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Utility theory

Syllabus objectives

1.2 Rational choice theory

1. Explain the meaning of the term 'utility function'.
2. Explain the axioms underlying utility theory and the expected utility theorem.
3. Explain how the following economic characteristics of investors can be expressed mathematically in a utility function:
   - non-satiation
   - risk aversion, risk-neutrality and risk-seeking
   - declining or increasing absolute and relative risk aversion.
4. Discuss the economic properties of commonly used utility functions.
5. Discuss how a utility function may depend on current wealth and discuss state-dependent utility functions.
6. Perform calculations using commonly used utility functions to compare investment opportunities.
8. Analyse simple insurance problems in terms of utility theory.

0 Introduction

This chapter focuses on utility theory as applied to investment choices.

In economics, 'utility' is the satisfaction that an individual obtains from a particular course of action.

In Section 1 we introduce utility functions and the *expected utility theorem*. This provides a means by which to model the way individuals make investment choices.

Section 2 describes the properties that are normally considered desirable in utility functions to ensure that they reflect the actual behaviour of investors. Chief amongst these are:

- *non-satiation*, a preference for more over less, and
- *risk aversion*, a dislike of risk.

These ideas underlie the rest of the course.

Section 3 considers various methods of measuring risk aversion and the way in which risk aversion might vary with wealth. The concepts of *absolute risk aversion* and *relative risk aversion* are discussed.

Section 4 introduces some commonly used examples of utility functions, namely the *quadratic*, *log*, and *power* utility functions, and discusses the properties of each.

Section 5 describes how to deal with situations in which a single utility function is inappropriate. In such instances, it may be necessary to vary either the parameters or the form of the utility function according to the particular situation to be modelled. This leads to the idea of *state-dependent* utility functions.

In order to use the expected utility theory, we need an explicit utility function. In Section 6 we look at how we might go about constructing such a utility function.

Section 7 then uses these utility functions to solve problems involving insurance premiums. In particular, determining the maximum premium a policyholder is willing to pay and the minimum amount an insurer should charge.

Expected utility theory can be useful, but it is not without problems. In Section 8, we therefore consider the limitations of the expected utility theory for risk management purposes – in particular, the need to know the precise form and shape of the individual's utility function.

1 Utility theory

1.1 Introduction

In this section we use utility theory to consider situations that involve uncertainty, as will normally be the case where investment choices are concerned.

Uncertainty

In what follows, we assume any asset that yields uncertain outcomes or returns, *ie* any *risky* asset, can be characterised as a set of objectively known probabilities defined on a set of possible outcomes. For example, Equity A might offer a return to Investor X of either -4 % or +8% in the next time period, with respective probabilities of ¼ and ¾.

Question

Each year, Mr A is offered the opportunity to invest £1,000 in a risk fund. If successful, at the end of the year he will be given back £2,000. If unsuccessful, he will be given back only £500. There is a 50% chance of either outcome. Calculate the expected rate of return per annum on the investment.

Solution

We can calculate the expected rate of return as follows:

$$\frac{(0.5\times 2{,}000)+(0.5\times 500)}{1{,}000}-1=25\%$$

Given the uncertainty involved, the rational investor cannot maximise utility with complete certainty. We shall see that the rational investor will instead attempt to maximise *expected* utility by choosing between the available risky assets.

Utility functions

In the application of utility theory to finance and investment choice, it is assumed that a numerical value called the *utility* can be assigned to each possible value of the investor's wealth by what is known as a 'preference function' or 'utility function'.

Utility functions show the level of utility associated with different levels of wealth. For example, Investor X might have a utility function of the form:

$$U(w) = \log(w)$$

where w is the current or future wealth.

1.2 The expected utility theorem

Introduction

The theorem has two parts.

1. **The expected utility theorem states that a function, $U(w)$, can be constructed as representing an investor's utility of wealth, w, at some future date.**

2. **Decisions are made in a manner to maximise the expected value of utility given the investor's particular beliefs about the probability of different outcomes.**

In situations of uncertainty it is impossible to maximise utility with complete certainty. For example, suppose that Investor X invests a proportion a of his wealth in Equity A and places the rest in a non-interest-bearing bank account. Then his wealth in the next period cannot be predicted with complete certainty and hence neither can his utility.

It is possible, however, to say what his *expected* wealth equals. Likewise if the functional form of $U(w)$ is known, then it is possible to calculate his *expected* utility. The expected utility theorem says that when making a choice an individual should choose the course of action that yields the highest expected *utility* – and *not* the course of action that yields the highest expected wealth, which will usually be different.

Question

Derive an expression for the expectation of Investor X's next-period wealth if he invests a proportion a of his current wealth w in Equity A (which pays –4% or +8%, with respective probabilities ¼ and ¾) and the rest in a non-interest-bearing bank account.

Solution

$$\begin{aligned}E(w) &= (1-a)w + aw\left[0.25\times 0.96 + 0.75\times 1.08\right] \\ &= (1-a)w + 1.05aw \\ &= (1+0.05a)w\end{aligned}$$

The answer can also be arrived at directly by noting that the expected next-period wealth will be the initial wealth w, plus the expected return of 5% on the investment aw.

Calculating the expected utility

Suppose a risky asset has a set of N possible outcomes for wealth $(w_1,\ldots,w_N)$, each with associated probabilities of occurring of $(p_1,\ldots,p_N)$, then the *expected utility* yielded by investment in this risky asset is given by:

$$E(U) = \sum_{i=1}^{N} p_i\, U(w_i)$$

So the expected utility is a weighted average of the utilities associated with each possible individual outcome.

Question

State an expression for the expectation of the next-period utility of Investor X, again assuming that he invests a proportion a in Equity A and the rest in a non-interest-bearing bank account. He has the utility function $U(w) = \log(w)$.

Solution

$$E\left[U\left(w\right)\right]=0.25\left\{\log\left((1-0.04a)w\right)\right\}+0.75\left\{\log\left((1+0.08a)w\right)\right\}$$

Note that a *risk-free asset* is a special case of a risky asset that has a probability of one associated with the *certain* outcome, and zero probability associated with all other outcomes.

By combining an investor's beliefs about the set of available assets with a suitable utility function, we can determine the optimal investment portfolio for the investor, *ie* that which maximises expected utility in that period.

Question

Investor A has an initial wealth of $100, which is currently invested in a non-interest-bearing account, and a utility function of the form:

$$U(w)=\log(w)$$

where w is the investor's wealth at any time.

Investment Z offers a return of -18% or +20% with equal probability.

(i) What is Investor A's expected utility if nothing is invested in Investment Z?

(ii) What is Investor A's expected utility if they're entirely invested in Investment Z?

(iii) What proportion a of wealth should be invested in Investment Z to maximise expected utility? What is Investor A's expected utility if they invest this proportion in Investment Z?

Solution

(i) If nothing is invested in Investment Z, the expected utility is:

$$\log(100) = 4.605$$

(ii) If Investor A is entirely invested in Investment Z, the expected utility is:

$$0.5 \times \log(0.82 \times 100) + 0.5 \times \log(1.2 \times 100) = 4.597$$

(iii) If a proportion a of wealth is invested in Investment Z, the expected utility is given by:

$$E\left[U(w)\right]=0.5\left\{\log\left(100(1-0.18a)\right)\right\}+0.5\left\{\log\left(100(1+0.2a)\right)\right\}$$
$$=0.5\left\{\log\left(100-18a\right)\right\}+0.5\left\{\log\left(100+20a\right)\right\}$$

We differentiate with respect to a to find a maximum:

$$\frac{dE\left[U(w)\right]}{da}=0.5\times\frac{-18}{100-18a}+0.5\times\frac{20}{100+20a}$$
$$=\frac{-9}{100-18a}+\frac{10}{100+20a}$$

We then set equal to zero:

$$\frac{9}{100-18a}=\frac{10}{100+20a}$$

Solving, we find $a=0.2777$.

Checking to see if this gives a maximum:

$$\frac{d^2E\left[U(w)\right]}{da^2}=\frac{+9(-18)}{(100-18a)^2}+\frac{-10(20)}{(100+20a)^2}$$

This gives a negative value so it is a maximum.

Finding the expected utility from investing 27.77% in Investment Z:

$$E\left[U(w)\right]=0.5\left\{\log\left(100-18\times 0.2777\right)\right\}+0.5\left\{\log\left(100+20\times 0.2777\right)\right\}$$
$$=4.6066$$

1.3 Derivation of the expected utility theorem

The expected utility theorem can be derived formally from the following four axioms.

In other words, an investor whose behaviour is consistent with these axioms will always make decisions in accordance with the expected utility theorem.

1. Comparability

An investor can state a preference between all available certain outcomes.

In other words, for any two certain outcomes A and B, either:

A is preferred to B,

B is preferred to A,

or the investor is indifferent between A and B.

These preferences are sometimes denoted by:

$$U(\mathrm{A}) > U(\mathrm{B}),\ U(\mathrm{B}) > U(\mathrm{A}) \text{ and } U(\mathrm{A}) = U(\mathrm{B})$$

Note that A and B are examples of what we previously referred to as w_i.

2. Transitivity

If A is preferred to B and B is preferred to C, then A is preferred to C.

ie $\quad U(\mathrm{A}) > U(\mathrm{B}) \text{ and } U(\mathrm{B}) > U(\mathrm{C}) \quad \Rightarrow \quad U(\mathrm{A}) > U(\mathrm{C})$

Also:

$$U(\mathrm{A}) = U(\mathrm{B}) \text{ and } U(\mathrm{B}) = U(\mathrm{C}) \quad \Rightarrow \quad U(\mathrm{A}) = U(\mathrm{C})$$

This implies that investors are consistent in their rankings of outcomes.

3. Independence

If an investor is indifferent between two certain outcomes, A and B, then they are also indifferent between the following two gambles:

(i) A with probability *p* and C with probability (1 – *p*)

(ii) B with probability *p* and C with probability (1 – *p*).

Hence, if $U(\mathrm{A}) = U(\mathrm{B})$ (and of course $U(\mathrm{C})$ is equal to itself), then:

$$p\,U(\mathrm{A}) + (1-p)\,U(\mathrm{C}) = p\,U(\mathrm{B}) + (1-p)\,U(\mathrm{C})$$

Thus, the choice between any two certain outcomes is independent of all other certain outcomes.

4. Certainty equivalence

Suppose that A is preferred to B and B is preferred to C. Then there is a unique probability, *p*, such that the investor is indifferent between B and a gamble giving A with probability *p* and C with probability (1 – *p*).

Thus if:

$$U(\mathrm{A}) > U(\mathrm{B}) > U(\mathrm{C})$$

Then there exists a unique p ($0 < p < 1$) such that:

$$pU(A) + (1-p)U(C) = U(B)$$

B is known as the 'certainty equivalent' of the above gamble.

It represents the certain outcome or level of wealth that yields the same certain utility as the expected utility yielded by the gamble or lottery involving outcomes A and C.

The four axioms listed above are not the only possible set of axioms, but they are the most commonly used.

Question

Suppose that an investor is asked to choose between various pairs of strategies and responds as follows:

Choose between:	*Response*
B and D	B
A and D	D
C and D	indifferent
B and E	B
A and C	C
D and E	indifferent

Assuming that the investor's preferences satisfy the four axioms discussed above, how does the investor rank the five investments A to E?

Solution

From the responses we can note immediately that:

B > D,   D > A,   C = D,   B > E,   C > A,   D = E

Hence, transitivity then implies that:

B > D > A
C = D = E

And so we have that: B > C = D = E > A

2 The expression of economic characteristics in terms of utility functions

In mainstream finance theory, investors' preferences are assumed to be influenced by their attitude to risk. We need to consider, therefore, how an investor's risk-return preference can be described by the form of their utility function.

The mathematical form of utility functions is normally assumed to satisfy desirable properties that accord with everyday observation about how individuals typically act in the face of uncertainty.

2.1 Non-satiation

As a basis to understanding risk attitudes, let us first assume that people prefer more wealth to less. This is known as the principle of non-satiation and can be expressed as:

$$U'(w) > 0$$

This is clearly analogous to individuals preferring more to less of a good or service in the standard choice between different bundles of goods in situations of certainty.

The derivative of utility with respect to wealth is often referred to as the *marginal utility of wealth*. Non-satiation is therefore equivalent to an assumption that the marginal utility of wealth is strictly positive.

2.2 Risk aversion

Attitudes to risk can now be expressed in terms of the properties of utility functions.

In particular, we can choose the form of the utility function that we use to model an individual's preferences according to whether or not the individual concerned likes, dislikes or is indifferent to risk.

Risk-averse investor

A *risk-averse* investor values an incremental increase in wealth less highly than an incremental decrease and will reject a fair gamble.

A *fair gamble* is one that leaves the expected wealth of the individual unchanged. Equivalently, it can be defined as a gamble that has an overall expected value of zero.

Question

Suppose that an unbiased coin is tossed once. Determine the fairness of a gamble in which you receive \$1 if it lands heads up but lose \$1 if it lands tails up.

Solution

The gamble is fair because your expected gain from accepting the gamble is zero and your expected wealth remains unchanged (though your actual wealth will of course change by \$1).

