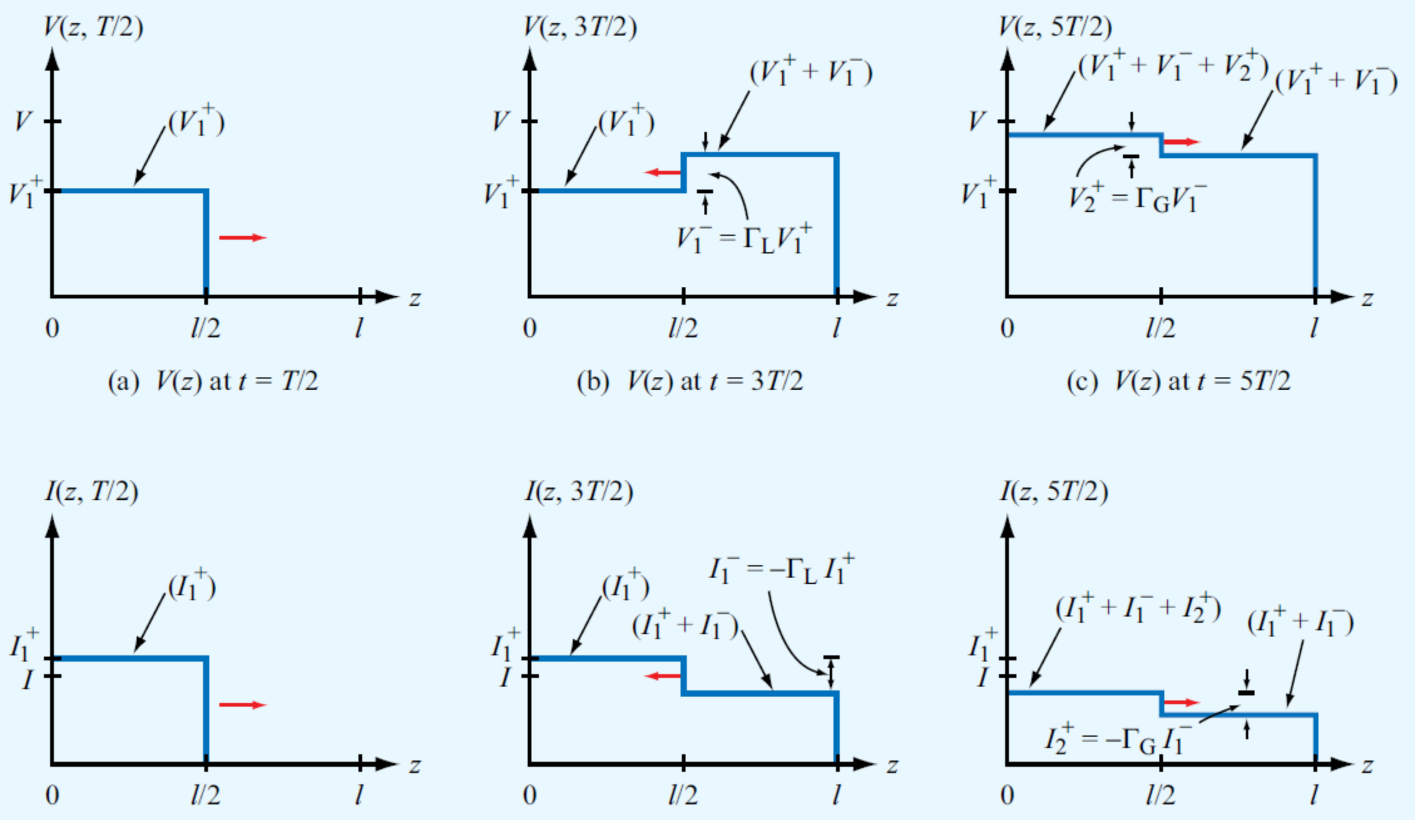
$$V_\infty = V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \cdots = V_1^+\left[1 + \Gamma_L + \Gamma_L\Gamma_g + \Gamma_L^2\Gamma_g + \Gamma_L^2\Gamma_g^2 + \Gamma_L^3\Gamma_g^2 + \cdots\right]$$

$$V_\infty = V_1^+(1 + \Gamma_L)\left(1 + \Gamma_L\Gamma_g + \Gamma_L^2\Gamma_g^2 + \cdots\right) = V_1^+ \frac{1 + \Gamma_L}{1 - \Gamma_L\Gamma_g} = \frac{R_L}{R_g + R_L} V_g$$