Equivalently, the overall expected value of the gamble is equal to:

$$\tfrac{1}{2} \times (+\$1) + \tfrac{1}{2} \times (-\$1) = 0$$

Risk-averse investors derive less additional utility from the prospect of a possible gain than they lose from the prospect of an identical loss with the same probability of occurrence. Consequently they will not accept a fair gamble. They may, however, be willing to trade off lower expected wealth in return for a reduction in the variability of wealth. This is the basic principle underlying insurance.

It is normally assumed that investors are risk-averse and consequently that they will accept additional risk from an investment only if it is associated with a higher level of expected return. Hence, the importance of the trade-off between risk and return that is a feature of most investment decisions.

For a risk-averse investor, the utility function condition is:

$$\boldsymbol{U''(w) < 0}$$

In other words, for a risk-averse investor, utility is a (strictly) concave function of wealth, as shown in Figure 2.1.

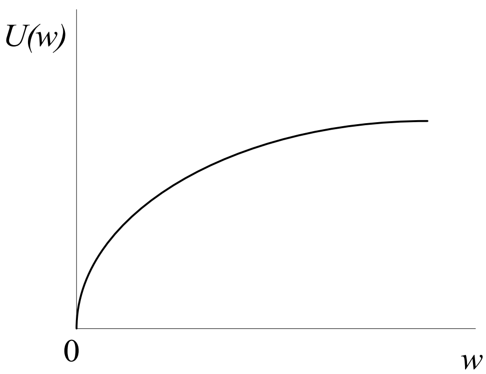


Figure 2.1 – A concave utility function for a risk-averse investor

This concavity condition means that the marginal utility of wealth (strictly) *decreases* with the level of wealth and consequently each additional dollar, say, adds less satisfaction to the investor than the previous one.

Question

Suppose that a risk-averse investor with wealth w is faced with the gamble described in the previous question. Show that this investor will derive less additional utility from the possible gain than that lost from the possible loss and hence that risk aversion is consistent with the condition $U''(w) < 0$.

Solution

The investor's certain utility if the gamble is rejected is $U(w)$. The investor's expected utility obtained by accepting the gamble is given by:

$$E(U) = ½U(w-1) + ½U(w+1)$$

The gamble is therefore rejected if:

$$½\,U(w-1) + ½\,U(w+1) \;<\; U(w)$$

$$\Leftrightarrow \qquad U(w-1) + U(w+1) \;<\; 2\,U(w)$$

$$\Leftrightarrow \qquad U(w+1) - U(w) \;<\; U(w) - U(w-1)$$

ie if the additional utility from winning the gamble is less than the loss of utility from losing the gamble. This will be the case if $U''(w) < 0$, which is by definition the case for a risk-averse investor.

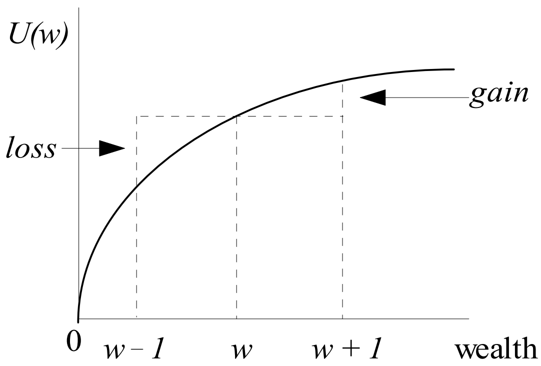


Figure 2.2 – Losses and gains in utility

Risk-seeking investor

A *risk-seeking* investor values an incremental increase in wealth more highly than an incremental decrease and will seek a fair gamble. The utility function condition is:

$$\boldsymbol{U''(w) > 0}$$

A risk-seeking or risk-loving investor will accept any fair gamble and may even accept some unfair gambles (that reduce expected wealth) because the potential increase in utility resulting from the possible gain exceeds the potential decrease in utility associated with the corresponding loss.

Question

What is the shape of the utility function of a risk-seeking investor?

Solution

A risk-seeking investor has a convex utility function, because $U''(w) > 0$. The utility function looks as follows:

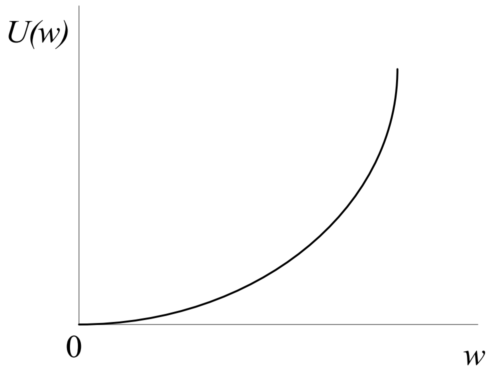


Figure 2.3 – A convex utility function

Risk-neutral investor

A *risk-neutral* investor is indifferent between a fair gamble and the *status quo*. In this case:

$$U''(w) = 0$$

Question

What can we say about the marginal utility of wealth of a risk-neutral investor?

Solution

For a risk-neutral investor, $U''(w) = 0$. Thus, $U'(w)$ is constant and so the marginal utility of wealth must itself be constant (and positive assuming non-satiation), so that each additional \$1 leads to the same change in utility, regardless of wealth.

The utility function of a risk-neutral investor is a *linear* function of wealth. Assuming non satiation, then $U'(w) > 0$ and so $U(w)$ increases with w for all w. Thus, the maximisation of expected utility is equivalent to the maximisation of expected *wealth*, in the sense that it will always lead to the same choices being made.

Question

(i) Ignoring any pleasure derived from gambling, a risk-averse person will:

A never gamble
B accept fair gambles
C accept fair gambles and some gambles with an expected loss
D none of the above

(ii) Ignoring any pleasure derived from gambling, a risk-neutral person will:

I always accept fair gambles
II always accept unfair gambles
III always accept better than fair gambles

A I and II are true
B II and III are true
C I only is true
D III only is true

(iii) A risk-loving person will:

I always accept a gamble
II always accept unfair gambles
III always accept fair gambles

A I and II are true
B II and III are true
C I only is true
D III only is true

Solution

(i) *A risk-averse person will not accept fair gambles*. However, they might accept a gamble where they expected, on average, to win. This would happen if the expected profit from gambling was sufficient to compensate them for taking on the risk. Therefore the correct answer is D.

(ii) *A risk-neutral person will be indifferent to accepting a fair gamble*, but will accept better than fair gambles. Therefore statement III is always true. Statement II is false. Statement I is false (we can't be certain that a person who is indifferent to the gamble will always accept it). The answer is thus D.

(iii) *A risk-loving person will be happy to accept fair gambles*. A risk-loving person will also accept *some* unfair gambles. However, if the odds are very unfair, even a risk-loving person will not accept a gamble. Thus the correct answer is D.

Note that this solution shows how risk aversion/neutrality/loving can be defined in terms of a person's attitude towards a fair gamble.

3 Measuring risk aversion

3.1 Introduction

In practice, we normally assume that an investor is risk-averse and by looking at the sign of $U''(w)$ we can deduce whether or not this is in fact the case.

The way risk aversion changes with wealth may also be of interest.

The degree of risk aversion is likely to vary with the investor's existing level of wealth. For example, we might imagine that wealthy investors are less concerned about risk.

3.2 Risk aversion and the certainty equivalent

Consider the certainty equivalent of a gamble. For a risk-averse individual this is higher than the actual likelihood of the outcome, *ie* the individual would need to receive odds higher than expected to accept this gamble.

Alternative definitions of the certainty equivalent

Note that we can distinguish between two different types of certainty equivalents depending upon the situation that we are considering, namely:

- The certainty equivalent of the portfolio consisting of the combination of the existing wealth *w and* the gamble *x*, which we can denote c_w.
- The certainty equivalent of the gamble *x* alone, c_x, which will also depend upon the existing level of wealth.

Thus, for a fair gamble and a risk-averse investor, it must be the case that $c_w < w$ and $c_x < 0$.

1. 'Additive' or 'absolute' gamble

Consider a gamble with outcomes represented by a random variable *x*, in which the sums won or lost are *fixed absolute amounts*. The actual sums won or lost are therefore independent of the value of initial wealth *w*. If the investor accepts the gamble, the resulting total wealth is *w + x.*

The certainty equivalent of the combined portfolio of initial wealth plus gamble, c_w, is then defined as the certain level of wealth that solves:

$$U(c_w) = E\left[U\left(w+x\right)\right]$$

The certainty equivalent of the gamble itself is equal to:

$$c_x = c_w - w$$

because we require that $U(w+c_x)=U(c_w)$ and U is a strictly increasing function.

c_x is negative for a fair gamble and its absolute value represents the maximum sum that the risk-averse investor would pay to avoid the risk.

Question

Suppose that an unbiased coin is tossed once, and a gamble exists in which an investor receives \$1 if it lands heads up but loses \$1 if it lands tails up. Further assume that:

- the investor has initial wealth of \$10 and
- a utility function of the form $U(w)=\sqrt{w}$.

Determine the investor's certainty equivalent for this gamble.

Solution

With an initial wealth of 10, the expected utility of total wealth is given by:

$$½\times\left[\sqrt{11}+\sqrt{9}\right]=3.1583$$

The certainty equivalent, c_w , of the initial wealth plus the gamble satisfies:

$$U(c_w)=3.1583$$

$$\Rightarrow \qquad \sqrt{c_w}=3.1583 \qquad \Rightarrow \qquad c_w = 3.1583^2 = 9.9749$$

Hence, the certainty equivalent of the gamble itself is:

$$c_x = c_w - w = 9.9749 - 10 = -0.0251$$

Note that c_x is negative. This means we would have to pay the investor to accept the gamble. Equivalently, the investor would be prepared to pay 0.0251 to avoid the gamble.

2. 'Multiplicative' or 'proportional' gamble

This is a gamble, with outcomes represented by a random variable *y*, in which the sums won or lost are all expressed as *proportions* of the initial wealth. If the investor accepts the gamble they therefore end up with a final wealth of $w \times y$. For example, in a fair gamble of this type, the investor might win 15% of their initial wealth (y = 1.15) with probability ¼ and lose 5% (y = 0.95) with a probability of ¾. Note that in this case, the actual sums won or lost therefore depend directly upon the value of initial wealth *w*, *ie* a larger *w* produces larger wins or losses.

In this instance, the certainty equivalent of total wealth including the proceeds of the gamble can be defined as the level of wealth that satisfies:

$$U(c_w) = E\left[U\left(w \times y\right)\right]$$

The certainty equivalent of the gamble alone is defined as before and is again negative for a fair gamble for a risk-averse investor.

The certainty equivalent and absolute risk aversion

If the absolute value of the certainty equivalent decreases with increasing wealth, the investor is said to exhibit declining *absolute* risk aversion. If the absolute value of the certainty equivalent increases, the investor exhibits increasing *absolute* risk aversion.

Here we have in mind:

- an *additive* gamble
- the certainty equivalent of the gamble alone, c_x .

If the investor's preferences exhibit decreasing (increasing) absolute risk aversion (ARA), then the absolute value of c_x decreases (increases) and the investor is prepared to pay a smaller (larger) absolute amount in order to avoid the risk associated with the gamble.

ie increasing / decreasing ARA $\Leftrightarrow$ increasing / decreasing $|c_x|$

The certainty equivalent and relative risk aversion

If the absolute value of the certainty equivalent decreases (increases) as a *proportion* of total wealth as wealth increases the investor is said to exhibit declining (increasing) *relative* risk aversion.

Here we are looking at:

- a multiplicative gamble
- the certainty equivalent of the gamble alone as a proportion of initial wealth, *ie* $\dfrac{c_x}{w}$.

If the investor's preferences exhibit decreasing (increasing) relative risk aversion (RRA), then the absolute value of c_x/w decreases (increases) with wealth *w.*

ie increasing / decreasing RRA $\Leftrightarrow$ increasing / decreasing $\left|\dfrac{c_x}{w}\right|$

Example

We shall see below that the log utility function $U(w) = \log(w)$ exhibits:

- decreasing absolute risk aversion
- constant relative risk aversion.

To see these results, consider an individual with an initial wealth of \$100, who faces a fair gamble that offers an equal chance of winning or losing \$20. In this case:

$$\begin{aligned} U(c_w) &= E\left[U(w+x)\right] \\ &= \frac{1}{2}\left[\log 120 + \log 80\right] \\ &= 4.5848 \end{aligned}$$

$\Rightarrow \quad c_w = e^{4.5848} = 97.980$ and $c_x = 97.980 - 100 = -2.020$

If instead the individual's initial wealth is \$200, then:

$$U(c_w) = \frac{1}{2}\left[\log 220 + \log 180\right] = 5.2933$$

$\Rightarrow \quad c_w = e^{5.2933} = 198.997$

and: $\quad c_x = 198.997 - 200 = -1.003$

The absolute value of c_x, the certainty equivalent of the (fair) gamble alone, has decreased with wealth (for a gamble with *fixed absolute proceeds*), as is the case with decreasing absolute risk aversion.

Let us now consider the case of a multiplicative gamble. Suppose the individual is offered an equal chance of winning or losing 20% of their initial wealth. If the initial wealth is \$100, then the investor could win or lose \$20. This is equivalent to our first example. We have seen that $c_w = 97.980$ and that $c_x = -2.020$. We can also find that:

$$\frac{c_x}{w} = \frac{-2.020}{100} = -0.0202$$

If the initial wealth is \$200, the investor could win or lose 20%, *ie* \$40. Then:

$$U(c_w) = \frac{1}{2}\left[\log 240 + \log 160\right] = 5.2779$$

$\Rightarrow \quad c_w = e^{5.2779} = 195.959$

And: $\quad c_x = 195.959 - 200 = -4.041$

Thus:

$$\frac{c_x}{w} = \frac{-4.041}{200} = -0.0202$$

ie the absolute value of c_x/w, the certainty equivalent of the (fair) gamble as a proportion of initial wealth, is invariant to wealth (for a gamble with *fixed percentage proceeds*), corresponding to constant relative risk aversion.

3.3 Risk aversion and the utility function

Absolute and relative risk aversion can be expressed in terms of the utility function as follows.

Absolute risk aversion is measured by the function

$$A(w) = \frac{-U''(w)}{U'(w)}$$

Relative risk aversion is measured by the function

$$R(w) = -w\frac{U''(w)}{U'(w)}$$

These are often referred to as the *Arrow-Pratt* measures of absolute risk aversion and relative risk aversion.

The above results concerning the relationship between the certainty equivalent and the measures of risk aversion arise because it can be shown that the:

- absolute value of the certainty equivalent of a fair gamble is proportional to $\dfrac{-U''(w)}{U'(w)}$
- absolute value of the certainty equivalent of a fair gamble expressed as a proportion of the investor's wealth is proportional to $-w\dfrac{U''(w)}{U'(w)}$.

The following table shows the relationships between the first derivatives of the above functions and declining or increasing absolute and relative risk aversion.

	Absolute risk aversion	***Relative risk aversion***
Increasing	$A'(w) > 0$	$R'(w) > 0$
Constant	$A'(w) = 0$	$R'(w) = 0$
Decreasing	$A'(w) < 0$	$R'(w) < 0$

3.4 Risk aversion and the investment choice

The way that risk aversion changes with wealth can be expressed in terms of the amount of wealth held as risky assets.

Investors who hold an increasing *absolute* amount of wealth in risky assets as they get wealthier exhibit declining *absolute* risk aversion. Investors who hold an increasing *proportion* of their wealth in risky assets as they get wealthier exhibit declining *relative* risk aversion.

In practice, it is often assumed that as wealth increases, so the absolute amount that a typical investor is willing to invest in risky assets will increase, *ie* that absolute risk aversion decreases with wealth.

It is not so clear cut as to whether we would expect the *proportion* of risky assets to increase or decrease. Consequently the assumption of constant relative risk aversion is sometimes made.

4 Some commonly used utility functions

4.1 The quadratic utility function

The general form of the quadratic utility function is

$$U(w) = a + bw + cw^2$$

Since adding a constant to a utility function or multiplying it by a constant will not affect the decision-making process, we can write the general form simply as:

$$U(w) = w + dw^2$$

Thus:

$$U'(w) = 1 + 2dw$$

and $$U''(w) = 2d$$

Therefore, if the quadratic utility function is to satisfy the condition of diminishing marginal utility of wealth (risk aversion), we must have $d < 0$.

The consequence of this is that the quadratic utility function can only satisfy the condition of non-satiation over a limited range of w:

$$-\infty < w < -\frac{1}{2d}$$

This constraint on the range of possible values for w is a significant limitation of using quadratic utility functions.

The absolute and relative risk aversion measures are given by:

$$A(w) = \frac{-U''(w)}{U'(w)} = \frac{-2d}{1+2dw}$$

$$A'(w) = \frac{4d^2}{(1+2dw)^2} > 0$$

and:

$$R(w) = w\frac{-U''(w)}{U'(w)} = \frac{-2dw}{1+2dw}$$

$$R'(w) = \frac{-2d}{1+2dw} + \frac{4d^2w}{(1+2dw)^2} = \frac{-2d}{(1+2dw)^2} > 0$$

Thus the quadratic utility function exhibits both increasing absolute and relative risk aversion.

Question

Draw the quadratic utility function over the range $0 \le w < -1/2d$ and show why it is valid only for $w < -1/2d$, for a non-satiated risk-averse investor.

Solution

For non-satiation we require:

$$U'(w) > 0, \ \textit{ie}\ 1 + 2dw > 0$$

$$\Leftrightarrow \qquad 2dw > -1$$

$$\Leftrightarrow \qquad dw > -½$$

$$\Leftrightarrow \qquad w < -1/2d, \text{ as } d < 0 \text{ for a risk-averse investor.}$$

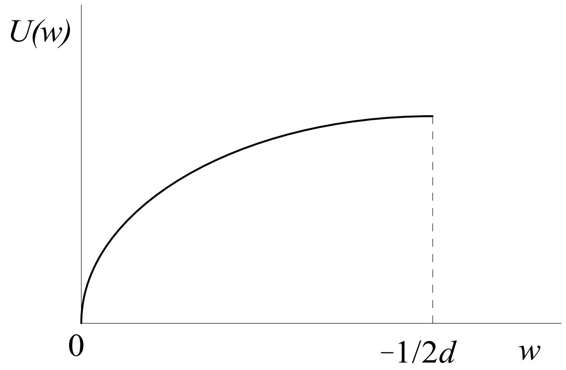


Figure 2.4 – The range of the quadratic utility function

4.2 The log utility function

The form of the log utility function is:

$$U(w) = \ln(w) \qquad (w > 0)$$

Thus:

$$U'(w) = \frac{1}{w}$$

and:

$$U''(w) = -\frac{1}{w^2}$$

Thus the log utility function satisfies the principle of non-satiation and diminishing marginal utility of wealth.

This is because we have assumed that the log utility function is defined only for positive values of w and $U'(w) = \dfrac{1}{w} > 0$, *ie* non-satiation and $U''(w) = \dfrac{-1}{w^2} < 0$, *ie* diminishing marginal utility of wealth, for $w > 0$.

The absolute and relative risk aversion measures are given by:

$$A(w) = \frac{-U''(w)}{U'(w)} = \frac{1}{w}$$

$$A'(w) = -\frac{1}{w^2} < 0$$

and:

$$R(w) = w\frac{-U''(w)}{U'(w)} = 1$$

$$R'(w) = 0$$

Thus the log utility function exhibits declining absolute risk aversion and constant relative risk aversion. This is consistent with an investor who keeps a constant proportion of wealth invested in risky assets as they get richer.

This investor will also invest an increasing *absolute* amount of wealth in risky assets.

Utility functions exhibiting constant relative risk aversion are said to be 'iso-elastic'.

Iso-elastic means that the elasticity of the marginal utility of wealth is constant with respect to wealth.

The use of iso-elastic utility functions simplifies the determination of an optimal strategy for a multi-period investment decision, because it allows for a series of so-called 'myopic' decisions. What this means is that the decision at the start of each period only considers the possible outcomes at the end of that period and ignores subsequent periods.

Thus, the individual's utility maximisation choice in each period is independent of all subsequent periods. The decision is said to be 'myopic' because it is short-sighted, *ie* it does not need to look to future periods.

4.3 The power utility function

The form of the power utility function is:

$$U(w) = \frac{w^\gamma - 1}{\gamma} \qquad (w > 0)$$

Thus:

$$U'(w) = w^{\gamma-1}$$

and:

$$U''(w) = (\gamma-1)w^{\gamma-2}$$

Thus for the power utility function to satisfy the principle of non-satiation and diminishing marginal utility of wealth we require $\gamma < 1$.

The absolute and relative risk aversion measures are given by:

$$A(w) = \frac{-U''(w)}{U'(w)} = -\frac{(\gamma-1)}{w}$$

$$A'(w) = \frac{(\gamma-1)}{w^2} < 0$$

and:

$$R(w) = w\frac{-U''(w)}{U'(w)} = -(\gamma-1)$$

$$R'(w) = 0$$

Thus, like the log utility function, the power utility function exhibits declining absolute risk aversion and constant relative risk aversion.

It is therefore also iso-elastic.

The power utility function, in the form given above, is one of a wider class of commonly used functions known as HARA (hyperbolic absolute risk aversion) functions. γ is the risk aversion coefficient.

This is because, for such functions, the absolute risk aversion is a hyperbolic function of wealth *w*. For example, in the case of the log utility function:

$$w \times A(w) = \text{constant}$$

Hence, a plot of *A(w)* against *w* describes a rectangular hyperbola.

Question

Suppose Investor A has a *power* utility function with $\gamma = 1$, whilst Investor B has a power utility function with $\gamma = 0.5$.

(i) Which investor is more risk-averse (assuming that $w > 0$)?

(ii) Suppose that Investor B has an initial wealth of 100 and is offered the opportunity to buy Investment X for 100, which offers an equal chance of a payout of 110 or 92. Will the Investor B choose to buy Investment X?

Solution

(i) ***Which investor is more risk-averse?***

Investor B is more risk-averse because they have a lower risk aversion coefficient γ. We can show this by deriving the absolute risk aversion and relative risk aversion measures for each investor.

For Investor A:

$$A(w) = R(w) = 0$$

ie Investor A is *risk-neutral*.

For Investor B:

$$A(w) = \frac{1}{2w} > 0, \quad R(w) = \frac{1}{2} > 0$$

Hence, Investor B is *strictly risk-averse* for all $w > 0$.

(ii) ***Will Investor B buy Investment X?***

If Investor B buys X, then they will enjoy an expected utility of:

$$0.5\left[2\left(\sqrt{110}-1\right)+2\left(\sqrt{92}-1\right)\right] = 18.08$$

If, however, they do not buy X, then their expected (and certain) utility is:

$$2\left(\sqrt{100}-1\right) = 18$$

Thus, as buying X yields a higher expected utility, the investor ought to buy it.

4.4 Other utility functions

As evidenced from the above, many different utility functions have appeared in literature whose role is to describe the manner in which an investor derives utility from given choices. None of the utility functions described above allows much freedom in calibrating the function used to reflect a particular investor's preferences.

Question

Consider the following utility function:

$$U(w) = -e^{-aw}, \quad a>0$$

Derive expressions for the absolute risk aversion and relative risk aversion measures. What does the latter indicate about the investor's desire to hold risky assets?

Solution

The utility function $U(w) = -e^{-aw}$ is such that:

$$U'(w) = ae^{-aw} \quad \text{and} \quad U''(w) = -a^2e^{-aw}$$

Thus:

$$A(w) = \frac{-U''(w)}{U'(w)} = a > 0 \quad \text{and} \quad A'(w) = 0$$

and:

$$R(w) = \frac{-wU''(w)}{U'(w)} = aw > 0 \quad \text{and} \quad R'(w) = a > 0$$

Hence, as the absolute risk aversion is constant and independent of wealth the investor must hold the same *absolute* amount of wealth in risky assets as wealth increases. Both this, and the fact that the relative risk aversion increases with wealth, are consistent with a *decreasing proportion* of wealth being held in risky assets as wealth increases.

5 The variation of utility functions with wealth

5.1 Introduction

A further extension of the utility function is to consider wealth. It may not be possible to model an investor's behaviour over all possible levels of wealth with a single utility function. An obvious example is the quadratic utility function described above, which only satisfies the non-satiation condition over a limited wealth range. Sometimes this problem can be dealt with by using utility functions with the same functional form but different parameters over different ranges of wealth.

For example, the power utility function could be used to model preferences over all wealth levels, but with the value of the risk aversion coefficient γ changing with wealth. Sometimes, however, it may be necessary to go even further and use different functional forms over different ranges – by constructing *state-dependent utility functions*.

5.2 State-dependent utility functions

State-dependent utility functions can be used to model the situation where there is a discontinuous change in the state of the investor at a certain level of wealth.

They reflect the reality that the usefulness of a good or service to an individual, including wealth, may vary according to the circumstances of the individual. For example, the value of an umbrella depends upon whether or not we believe that it is going to rain over the next few hours or days. In a similar way, the utility that we derive from wealth may also reflect both our existing financial state and our more general circumstances in a way that cannot be captured by a simple functional form. We may therefore need to model preferences using a sophisticated utility function constructed by combining one or more of the standard functions discussed above – so that a different utility function effectively applies over different levels of wealth. A utility function of this kind may involve discontinuities and/or kinks.

Such a situation arises when we consider an insurance company that will become insolvent if the value of its assets falls below a certain level. At asset levels just above the insolvency position, the company will be highly risk-averse and this can be modelled by a utility function that has a discontinuity at the insolvency point.

However, the consequence of applying the same utility function when the company has just become insolvent would be that the company would be prepared to accept a high probability of losing its remaining assets for a chance of regaining solvency.

In other words, at the point of becoming (technically) insolvent the company is very risk-averse, being very keen to avoid this happening. Should it become (technically) insolvent, however, given that the damage (to its reputation or otherwise) has already been done, it may then be willing to take more risks in order to regain solvent status.

This is unlikely to reflect reality and so a different form of utility function would be required to model the company's behaviour in this state.

Question

Draw the utility function of the above company.