Reflections at both ends: example

- $R_g = 4Z_0$ and $R_L = 2Z_0$

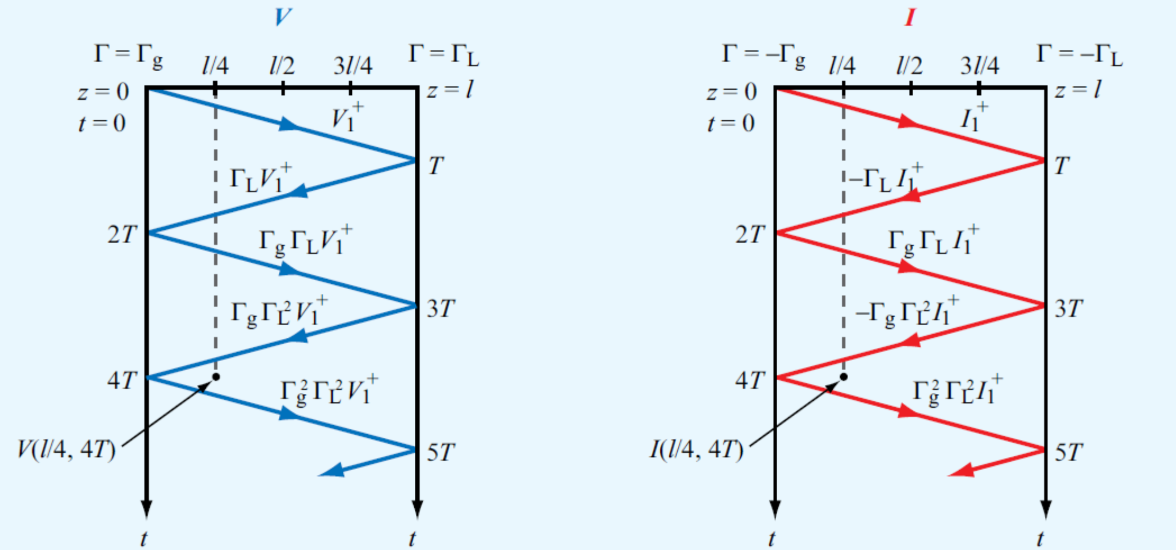
$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{1}{3} \qquad \Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} = \frac{3}{5}$$



(a) $V(z)$ at $t = T/2$ (b) $V(z)$ at $t = 3T/2$ (c) $V(z)$ at $t = 5T/2$

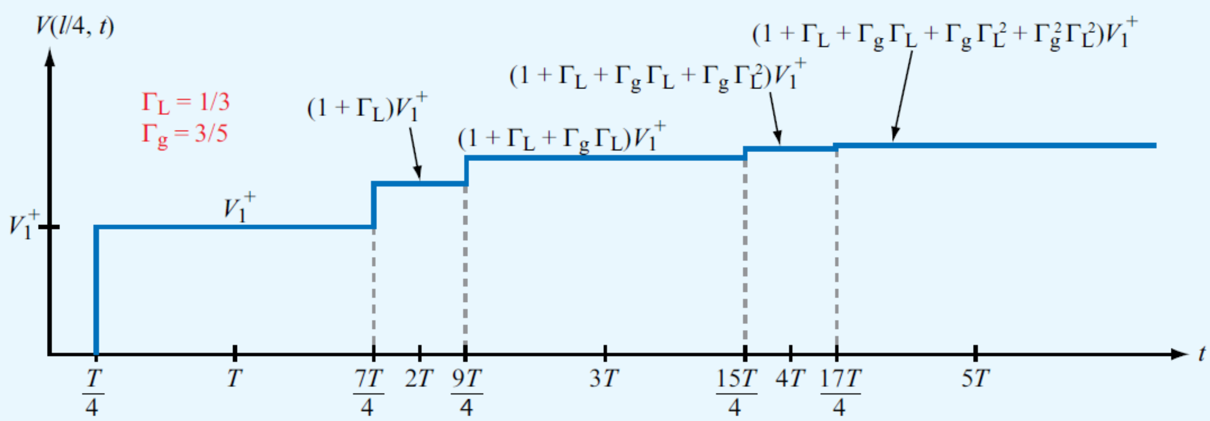
(d) $I(z)$ at $t = T/2$ (e) $I(z)$ at $t = 3T/2$ (f) $I(z)$ at $t = 5T/2$