Solution

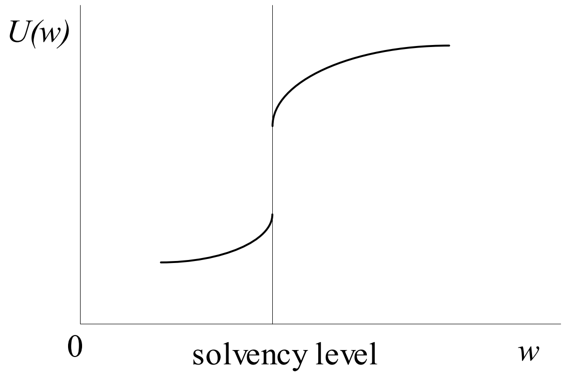


Figure 2.5 – A state-dependent utility function

ie the company is extremely risk-averse when just solvent, so the curve has a rapidly changing gradient. At the solvency level the curve is vertical as any slight increase in wealth leads to a large jump in utility.

Utility functions can also depend on states other than those, such as insolvency, which are determined by the level of wealth. Obvious examples for an individual include the differences between being healthy or sick, married or single. The state of an individual can also be affected by the anticipation of future events, *eg* if a legacy is expected.

Thus, under some circumstances an individual's utility might accurately be described by a function of the form $U(h,w)$ where h is an indicator of health.

6 Construction of utility functions

6.1 Introduction

In order to use a particular utility function, we need to calibrate the function so that it is appropriate to the particular individual to whom it applies. In other words, we need to find the values of the parameters, for example the value of γ in the power utility function, that apply to the individual.

One approach that has been proposed is to devise a series of questions that allow the shape of an individual's utility function to be roughly determined. A utility curve of a predetermined functional form can then be fitted by a least squares method to the points determined by the answers to the questions. The curve fitting is constrained by the requirement that the function has the desired economic properties (non-satiation, risk aversion and, perhaps, declining absolute risk aversion).

The student may be expected to show how a utility function can be constructed in general when there is a discrete set of outcomes and the axioms of this chapter apply.

6.2 Construction of utility functions by direct questioning

In theory, to determine an individual's utility function we could simply ask the individual what it is. However, in practice it is most unlikely that someone will be able to describe the mathematical form of their utility function.

6.3 Construction of utility functions by indirect questioning

An alternative procedure involves firstly, fixing two values of the utility function for the two extremes of wealth being considered. Secondly, the individual is asked to identify a certain level of wealth such that he or she would be indifferent between that certain level of wealth and a gamble that yields either of the two extremes with particular probabilities. The process is repeated for various scenarios until a sufficient number of plots is found.

Example

Suppose that we wish to determine the nature of an individual's utility function over the range of wealth $0 < w < 4$. One possible approach is to first fix $U(0) = 0$ and $U(4) = 1$. These are the first two points on the individual's utility function.

We could then ask the individual to identify the certain level of wealth, w' such that they would be indifferent between w' for certain and a gamble that yields each of 0 and 4 with equal probability, *ie* w' is the certainty equivalent of the gamble. The expected utility of the gamble is:

$$E[U] = \frac{1}{2}\left[U(0) + U(4)\right] = \frac{1}{2}[0+1] = \frac{1}{2}$$

If $w' = 1.8$ say, then we know that $U(1.8) = 0.5$ and thus have a third point on the utility function.

We could then repeat the exercise for a gamble involving equal probabilities of producing 1.8 and 4, which yields an expected utility of:

$$E[U]=\frac{1}{2}\left[U(1.8)+U(4)\right]=\frac{1}{2}\left[0.5+1\right]=0.75$$

If the certainty equivalent of this gamble is, say, 2.88, then we know that $U(2.88) = 0.75$, giving us a fourth point on the individual's utility function.

This process can be repeated until a sufficient number of points along the individual's utility curve have been identified and a plot of those points produced. Ordinary least squares regression or maximum likelihood methods can then be used to fit an appropriate functional form to the resulting set of values.

Another form of indirect questioning uses information on the premiums that a person is prepared to pay in order to gain an idea of the certainty equivalent of a particular risk.

Thus, we could ask a person what is the maximum that he would be prepared to pay for insurance with a given level of initial wealth and a given potential insurance situation. Points can then be derived on the utility function, which would give rise to the answers given. By repeating this questioning for different initial wealth levels, all the points on the person's utility function could be found.

Consider an example of a person with a house worth £100,000. Suppose that the owner is considering insurance against a variety of perils, each of which would destroy the house completely. These perils have different probabilities of occurring, and the owner has assessed the amount that they're prepared to pay to insure against each peril, as shown in the following table:

Peril	A	B	C	D
Loss (£K)	100	100	100	100
Probability	0.05	0.15	0.3	0.5
Premium owner is prepared to pay (£K)	20	40	60	80

We can use this table to find out some information about the owner's utility function:

- Fix two values of the utility function. For example, let us suppose that the owner derives utility of zero if they have no wealth, and utility of 1 if they suffer no loss at all, *ie* $U(100)=1,\ U(0)=0$, working in units of £1,000. This is legitimate, because by fixing two points we are just choosing a level and a scale for our measure of utility.

- Consider Peril A. The owner's utility of wealth with insurance will equal the expected utility of wealth without insurance, if they have paid the maximum premium they're prepared to pay. With insurance, the level of wealth is certain to be 80. Without it, it may be 100 with probability 0.95, or 0 with probability 0.05.

So:

$$U(80) = 0.05 \times U(0) + 0.95 \times U(100) = 0.95$$

and we have a point on the utility function.

Question

Show, using a similar argument, that $U(60) = 0.85$, and find two more points on the owner's utility function. Draw a rough sketch of the graph of the utility function.

Solution

Consider Peril B. With insurance against this peril, the owner's wealth will be 60. Without it, the level of wealth will either be 100 with probability 0.85 or 0 with probability 0.15. Equating the utilities of these two possibilities, we have

$$U(60) = 0.85 \times U(100) + 0.15 \times U(0) = 0.85 .$$

Similarly, considering Perils C and D, we obtain $U(40) = 0.7$ and $U(20) = 0.5$. So the utility function will look something like this:

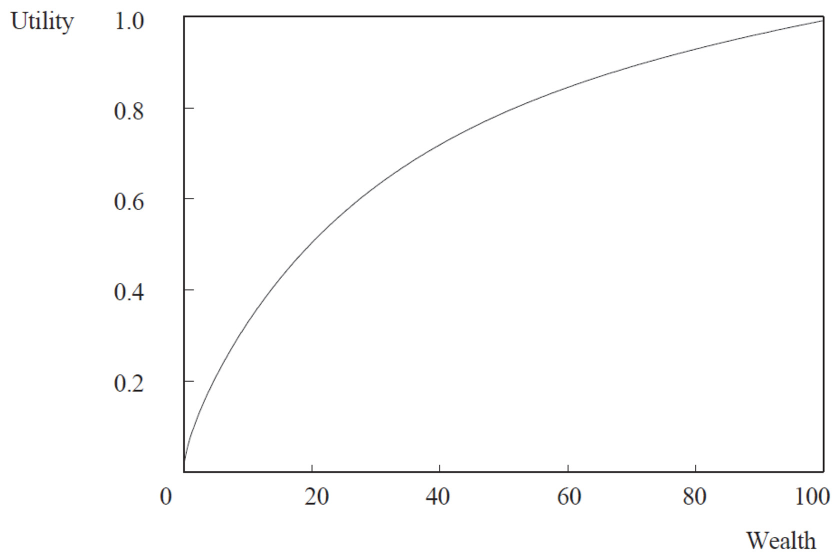


Figure 2.6 – A concave utility function

The complete utility function can be constructed by considering a large number of different scenarios, each of which contributes a point to the curve. Note that this particular function does appear to satisfy both the usual conditions $U'(w) > 0$ and $U''(w) < 0$.

7 Maximising utility through insurance

7.1 Introduction

Utility theory can be used to explain decisions such as purchasing insurance or buying a lottery ticket. Both of these activities are more likely to diminish the expected wealth of an individual. However, by purchasing insurance, one may be maximising expected utility.

A person who is risk averse will be prepared to pay more for insurance than the long-run average value of claims which will be made. Thus, insurance can be worthwhile for the risk averse policyholder even if the insurer has to charge a premium in excess of the expected value of claims in order to cover expenses and to provide a profit margin. An insurance contract is feasible if the minimum premium that the insurer is prepared to charge is less than the maximum amount that a potential policyholder is prepared to pay.

7.2 Finding the maximum premium

The maximum premium, P, which an individual will be prepared to pay in order to insure themselves against a random loss X is given by the solution of the equation:

$$E[U(a-X)] = U(a-P)$$

where a is the initial level of wealth.

Notice the similarity between this equation and the certainty equivalent relationship in Section 3.2. The individual is prepared to pay a certain amount P in order to avoid the uncertainty of the random loss X.

For example, consider an individual with a utility function of $U(x) = \sqrt{x}$ and current wealth of £15,000. Assume that this individual is at risk of suffering damages that are uniformly distributed up to 15,000. Then the individual's expected utility is:

$$E[U(a-X)] = \int_0^{15} \frac{1}{15}\sqrt{15-x}\,dx$$

$$= \left[\frac{-2}{3\times 15}(15-x)^{3/2} \right]_0^{15}$$

$$= 2.582$$

Then equating this to $U(a-P)$ gives:

$$U(a-P) = \sqrt{15-P} = 2.582$$

$$\Rightarrow \qquad P = 15 - 2.582^2 = 8.333$$

This individual would be willing to pay up to £8,833.33 for insurance that covers any loss.

This is well above the £7,500 expected loss.

7.3 Finding the minimum premium

The minimum insurance premium Q which an insurer should be prepared to charge for insurance against a risk with potential loss Y is given by the solution of the equation:

$$E[U(a+Q-Y)] = U(a)$$

where a is the initial wealth of the insurer.

Question

An insurer with initial wealth of £2,000 and a utility of $U(x)=\log(x)$ is designing a policy to cover damages of £500 that occur with probability 0.5.

Calculate the minimum premium that the insurer can charge for the policy.

Solution

From the equation above we have:

$$E\left[U(2{,}000+Q-Y)\right] = U(2{,}000)$$

and with $U(x)=\log(x)$ the expectation becomes:

$$\begin{aligned}
E\left[U(2{,}000+Q-Y)\right] &= 0.5\log(2{,}000+Q-500)+0.5\log(2{,}000+Q) \\
&= \log\left((1{,}500+Q)^{0.5}\right)+\log\left((2{,}000+Q)^{0.5}\right) \\
&= \log\left((1{,}500+Q)^{0.5}\times(2{,}000+Q)^{0.5}\right) \\
&= \log\left(\sqrt{(1{,}500+Q)\times(2{,}000+Q)}\right)
\end{aligned}$$

Equating this with $U(2{,}000)=\log(2{,}000)$ yields:

$$\log\left(\sqrt{(1{,}500+Q)\times(2{,}000+Q)}\right)=\log\left(2{,}000\right)$$

$$\Rightarrow \quad (1{,}500+Q)\times(2{,}000+Q)=2{,}000^2$$

$$\Rightarrow \quad Q^2+3{,}500Q-1{,}000{,}000=0$$

Resulting in:

$$Q=\frac{-3{,}500\pm\sqrt{3{,}500^2+4\times1{,}000{,}000}}{2}\approx -1{,}750\pm2{,}015.56$$

Taking the positive root gives a minimum premium of £265.56.

8 Limitations of utility theory

The expected utility theorem is a very useful device for helping our thinking about risky decisions, because it focuses attention on the types of trade-offs that have to be made. However, the expected utility theorem has several limitations that reduce its relevance for risk management purposes:

1. **To calculate expected utility, we need to know the precise form and shape of the individual's utility function. Typically, we do not have such information.**

   Even using the questioning techniques described earlier in this chapter, it is still optimistic to assume that it will be possible to construct a utility function that accurately reflects an individual's preferences.

   Usually, the best we can hope for is to identify a general feature, such as risk aversion, and to use the rule to identify broad types of choices that might be appropriate.

2. **The theorem cannot be applied separately to each of several sets of risky choices facing an individual.**

3. **For corporate risk management, it may not be possible to consider a utility function for the firm as though the firm was an individual.**

   The firm is a coalition of interest groups, each having claims on the firm. The decision process must reflect the mechanisms with which these claims are resolved and how this resolution affects the value of the firm. Furthermore, the risk management costs facing a firm may be only one of a number of risky projects affecting the firm's owners (and other claimholders). The expected utility theorem is not an efficient mechanism for modelling the *interdependence* of these sources of risk.

Alternative decision rules that can be used for risky choices include those under mean-variance portfolio theory and stochastic dominance.

Both of these topics are covered in later chapters.

New theories of non-rational investment behaviour, known as behavioural finance, are also covered in this course.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 2 Summary

The expected utility theorem

The *expected utility theorem* states that:

- a function, $U(w)$, can be constructed representing an investor's utility of wealth, w
- the investor faced with uncertainty makes decisions on the basis of maximising the *expected value* of utility.

Utility functions

The investor's risk-return preference is described by the form of their utility function. It is usually assumed that investors both:

- prefer more to less (*non-satiation*)
- are *risk-averse*.

Additionally, investors are sometimes assumed to exhibit decreasing *absolute risk aversion* – *ie* the *absolute* amount of wealth held in risky assets increases with wealth. In contrast, *relative risk aversion* indicates how the *proportion* of wealth held as risky assets varies with wealth.

Absolute and relative risk aversion are measured by the functions:

$$A(w) = \frac{-U''(w)}{U'(w)} \qquad\qquad R(w) = \frac{-wU''(w)}{U'(w)}$$

Amongst the utility functions commonly used to model investors' preferences are the:

- quadratic utility function
- log utility function
- power utility function.

State-dependent utility functions

Sometimes it may be inappropriate to model an investor's behaviour over all possible levels of wealth with a single utility function. This problem can be overcome either by using:

- utility functions with the same functional form but different parameters over different ranges of wealth, or
- *state-dependent utility functions*, which model the situation where there is a discontinuous change in the state of the investor at a certain level of wealth.

Construction of utility functions

One approach to constructing utility functions involves questioning individuals about their preferences.

The questioning may be direct or indirect. Indirect questioning may be framed in terms of how much an individual would be prepared to pay for insurance against various risks.

Maximum premium

With an initial wealth a, the maximum premium, P, that a policyholder would be willing to pay in order to avoid a potential loss, X, is given by:

$$E[U(a-X)] = U(a-P)$$

Minimum premium

With an initial wealth a, the minimum premium, Q, an insurer could charge to cover potential damages, Y, is given by:

$$E[U(a+Q-Y)] = U(a)$$

Limitations of utility theory

1. We need to know the precise form and shape of the individual's utility function.
2. The expected utility theorem cannot be applied separately to each of several sets of risky choices facing an individual.
3. For corporate risk management, it may not be possible to consider a utility function for the firm as though the firm was an individual.

Chapter 2 Practice Questions

2.1 An investor can invest in two assets, A and B:

Exam style

	A	B
expected return	6%	8%
variance	4%%	25%%

The correlation coefficient of the rate of return of the two assets is denoted by ρ and is assumed to take the value 0.5.

The investor is assumed to have an expected utility function of the form:

$$E_\alpha(U) = E(r_p) - \alpha \, Var(r_p)$$

where α is a positive constant and r_p is the rate of return on the assets held by the investor.

(i) Determine, as a function of α, the portfolio that maximises the investor's expected utility. [8]

(ii) Show that, as α increases, the investor selects an increasing proportion of Asset A. [1]
[Total 9]

2.2 Colin's preferences can be modelled by the utility function such that:

$$U'(w) = 3 - 2w, \ (w > 0).$$

(i) Determine the range of values over which this utility function can be satisfactorily applied.

(ii) Explain how Colin's holdings of risky assets will change as his wealth decreases.

(iii) Which of the following investments will he choose to maximise his expected utility?

Investment A		*Investment B*		*Investment C*	
outcome	*probability*	*outcome*	*probability*	*outcome*	*probability*
0.1	0.3	0	0.3	0.2	0.45
0.3	0.4	0.2	0.2	0.3	0.1
0.5	0.3	0.9	0.5	0.4	0.45

2.3 By considering the relationship $R(w) = w \times A(w)$, explain which of the following statements is true for a risk-averse individual.

1. If an investor's preferences display decreasing relative risk aversion then they must also display decreasing absolute risk aversion.

2. If an investor's preferences display decreasing absolute risk aversion then they must also display decreasing relative risk aversion.

2.4 Explain the four axioms that are required to derive the expected utility theorem.

2.5 Jenny has a quadratic utility function of the form $U(w) = w - 10^{-5}w^2$. She has been offered a job with Company X, in which her salary would depend upon the success or otherwise of the company. If it is successful, which will be the case with probability ¾, then her salary will be \$40,000, whereas if it is unsuccessful she will receive \$30,000.

Exam style

(i) Assuming that Jenny has no other wealth, state the salary range over which $U(w)$ is an appropriate representation of her individual preferences. [2]

(ii) Calculate the expected salary and the expected utility offered by the job. [2]

(iii) Suppose she was also to be offered a fixed salary by Company Z. Determine the minimum level of fixed salary that she would accept to work for Company Z in preference to Company X. [3]

(iv) Suppose that the owners of Company X are both risk-neutral and very keen that Jenny should join them and not Company Z. Determine whether the firm should agree to pay her a fixed wage, and, if so, how much. Comment briefly on your answer. [1]

[Total 8]

2.6 Suppose that Lance and Allan each have a log utility function and an initial wealth of 100 and 200 respectively. Both are offered a gamble such that they will receive a sum equal to 30% of their wealth should they win, whereas they will lose 10% of their wealth should they lose. The probability of winning is ¼.

(i) State whether or not the gamble is fair.

(ii) Calculate Lance's certainty equivalent for the gamble alone and comment briefly on your answer.

(iii) Repeat part (ii) in respect of Allan and compare your answer with that in part (ii).

(iv) Confirm that your comments in part (iii) apply irrespective of the individual's wealth.

2.7 Jayne's utility function can be described as $U(w) = \sqrt{w}$. She faces a potential loss of £100,000 in the event that her should house burn down, which has a probability of 0.01.

Exam style

(i) Calculate the maximum premium that Jayne would be prepared to pay to insure herself against the total loss of her house if her initial level of wealth was £140,000 and comment on your results. [3]

Suppose that UN Life plc has an initial wealth of £100 million and a utility function of the form $U(w) = w$.

(ii) Calculate the minimum premium UN Life plc would require in order to offer insurance to Jayne and comment on whether insurance is feasible in this instance. [3]

[Total 6]

2.8 An insurance company will be required to make a payout of £500 on a particular risk event, which is likely to occur with a probability of 0.4. The utility for any level of wealth, w, is given by:

Exam style

$$U(w) = 4,000 + 0.5w$$

The insurer's initial level of wealth is £6000. Calculate the minimum premium the insurer will require in order to take on the risk. [3]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 2 Solutions

2.1 (i) ***Maximising the investor's expected utility***

Assuming that all of the investor's money is invested, and hence the portfolio weights sum to 1, the expected return and variance of a portfolio consisting of a proportion x_A of wealth held in Asset A, and a proportion $1-x_A$ of wealth held in Asset B are:

$$E_P = x_A E_A + (1-x_A)E_B$$

$$= 0.06x_A + 0.08(1-x_A) \qquad [1½]$$

$$= 0.08 - 0.02x_A$$

and:

$$V_P = x_A^2 V_A + x_B^2 V_B + 2x_A x_B \sigma_A \sigma_B \rho_{AB}$$

$$= 0.0004x_A^2 + 0.0025(1-x_A)^2 + 0.0010x_A(1-x_A) \qquad [1½]$$

$$= 0.0019x_A^2 - 0.0040x_A + 0.0025$$

Therefore the investor's expected utility is:

$$E_\alpha(U) = E(r_p) - \alpha Var(r_p)$$

$$= 0.08 - 0.02x_A - \alpha\left(0.0019x_A^2 - 0.0040x_A + 0.0025\right) \qquad [2]$$

We can maximise this function of x_A by differentiating and setting to zero:

$$\frac{dE}{dx_A} = -0.02 - \alpha\left(0.0038x_A - 0.0040\right) = 0 \qquad [1]$$

$$\Leftrightarrow \qquad x_A = \frac{20\alpha - 100}{19\alpha} \qquad [½]$$

or:

$$x_A = \frac{20}{19} - \frac{100}{19\alpha} \qquad [½]$$

The second-order derivative is:

$$\frac{d^2E}{dx_A^2} = -0.0038\alpha < 0$$ [1]

which confirms that we have a maximum.

[Total 8]

(ii) ***Show that the investor selects an increasing proportion of Asset A***

Differentiating the formula for the optimal value of x_A in terms of α gives:

$$\frac{dx_A}{d\alpha} = \frac{100}{19\alpha^2} > 0$$ [1]

This confirms that as α increases, so x_A, the proportion of wealth held in Asset A, increases too.

2.2 (i) ***Range of wealth applicable***

Assuming *non-satiation*, which requires that $U'(w) > 0$, Colin's preferences can be modelled by this utility function provided that $0 < w < \frac{3}{2}$.

(ii) ***How Colin's holdings of risky assets vary with his wealth***

Differentiating the expression given in the question yields $U''(w) = -2$.

Thus, over the relevant range of *w*:

$$A(w) = \frac{2}{3-2w} > 0\,,\; A'(w) = \frac{4}{(3-2w)^2} > 0$$

and $$R(w) = \frac{2w}{3-2w} > 0\,,\; R'(w) = \frac{6}{(3-2w)^2} > 0$$

Hence, as Colin's wealth *decreases* the:

- *absolute amount* of his investment in risky assets will *increase* (as his absolute risk aversion decreases as his wealth decreases)
- *proportion* of his wealth that is invested in risky assets will *increase* (as his relative risk aversion decreases as his wealth decreases).

(iii) ***Colin's choice of investments***

Integrating the expression in the question gives Colin's utility function:

$$U(w) = a + 3w - w^2$$

As the properties of utility functions are invariant to linear transformations, we can set the arbitrary constant *a* equal to zero.

His expected utility from each of the investments is therefore as follows.

$$EU_A = 0.3\times U(0.1) + 0.4\times U(0.3) + 0.3\times U(0.5)$$

$$= 0.3\times 0.29 + 0.4\times 0.81 + 0.3\times 1.25$$

$$= 0.786$$

$$EU_B = 0.3\times U(0) + 0.2\times U(0.2) + 0.5\times U(0.9)$$

$$= 1.057$$

$$EU_C = 0.45\times U(0.2) + 0.1\times U(0.3) + 0.45\times U(0.4)$$

$$= 0.801$$

Thus, Colin will choose Investment B to maximise his expected utility.

2.3 The relationship between absolute risk aversion $A(w)$ and relative risk aversion $R(w)$ is such that:

$$R(w) = w\times A(w)$$

Differentiating with respect to wealth w gives:

$$\frac{\partial R}{\partial w} = A + w\frac{\partial A}{\partial w} \qquad (1)$$

Considering the first statement, Equation (1) tells us that if $\frac{\partial R}{\partial w}<0$ and so relative risk aversion is decreasing, then it must also be the case that $\frac{\partial A}{\partial w}<0$ (given that w and $A(w)$ are both positive for a risk-averse individual), *ie* $\frac{\partial R}{\partial w}<0 \Rightarrow \frac{\partial A}{\partial w}<0$.

An investor who displays decreasing relative risk aversion invests a larger proportion of wealth in risky assets as wealth increases. This also implies a larger monetary amount is invested in risky assets, ie decreasing absolute risk aversion.

Considering the second statement, then if $\frac{\partial A}{\partial w}<0$, it does not follow that $\frac{\partial R}{\partial w}$ is necessarily negative. This will depend upon the relative magnitudes of $A(w)$, w and $\frac{\partial A}{\partial w}$. Thus, $\frac{\partial A}{\partial w}<0$ does not imply that $\frac{\partial R}{\partial w}<0$.

An investor who displays decreasing absolute risk aversion invests a larger monetary amount in risky assets as wealth increases. This does not necessarily equate to a larger percentage of wealth.

Hence, the first statement is true, whereas the second statement is false.

2.4 The expected utility theorem can be derived formally from the following four axioms:

1. Comparability

An investor can state a preference between all available certain outcomes.

2. Transitivity

If A is preferred to B and B is preferred to C, then A is preferred to C.

3. Independence

If an investor is indifferent between two certain outcomes, A and B, then he is also indifferent between the following two gambles:

(i) A with probability p and C with probability $(1-p)$; and

(ii) B with probability p and C with probability $(1-p)$.

4. Certainty equivalence

Suppose that A is preferred to B and B is preferred to C. Then there is a unique probability, p, such that the investor is indifferent between B and a gamble giving A with probability p and C with probability $(1-p)$.

B is known as the *certainty equivalent* of the above gamble.

2.5 (i) ***Salary range of utility function***

If: $U(w) = w - 10^{-5}w^2$

then: $U'(w) = 1 - 2w \times 10^{-5}$ [½]

and: $U''(w) = -2 \times 10^{-5}$ [½]

Now in order for Jenny to:

- prefer more to less, we require that $U'(w) > 0$, which in this case will be true for all $w < ½ \times 10^5$, *ie* $w < 50,000$
- be risk-averse, we require that $U''(w) < 0$, which in this case will be true for all $w > 0$.