Bounce diagram



(a) Voltage bounce diagram

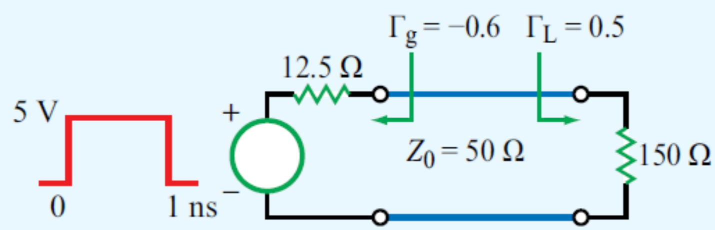
(b) Current bounce diagram



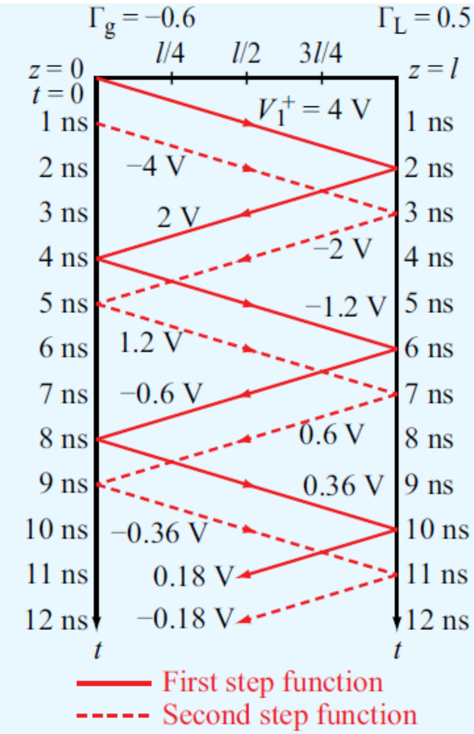
(c) Voltage versus time at $z = l/4$

Pulse propagation

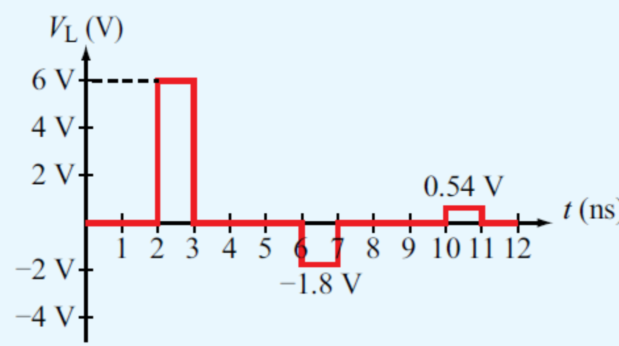
- Q. The transmission-line circuit of Fig. (a) is excited by a rectangular pulse of duration $\tau = 1\ ns$ that starts at $t = 0$. Establish the waveform of the voltage response at the load, given that the pulse amplitude is 5 V, the phase velocity is c, and the length of the line is 0.6 m.



(a) Pulse circuit



(b) Bounce diagram



(c) Voltage waveform at the load

- Solution

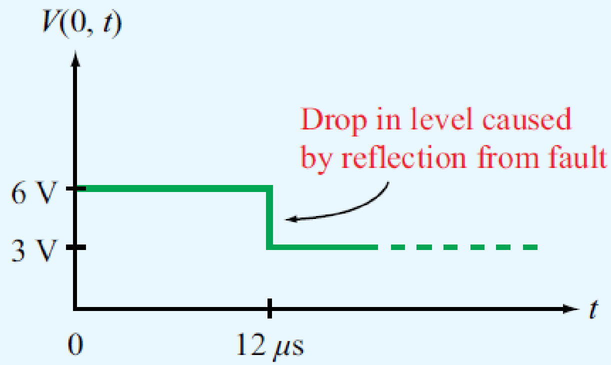
$$T = \frac{l}{c} = 2\ ns$$

$$\Gamma_L = \frac{150 - 50}{150 + 50} = 0.5 \qquad \Gamma_g = \frac{12.5 - 50}{12.5 + 50} = -0.6$$

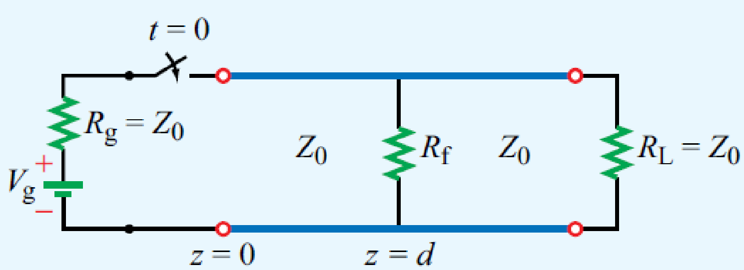
$$V_1^+ = \frac{Z_0 V_{01}}{R_g + Z_0} = \frac{50 \times 5}{12.5 + 50} = 4$$

Time-domain reflectometer (TDR)

- Q. If the voltage waveform shown in Fig. (a) is seen on an oscilloscope connected to the input of a 75 Ωmatched transmission line, determine (a) the generator voltage, (b) the location of the fault, and (c) the fault shunt resistance. The line's insulating material is Teflon with $\epsilon_r = 2.1$.



(a) Observed voltage at the sending end



(b) The fault at $z = d$ is represented by a fault resistance R_f

- Solution (a)

$$V_1^+ = \frac{Z_0 V_g}{R_g + Z_0} = \frac{V_g}{2} \qquad V_g = 2V_1^+ = 12\ V$$

- Solution (b)

$$\Delta t = \frac{2d}{u_p} = \frac{2d}{c/\sqrt{\epsilon_r}} = 12\ \mu s \qquad d = 1{,}242\ m$$

- Solution (c)

$$V_1^- = \Gamma_f V_1^+ = -3\ V \qquad \Gamma_f = -0.5$$

$$\Gamma_f = \frac{R_{Lf} - Z_0}{R_{Lf} + Z_0} \qquad R_{Lf} = 25\ \Omega$$

$$\frac{1}{R_{Lf}} = \frac{1}{R_f} + \frac{1}{Z_0} \qquad R_f = 37.5\ \Omega$$