Thus, the appropriate salary range is $w <$ \$50,000. [1]

[Total 2]

(ii) ***Expected salary and expected utility***

Her expected salary is given by:

$$¾ \times 40,000 + ¼ \times 30,000 = \$37,500$$ [1]

Her expected utility is given by:

$$\tfrac{3}{4} \times [(40{,}000 - 10^{-5}\times(40{,}000)^2] + \tfrac{1}{4} \times [(30{,}000 - 10^{-5}\times(30{,}000)^2] = 23{,}250$$ [1]

[Total 2]

(iii) ***Minimum fixed salary***

The minimum level of salary, x say, is equal to the certainty equivalent of the job offer from Company X. [½]

This is given by:

$$U(x) = 23{,}250$$

$$x - 10^{-5}x^2 = 23{,}250$$

$$-x + 10^{-5}x^2 + 23{,}250 = 0$$ [½]

Using the formula for solving quadratic equations we find:

$$x = 63{,}229 \text{ or } 36{,}771$$ [1½]

As the first of these values is greater than the maximum salary available when Company X is successful it can be disregarded. Hence the minimum level of fixed salary that she would accept to work for Company Z is $36,771. [½]

[Total 3]

(iv) ***Should Company X offer a fixed salary?***

Yes – if they are risk-neutral, then they should offer Jenny a fixed salary in preference to a variable one. Jenny is risk-averse and therefore derives additional utility from the certainty offered by a fixed salary. [½]

Therefore, Company X will be able to entice Jenny to work for them in return for a salary of just (or strictly speaking slightly above) $36,771, instead of the expected salary of $37,500 in (i). [½]

[Total 1]

2.6 (i) ***Is the gamble fair?***

For any given initial level of wealth w, the expected value of the gamble is given by:

$$\tfrac{1}{4} \times 1.3w + \tfrac{3}{4} \times 0.9w - w = 0$$

Thus, the gamble is fair.

(ii) ***Lance's certainty equivalent of the gamble alone***

Lance's expected utility should he undertake the gamble is given by:

$$E[U] = \tfrac{1}{4}\log(130) + \tfrac{3}{4}\log(90) = 4.59174$$

Thus, his certainty equivalent for the initial wealth and the gamble is given by:

$$U(c_w) = \log(c_w) = 4.59174$$

$$\Rightarrow \quad c_w = e^{4.59174} = 98.666$$

and the certainty equivalent for the gamble alone is given by:

$$c_x = c_w - w = -1.334$$

This is negative because he is risk-averse.

The negative value of c_x means that Lance would have to be paid to accept the gamble.

(iii) ***Allan's certainty equivalent of the gamble alone***

Allan's expected utility should he undertake the gamble is given by:

$$E[U] = \tfrac{1}{4}\log(260) + \tfrac{3}{4}\log(180) = 5.28489$$

His certainty equivalent for the gamble alone is:

$$c_x = e^{5.28489} - 200 = -2.668$$

Comparing the two answers, we can see that the two certainty equivalents are equal to the same proportion of each individual's initial wealth. This is because the log utility function is consistent with preferences that exhibit constant *relative* risk aversion.

(iv) ***Relative risk aversion***

The constancy of relative risk aversion with a log utility function can be confirmed by differentiating it, *ie*:

If: $\quad U(w) = \log(w)$

then: $\quad U'(w) = \dfrac{1}{w}$ and $U''(w) = -\dfrac{1}{w^2}$

Thus: $\quad R(w) = -w\dfrac{U''(w)}{U'(w)} = 1 \quad$ and $\quad R'(w) = 0$

So, the log utility function exhibits constant relative risk aversion irrespective of w – though the log utility function is of course defined only for $w > 0$.

2.7 (i) ***Jayne's maximum premium***

Let P be the maximum insurance premium Jayne is prepared to pay and X be the loss she faces. Then Jayne's utility with insurance is:

$$U = \sqrt{140{,}000 - P} \qquad [½]$$

Whereas her expected utility without insurance is:

$$E[U(140,000 - X)] = 0.99\sqrt{140,000} + 0.01\sqrt{40,000} = 372.424$$ [½]

Equating these two expressions gives:

$$\sqrt{140,000 - P} = 372.424$$

ie $P = 140,000 - 372.424^2 = 1,300$ [1]

The maximum premium of £1,300 exceeds the expected loss of £1,000. This is because Jayne is risk-averse. [1]

[Total 3]

(ii) ***UN Life's minimum premium***

Let Q be the minimum premium required by UN Life, then its utility without insurance is 100,000,000. [½]

Whereas its expected utility with insurance is:

$$E[U(100m + Q - X)] = 0.99 \times (100,000,000 + Q) + 0.01 \times (99,900,000 + Q)$$

$$= 99,999,000 + Q$$ [½]

Equating these two expressions gives:

$$100,000,000 = 99,999,000 + Q$$

ie $Q = 1,000$ [1]

So, the minimum premium required by UN Life is less than the maximum premium Jayne is prepared to pay, which means that the insurance contract is feasible. [1]

Notice that the minimum insurance premium that the insurance company will accept is equal to the expected value of the claim. This is because the insurance company is risk-neutral.

[Total 3]

2.8 The minimum premium Q is given by the equation:

$$E\left[U(a + Q - Y)\right] = U(a)$$ [1]

where a is the initial wealth and Y is the payout. In this case we have:

$$E\left[U(6,000 + Q - Y)\right] = 0.4U(6,000 + Q - 500) + 0.6U(6,000 + Q)$$

$$= 0.4(4,000 + 0.5(5,500 + Q)) + 0.6(4,000 + 0.5(6,000 + Q))$$ [1]

$$= 0.5Q + 6,900$$

Equating this to $U(6,000) = 4,000 + 0.5 \times 6,000 = 7,000$ leads to:

$$0.5Q + 6,900 = 7,000$$

$\Rightarrow \quad Q = 200$ [1]

[Total 3]

4

Measures of investment risk

Syllabus objectives

2.1 Properties of risk measures

2.1.1 Define the following measures of investment risk:

- variance of return
- downside semi-variance of return
- shortfall probabilities
- Value at Risk (VaR) / TailVaR.

2.1.2 Describe how the risk measures listed in 2.1.1 above are related to the form of an investor's utility function.

2.1.3 Perform calculations using the risk measures listed in 2.1.1 above to compare investment opportunities.

2.1.4 Explain how the distribution of returns and the thickness of tails will influence the assessment of risk.

2.2 Risk and insurance companies

2.2.1 Describe how insurance companies help to reduce or remove risk.

2.2.2 Explain what is meant by the terms 'moral hazard' and 'adverse selection'.

0 Introduction

In financial economics, it is often assumed that the key factors influencing investment decisions are 'risk' and 'return'. In practice, return is almost always interpreted as the *expected* investment return. However, there are many possible interpretations and different ways of measuring investment risk, of which the variance is just one, each of which corresponds to a different utility function.

This chapter outlines a small number of such measures, together with their relative merits, and then moves on to discuss how insurance can be used to reduce the impact of risk.

1 Measures of risk

1.1 Introduction

Most mathematical investment theories of investment risk use variance of return as the measure of risk.

Examples include (mean-variance) portfolio theory and the capital asset pricing model, both of which are discussed later in this course.

However, it is not obvious that variance necessarily corresponds to investors' perception of risk, and other measures have been proposed as being more appropriate.

Some investors might not be concerned with the mean and variance of returns, but simpler things such as the maximum possible loss. Alternatively, some investors might be concerned not only with the mean and variance of returns, but also more generally with other higher moments of returns, such as the *skewness* of returns. For example, although two risky assets might yield the same expectation and variance of future returns, if the returns on Asset A are positively skewed, whilst those on Asset B are symmetrical about the mean, then Asset A might be preferred to Asset B by some investors.

1.2 Variance of return

For a continuous distribution, **variance of return is defined as:**

$$\int_{-\infty}^{\infty} (\mu - x)^2 f(x)\, dx$$

where μ is the mean return at the end of the chosen period and $f(x)$ is the probability density function of the return.

'Return' here means the proportionate increase in the market value of the asset, *eg* $x = 0.05$ if the asset value has increased by 5% over the period.

The units of variance are '%%', which means 'per 100 per 100'.

eg $(4\%)^2 = 16\%\% = 0.16\% = 0.0016$

Question

Investment returns (% *pa*), X, on a particular asset are modelled using a probability distribution with density function:

$$f(x) = 0.00075\left(100 - (x-5)^2\right) \quad \text{where } -5 \le x \le 15$$

Calculate the mean return and the variance of return.

Solution

The density function is symmetrical about $x=5$. Hence the mean return is 5%. Alternatively, this could be found by integrating as follows:

$$E[X]=0.00075\int_{-5}^{15}x\left(100-(x-5)^2\right)dx$$

$$=0.00075\int_{-5}^{15}75x+10x^2-x^3dx$$

$$=0.00075\left[\frac{75}{2}x^2+\frac{10}{3}x^3-\frac{1}{4}x^4\right]_{-5}^{15}$$

$$=0.00075[7,031.25-364.5833]$$

$$=5$$

ie 5% *pa*.

The variance is given by:

$$Var(X)=0.00075\int_{-5}^{15}(5-x)^2\left(100-(x-5)^2\right)dx$$

$$=0.00075\int_{-5}^{15}100(x-5)^2-(x-5)^4dx$$

$$=0.00075\left[\frac{100}{3}(x-5)^3-\frac{1}{5}(x-5)^5\right]_{-5}^{15}$$

$$=0.00075[13,333.33-(-13,333.33)]$$

$$=20$$

ie 20%% *pa*.

Alternatively, we can calculate the variance using the formula: $Var(X)=E[X^2]-(E[X])^2$, where $E[X^2]$ can be found by integration to be 45%%.

For a discrete distribution, variance of return is defined as:

$$\sum_x(\mu-x)^2P(X=x)$$

where μ is the mean return at the end of the chosen period.

Question

Investment returns (% *pa*), X, on a particular asset are modelled using the probability distribution:

X	*Probability*
–7	0.04
5.5	0.96

Calculate the mean return and variance of return.

Solution

The mean return is given by:

$$E[X] = -7 \times 0.04 + 5.5 \times 0.96 = 5$$

ie 5% *pa*.

The variance of return is given by:

$$Var(X) = (5-(-7))^2 \times 0.04 + (5-5.5)^2 \times 0.96 = 6$$

ie 6%% *pa*.

Alternatively, we can calculate the variance using the formula: $Var(X) = E[X^2] - (E[X])^2$, where $E[X^2]$ is 31%%.

Variance has the advantage over most other measures in that it is mathematically tractable, and the mean-variance framework discussed in a later chapter leads to elegant solutions for optimal portfolios. Albeit easy to use, the mean-variance theory has been shown to give a good approximation to several other proposed methodologies.

Mean-variance portfolio theory can be shown to lead to optimum portfolios if investors can be assumed to have quadratic utility functions or if returns can be assumed to be normally distributed.

In an earlier chapter we discussed how the aim of investors is to maximise their expected utility. The mean-variance portfolio theory discussed in a later chapter assumes that investors base their investment decisions solely on the mean and variance of investment returns. This assumption is consistent with the maximisation of expected utility provided that the investor's expected utility depends only on the mean and variance of investment returns.

It can be shown that this is the case if:

- the investor has a quadratic utility function, and/or
- investment returns follow a distribution that is characterised fully by its first two moments, such as the normal distribution.

If, however, neither of these conditions holds, then we cannot assume that investors make choices solely on the basis of the mean and variance of return. For example, with more complex utility functions and non-normal return distributions investors may need to consider other features of the distribution of returns, such as skewness and kurtosis.

Question

Define both the skewness and the fourth central moment (called the *kurtosis*) of a continuous probability distribution.

Solution

The *skewness* of a continuous probability distribution is defined as the third central moment.

It is a measure of the extent to which a distribution is asymmetric about its mean. For example, the normal distribution is symmetric about its mean and therefore has zero skewness, whereas the lognormal distribution is positively skewed.

The *kurtosis* of a continuous probability distribution is defined as the fourth central moment.

It is a measure of how likely extreme values are to appear (*ie* those in the tails of the distribution).

1.3 Semi-variance of return

The main argument against the use of variance as a measure of risk is that most investors do not dislike uncertainty of returns as such; rather they dislike the possibility of low returns.

For example, all rational investors would choose a security that offered a chance of either a 10% or 12% return in preference to one that offered a certain 10%, despite the greater uncertainty associated with the former.

One measure that seeks to quantify this view is downside semi-variance (also referred to as simply semi-variance). For a continuous random variable, **this is defined as:**

$$\int_{-\infty}^{\mu} (\mu - x)^2 f(x)\, dx$$

Semi-variance is not easy to handle mathematically, and it takes no account of variability above the mean. Furthermore, if returns on assets are symmetrically distributed, semi-variance is proportional to variance.

Question

Investment returns (% *pa*), X, on a particular asset are modelled using a probability distribution with density function:

$$f(x) = 0.00075\left(100-(x-5)^2\right) \quad \text{where } -5 \le x \le 15$$

Calculate the downside semi-variance of return.

Solution

We saw in an earlier question that the variance of investment returns for this asset is 20%%. Since the continuous distribution $f(x)$ is symmetrical, the downside semi-variance is half the variance, *ie* 10%%.

For a discrete random variable, the downside semi-variance is defined as:

$$\sum_{x<\mu} (\mu - x)^2 P(X = x)$$

Question

Investment returns (% *pa*), X, on a particular asset are modelled using the probability distribution:

X	*probability*
-7	0.04
5.5	0.96

Calculate the downside semi-variance of return.

Solution

We saw in an earlier question that the mean investment return for this asset is 5%. So the downside semi-variance is given by:

$$\sum_{x<5} (5-x)^2 P(X=x) = (5-(-7))^2 \times 0.04 = 5.76$$

ie 5.76%% *pa*.

1.4 Shortfall probabilities

A shortfall probability measures the probability of returns falling below a certain level. For continuous variables, the risk measure is given by:

$$\textbf{Shortfall probability} = \int_{-\infty}^{L} f(x)\,dx$$

where L is a chosen benchmark level.

For discrete random variables, the risk measure is given by:

$$\text{Shortfall probability} = \sum_{x<L} P(X = x)$$

The benchmark level, *L*, can be expressed as the return on a benchmark fund if this is more appropriate than an absolute level. In fact, any of the risk measures discussed can be expressed as measures of the risk relative to a suitable benchmark which may be an index, a median fund or some level of inflation.

L could alternatively relate to some pre-specified level of surplus or fund solvency.

Question

Investment returns (% *pa*), X, on a particular asset are modelled using a probability distribution with density function:

$$f(x) = 0.00075\left(100-(x-5)^2\right) \quad \text{where } -5 \le x \le 15$$

Calculate the shortfall probability where the benchmark return is 0% *pa*.

Solution

The shortfall probability is given by:

$$P(X<0) = 0.00075\int_{-5}^{0} 100-(x-5)^2\,dx$$

$$= 0.00075\left[100x - \frac{1}{3}(x-5)^3\right]_{-5}^{0}$$

$$= 0.00075\left[41.6667-(-166.6667)\right]$$

$$= 0.15625$$

Question

Investment returns (% *pa*), X, on a particular asset are modelled using the probability distribution:

X	*probability*
-7	0.04
5.5	0.96

Calculate the shortfall probability where the benchmark return is 0% *pa*.

Solution

The shortfall probability is given by:

$$P(X < 0) = 0.04$$

The main advantages of the shortfall probability are that it is easy to understand and calculate.

The main drawback of the shortfall probability as a measure of investment risk is that it gives no indication of the magnitude of any shortfall (being independent of the extent of any shortfall).

For example, consider two securities that offer the following combinations of returns and associated probabilities:

Investment A: 10.1% with probability of 0.9 and 9.9% with probability of 0.1

Investment B: 10.1% with probability of 0.91 and 0% with probability of 0.09

An investor who chooses between them purely on the basis of the shortfall probability based upon a benchmark return of 10% would choose Investment B, despite the fact that it gives a much bigger shortfall than Investment A if a shortfall occurs.

1.5 Value at Risk

***Value at Risk (VaR)* generalises the likelihood of underperforming by providing a statistical measure of downside risk.**

For a continuous random variable, **Value at Risk can be determined as:**

$$VaR(X) = -t \quad \text{where} \quad P(X < t) = p$$

VaR represents the maximum potential loss on a portfolio over a given future time period with a given degree of confidence, where the latter is normally expressed as $1-p$. So, for example, a 99% one-day VaR is the maximum loss on a portfolio over a one-day period with 99% confidence, *ie* there is a 1% probability of a greater loss.

Note that Value at Risk is a 'loss amount'. Therefore:

- a positive Value at Risk (a negative t) indicates a loss
- a negative Value at Risk (a positive t) indicates a profit
- Value at Risk should be expressed as a *monetary* amount and not as a percentage.

Question

Investment returns (% *pa*), X, on a particular asset are modelled using a probability distribution with density function:

$$f(x) = 0.00075\left(100-(x-5)^2\right) \quad \text{where } -5 \le x \le 15$$

Calculate the VaR over one year with a 95% confidence limit for a portfolio consisting of £100*m* invested in the asset.

Solution

We start by finding t, where $P(X<t)=0.05$:

$$\Rightarrow \qquad 0.00075\int_{-5}^{t} 100-(x-5)^2\,dx = 0.05$$

$$\Rightarrow \qquad 0.00075\left[100x-\frac{1}{3}(x-5)^3\right]_{-5}^{t} = 0.05$$

Since the equation in the brackets is a cubic in t, we are going to need to solve this equation numerically, by trial and error.

$$t=-3 \quad \Rightarrow \quad 0.00075\left[100x-\frac{1}{3}(x-5)^3\right]_{-5}^{-3} = 0.028$$

and $$t=-2 \quad \Rightarrow \quad 0.00075\left[100x-\frac{1}{3}(x-5)^3\right]_{-5}^{-2} = 0.06075$$

Interpolating between the two gives: $t=-3+\dfrac{0.05-0.028}{0.06075-0.028}=-2.3$

In fact, the true value is $t=-2.293$. Since t is a percentage investment return per annum, the 95% Value at Risk over one year on a £100*m* portfolio is $£100m \times 2.293\% = £2.293m$. This means that we are 95% certain that we will not lose more than £2.293*m* over the next year.

For a discrete random variable, VaR is defined as:

$$VaR(X) = -t \quad \text{where } t = \max\{x : P(X<x) \le p\}$$

Question

Investment returns (% *pa*), X, on a particular asset are modelled using the probability distribution:

X	*Probability*
-7	0.04
5.5	0.96

Calculate the 95% VaR over one year with a 95% confidence limit for a portfolio consisting of £100*m* invested in the asset.

Solution

We start by finding t, where $t = \max\{x : P(X < x) \leq 0.05\}$.

Now $P(X < -7) = 0$ and $P(X < 5.5) = 0.04$. Therefore $t = 5.5$.

Since t is a percentage investment return per annum, the 95% Value at Risk over one year on a £100*m* portfolio is $£100m \times -5.5\% = -£5.5m$. This means that we are 95% certain that we will not make profits of less than £5.5*m* over the next year.

VaR can be measured either in absolute terms or relative to a benchmark. Again, VaR is based on assumptions that may not be immediately apparent.

The problem is that in practice VaR is often calculated assuming that investment returns are normally distributed.

Question

Calculate the 97.5% VaR over one year for a portfolio consisting of £200*m* invested in shares. Assume that the return on the portfolio of shares is normally distributed with mean 8% *pa* and standard deviation 8% *pa*.

Solution

We start by finding t, where:

$$P(X < t) = 0.025, \text{ where } X \sim N(8, 8^2)$$

Standardising gives:

$$P\left(Z < \frac{t-8}{8}\right) = \Phi\left(\frac{t-8}{8}\right) = 0.025$$

Now $\Phi(-1.96)=0.025$ from page 162 of the *Tables*, so: $\dfrac{t-8}{8}=-1.96 \;\Rightarrow\; t=-7.68$.

Since t is a percentage investment return per annum, the 97.5% Value at Risk over one year on a £200*m* portfolio is $£200m \times 7.68\% = £15.36m$. This means that we are 97.5% certain that we will not lose more than £15.36*m* over the next year.

Portfolios exposed to credit risk, systematic bias or derivatives may exhibit non-normal distributions. The usefulness of VaR in these situations depends on modelling skewed or fat-tailed distributions of returns, either in the form of statistical distributions (such as the Gumbel, Frechet or Weibull distributions) or via Monte Carlo simulations. However, the further one gets out into the 'tails' of the distributions, the more lacking the data and, hence, the more arbitrary the choice of the underlying probability becomes.

Hedge funds are a good example of portfolios exposed to credit risk, systematic bias and derivatives. These are private collective investment vehicles that often adopt complex and unusual investment positions in order to make high investment returns. For example, they will often short-sell securities and use derivatives.

If the portfolio in the previous question was a hedge fund then modelling the return using a normal distribution may no longer be appropriate. A different distribution could be used to assess the lower tail but choosing this distribution will depend on the data available for how hedge funds have performed in the past. This data may be lacking or include survivorship bias, *ie* hedge funds that do very badly may not be included.

The Gumbel, Frechet and Weibull distributions are three examples of extreme value distributions, which are used to model extreme events.

The main weakness of VaR is that it does not quantify the size of the 'tail'. Another useful measure of investment risk therefore is the *Tail Value at Risk*.

1.6 Tail Value at Risk (TailVaR) and expected shortfall

Closely related to both shortfall probabilities and VaR are the TailVaR (or TVaR) and expected shortfall measures of risk.

The risk measure can be expressed as the expected shortfall below a certain level.

For a continuous random variable, the expected shortfall is given by:

$$\textbf{Expected shortfall} = E\left[\max(L-X,0)\right] = \int_{-\infty}^{L} (L-x)f(x)\,dx$$

where L is the chosen benchmark level.

If L is chosen to be a particular percentile point on the distribution, then the risk measure is known as the TailVaR.

The $(1-p)$ TailVaR is the expected shortfall in the p *th* lower tail. So, for the 99% confidence limit, it represents the expected loss *in excess of* the 1% lower tail value.

Question

Investment returns (% *pa*), X, on a particular asset are modelled using a probability distribution with density function:

$$f(x) = 0.00075\left(100-(x-5)^2\right) \quad \text{where } -5 \le x \le 15$$

Calculate the 95% TailVaR over one year for a portfolio consisting of £100*m* invested in the asset.

Solution

In a previous question, we calculated the 95% VaR for this portfolio to be £2.293m based on an investment return of –2.293%.

The expected shortfall in returns below –2.293% is given by:

$$E[\max(-2.293-X,0)] = 0.00075\int_{-5}^{-2.293}(-2.293-x)\left(100-(x-5)^2\right)dx$$

$$= 0.00075\int_{-5}^{-2.293}\left(-171.975-97.93x-7.707x^2+x^3\right)dx$$

$$= 0.00075\left[-171.975x-48.965x^2-2.569x^3+0.25x^4\right]_{-5}^{-2.293}$$

$$= 0.0462$$

On a portfolio of £100*m*, the 95% TailVaR is $£100m \times 0.000462 = £0.0462m$. This means that the expected loss *in excess of* £2.293*m* is £46,200.

For a discrete random variable, the expected shortfall is given by:

$$\text{Expected shortfall} = E\left[\max(L-X,0)\right] = \sum_{x<L}(L-x)P(X=x)$$

Question

Investment returns (% *pa*), X, on a particular asset are modelled using the probability distribution:

X	*Probability*
–7	0.04
5.5	0.96

Calculate the 95% TailVaR over one year for a portfolio consisting of £100*m* invested in the asset.

Solution

In a previous question, we calculated the 95% VaR for this portfolio to be –£5.5m based on an investment return of 5.5%.

The expected shortfall in returns below 5.5% is given by:

$$E[\max(5.5-X,0)] = \sum_{x<5.5}(5.5-x)P(X=x)$$

$$=(5.5-(-7))\times 0.04 = 0.5$$

On a portfolio of £100*m*, the 95% TailVaR is $£100m \times 0.005 = £0.5m$. This means that the expected reduction in profits *below* £5.5*m* is £0.5*m*.

However, TailVaR can also be expressed as the expected shortfall conditional on there being a shortfall.

To do this, we would need to take the expected shortfall formula and divide by the shortfall probability.

Other similar measures of risk have been called:

- **expected tail loss**
- **tail conditional expectation**
- **conditional VaR**
- **tail conditional VaR**
- **worst conditional expectation.**

They all measure the risk of underperformance against some set criteria. It should be noted that the characteristics of the risk measures may vary depending on whether the variable is discrete or continuous in nature.

Downside risk measures have also been proposed based on an increasing function of $(L-x)$, rather than $(L-x)$ itself in the integral above.

In other words, for continuous random variables, we could use a measure of the form:

$$\int_{-\infty}^{L} g(L-x)f(x)\,dx$$

Two particular cases of note are when:

1. $g(L-r)=(L-r)^2$ – this is the so-called *shortfall variance*
2. $g(L-r)=(L-r)$ – the *average or expected shortfall* measure defined above.

Note also that if $g(x)=x^2$ and $L = \mu$, then we have the semi-variance measure defined above.

Shortfall measures are useful for monitoring a fund's exposure to risk because the expected underperformance relative to a benchmark is a concept that is apparently easy to understand. As with semi-variance, however, no attention is paid to the distribution of outperformance of the benchmark, *ie* returns in excess of L are again completely ignored.

Question

Consider an investment whose returns follow a continuous uniform distribution over the range 0% to 10% *pa*.

(i) Write down the probability density function for the investment returns.

(ii) Calculate the mean investment return.

(iii) Calculate the variance and semi-variance measures of investment risk.

(iv) Calculate the shortfall probability and the expected shortfall based on a benchmark level of 3% *pa*.

Solution

Useful information about the continuous uniform distribution can be found on page 13 of the *Tables*, including the form of its probability density function, and formulae for its mean and variance.

(i) ***Probability density function***

Working in % units, the investment return follows a $U(0,10)$ distribution. So the probability density function is $f(x)=\dfrac{1}{10}$ for $0 \le x \le 10$ and 0 otherwise.

If we work with the returns expressed in decimals instead, then $f(x)=10$ *for* $0 \le x \le 0.10$ *and 0 otherwise.*

(ii) ***Mean***

The mean investment return is:

$$\frac{1}{2}(0+10)=5$$

ie 5% *pa*.

(iii) ***Variance and semi-variance***

The variance is given by:

$$\frac{1}{12}(10-0)^2=8.33$$

ie 8.33%% *pa.*

Alternatively, we could evaluate the variance using the integral:

$$\int_0^{10} \frac{(5-x)^2}{10}\,dx$$

Since the uniform distribution is symmetric, the semi-variance is equal to half the variance, *ie* 4.17%% *pa*.

Alternatively, we could evaluate the semi-variance using the integral:

$$\int_0^{5} \frac{(5-x)^2}{10}\,dx$$

(iv) ***Shortfall probability and expected shortfall***

The shortfall probability is given by:

$$SP = \int_0^3 \frac{1}{10}\,dx = \left[\frac{x}{10}\right]_0^3 = 0.3$$

The expected shortfall is given by:

$$ES = \int_0^3 \frac{(3-x)}{10}\,dx = \left[\frac{1}{10}\left(3x-0.5x^2\right)\right]_0^3 = 0.45\,\%$$

2 Relationship between risk measures and utility functions

An investor using a particular risk measure will base their decisions on a consideration of the available combinations of risk and expected return. Given a knowledge of how this trade-off is made it is possible, in principle, to construct the investor's underlying utility function. Conversely, given a particular utility function, the appropriate risk measure can be determined.

For example, if an investor has a quadratic utility function, the function to be maximised in applying the expected utility theorem will involve a linear combination of the first two moments of the distribution of return.

In other words, if an investor has a quadratic utility function then their attitude towards risk and return can be expressed purely in terms of the mean and variance of investment opportunities.

Thus variance of return is an appropriate measure of risk in this case.

Question

(i) State the expected utility theorem.

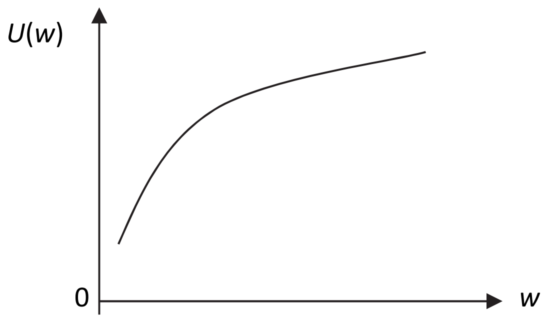
(ii) Draw a typical utility function for a non-satiated, risk-averse investor.

Solution

(i) The *expected utility theorem* states that:

- a function, $U(w)$, can be constructed representing an investor's utility of wealth, w
- the investor faced with uncertainty makes decisions on the basis of maximising the *expected value* of utility.

(ii)



Non-satiated investors prefer more wealth to less and so the graph slopes upwards, *ie* $U'(w) > 0$.

Risk-averse investors have diminishing marginal utility of wealth and so the slope of the graph decreases with w, *ie* $U''(w) < 0$.

If expected return and semi-variance below the expected return are used as the basis of investment decisions, it can be shown that this is equivalent to a utility function that is quadratic below the expected return and linear above.

Thus, this is equivalent to the investor being risk-averse below the expected return and risk-neutral for investment return levels above the expected return. Hence, no weighting is given to variability of investment returns above the expected return.

Use of a shortfall risk measure corresponds to a utility function that has a discontinuity at the minimum required return.

This therefore corresponds to the state-dependent utility functions discussed in a previous chapter.

Question

What is meant by a state-dependent utility function?

Solution

Sometimes it may be inappropriate to model an investor's behaviour over all possible levels of wealth with a single utility function. This problem can be overcome by using *state-dependent* utility functions, which model the situation where there is a discontinuous change in the state of the investor at a certain level of wealth.

3 Risk and insurance companies

3.1 Introduction

Individuals and corporations face risks resulting from unexpected events.

Risk-averse individuals can buy insurance to remove their exposure to risks. Being risk-averse, they will be willing to pay more for insurance than the expected cost of claims. The insurance company is willing to offer insurance primarily because it is able to spread its risks.

3.2 What to insure

Two considerations must be taken into account when assessing the effect of a risk: its likelihood and its severity. In a formal scenario, a risk matrix or graph is used as shown below.

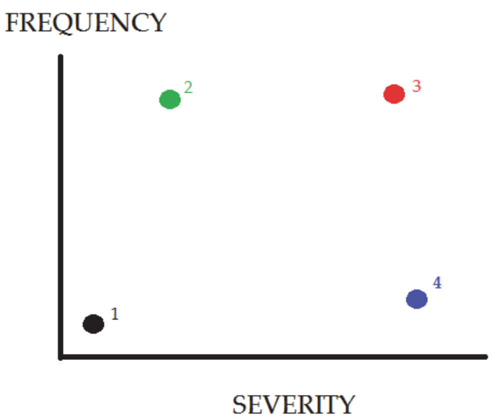


Figure 4.1

The figure above shows the frequency-severity dynamics of four possible events.

- **Event 1 is a low-frequency-low-severity event. Such an event does not warrant any worry to a corporation or individual.**

  For example, a solar eclipse occurs with low frequency, but is also accompanied by a low level of severity!

- **Event 2 is a high-frequency-low-severity event. Such an event would occur many times but at a low cost each time. The overall cost, due to high frequency, may be damaging. These types of events may need to be assessed on how they can be controlled.**

  For example, a smashed mobile phone screen is common but is typically limited in its extent.

- **Event 3 is a high-frequency-high-severity. Such events are to be avoided.**

  For example, car accidents involving fatalities happen relatively frequently and have a high severity.

- **Event 4 is a low-frequency-high-severity event. Such events tend to be insured.**

  For example, earthquake or hurricane damage.

In general, low severity events (such as events 1 and 2) are not generally insurable as the cost of management per claim is too expensive. However micro-insurance for poorer clients and new technologies that enable insuring small items over a short term (such as gadgets during a holiday) have been gaining ground.

3.3 Pooling resources

Insurance reduces the variability of losses due to adverse outcomes by pooling resources.

Consider a simple scenario of a property that has one hundredth chance of suffering £10,000 in damages (and 99% of no damages). The expected cost is 1% of £10,000 which is £100 while the VaR(99.5%) is £10,000.

Question

Explain why the 99.5% Value at Risk is £10,000.

Solution

Let X be the impact suffered by the property, so $P(X=0)=0.99$ and $P(X=-10{,}000)=0.01$. Then from the definition of Value at Risk for a discrete random variable we have:

$$VaR(X)=-t \text{ where } t=\max\{x:P(X<x)\leq p\}$$

But $P(X<-10{,}000)=0$ and $P(X<0)=0.01$, therefore:

$$t=\max\{x:P(X<x)\leq 0.005\}=-10{,}000$$

This means that the Value at Risk is £10,000.

If ten independent properties with similar characteristics are pooled together, the average cost is still £100 per property. At the extreme, the probability of all of them suffering the damage is 0.01^{10}. The VaR(99.5%) in this case is if one property suffers damage, that is £100 on average (using a Binomial distribution).

Question

Verify that the 99.5% Value at Risk in this case is £10,000.

Solution

Let X be the total impact suffered by the properties, so $X \sim Binomial(10,0.01)\times(-10{,}000)$. Then from the definition of Value at Risk for a discrete random variable we have:

$$VaR(X)=-t \text{ where } t=\max\{x:P(X<x)\leq p\}$$

If no properties suffer damage then $X = 0$. The probability of this is:

$$P(X=0)=\binom{10}{0}\times 0.01^0(1-0.01)^{10}=0.99^{10}=0.9044$$

So: $P(X<0)=1-0.9044=0.0956$

If exactly one property suffers damage then $X=-10{,}000$. The probability of this is:

$$P(X=-10{,}000)=\binom{10}{1}\times 0.01^1(1-0.01)^9=\frac{10!}{9!1!}0.01^1\times 0.99^9=0.0914$$

So: $P(X<-10{,}000)=1-0.9044-0.0914=0.0042$

Therefore:

$$t=\max\{x:P(X<x)\le 0.005\}=-10{,}000$$

This means that the Value at Risk is £10,000, which equates to one property suffering damage.

In pooling resources, an insurer attempts to group insureds (being corporations or individuals) within homogeneous groups. In the case of an individual with the ability to influence into which group they fall, adverse selection can occur. If the insurer is also risk averse, then the insurance premium needs to include a margin to compensate the insurer for taking on the risk.

3.4 Policyholder behaviour

Adverse selection **describes the fact that people who know that they are particularly bad risks are more inclined to take out insurance than those who know that they are good risks.**

It arises because customers typically know more about themselves than the insurance company knows.

Adverse selection is sometimes called 'self-selection' or 'anti-selection'.

To try and reduce the problems of adverse selection, insurance companies try to find out lots of information about potential policyholders. Policyholders can then be put in small, reasonably homogeneous pools and charged appropriate premiums.

Moral hazard **describes the fact that a policyholder may, because they have insurance, act in a way which makes the insured event more likely.**

This is because having the insurance provides less incentive to guard against the insured event happening. For example, while driving to work one day you realise that you forgot to lock the front door of your house. If you didn't have any household contents insurance, you might decide to go back and lock it. If you had adequate insurance, you might decide to carry on to work. This difference in behaviour *caused by the fact that you are insured* is an example of 'moral hazard'.

Moral hazard makes insurance more expensive. It may even push the price of insurance above the maximum premium that a person is prepared to pay.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 4 Summary

Measures of investment risk

Many investment models use *variance* of return as the measure of investment risk.

For a continuous random variable: $V = \int_{-\infty}^{\infty} (\mu - x)^2 f(x)\,dx$

For a discrete random variable: $V = \sum_x (\mu - x)^2 P(X = x)$

Variance has the advantage over most other measures that it:

- is mathematically tractable
- leads to elegant solutions for optimal portfolios, within the context of mean-variance portfolio theory.

The main argument against the use of variance as a measure of risk is that most investors do not dislike uncertainty of returns as such; rather they dislike the *downside risk* of low investment returns. Consequently, alternative measures of downside risk sometimes used include (in the continuous and then discrete cases):

- semi-variance of return: $\int_{-\infty}^{\mu} (\mu - x)^2 f(x)\,dx$ $\quad \sum_{x<\mu} (\mu - x)^2 P(X = x)$
- shortfall probability: $\int_{-\infty}^{L} f(x)\,dx$ $\quad \sum_{x<L} P(X = x)$

each of which ignores upside risk.

Value at Risk (VaR) represents the maximum potential loss on a portfolio over a given future time period with a given degree of confidence $(1-p)$. It is often calculated assuming that investment returns follow a normal distribution, which may not be an appropriate assumption.

For a continuous random variable, $VaR(X) = -t$, where $P(X < t) = p$.

For a discrete random variable, $VaR(X) = -t$, where $t = \max\{x : P(X < x) \le p\}$.

The expected shortfall, relative to a benchmark L is given by $E[\max(L - X, 0)]$.

For a continuous random variable, expected shortfall = $\int_{-\infty}^{L} (L - x) f(x)dx$.

For a discrete random variable, expected shortfall = $\sum_{x<L} (L - x) P(X = x)$.

When L is the VaR with a particular confidence level, the expected shortfall is known as *TailVaR*. TailVaR measures the expected loss *in excess* of the VaR.

It is also possible to calculate the expected shortfall and TailVaR *conditional* on a shortfall occurring by dividing through by the shortfall probability.

Relationship between risk measures and utility functions

If *expected return* and *variance* are used as the basis of investment decisions, it can be shown that this is equivalent to a quadratic utility function.

If *expected return* and *semi-variance* below the expected return are used as the basis of investment decisions, it can be shown that this is equivalent to a utility function that is quadratic below the expected return and linear above.

Use of a *shortfall risk measure* corresponds to a utility function that has a discontinuity at the minimum required return.

Using insurance to manage risk

Insurers decide which events to offer protection for based on the frequency and severity of the event.

The *pooling of resources* can be used to reduce an insurer's risk.

Adverse selection describes the fact that people who know that they are particularly bad risks are more inclined to take out insurance than those who know that they are good risks.

Moral hazard is the change in a policyholder's behaviour once insurance has been taken out, which makes the risk event more likely to occur.

Chapter 4 Practice Questions

4.1 Define the following measures of investment risk:

Exam style

(i) variance of return [1]

(ii) downside semi-variance of return [1]

(iii) shortfall probability [1]

(iv) Value at Risk. [1]
[Total 4]

4.2 Adam, Barbara and Charlie are all offered the choice of investing their entire portfolio in either a risk-free asset or a risky asset. The risk-free asset offers a return of 0% *pa*, whereas the returns on the risky asset are uniformly distributed over the range –5% to +10% *pa*. Assuming that each individual makes their investment choice in order to minimise their expected shortfall, and that they have benchmark returns of –2%, 0% and +2% *pa* respectively, who will choose which investment? Comment briefly on your answer.

4.3 (i) Define 'shortfall probability' for a continuous random variable.

(ii) An investor holds an asset that produces a random rate of return, R, over the course of a year. Calculate the shortfall probability using a benchmark rate of return of 1%, assuming:

(a) R follows a lognormal distribution with $\mu = 5\%$ and $\sigma^2 = (5\%)^2$

(b) R follows an exponential distribution with a mean return of 5%.

(iii) Explain with the aid of a simple numerical example the main limitation of the shortfall probability as a basis for making investment decisions.

4.4 Consider a zero-coupon corporate bond that promises to pay a return of 10% next period. Suppose that there is a 10% chance that the issuing company will default on the bond payment, in which case there is an equal chance of receiving a return of either 5% or 0%.

Exam style

(i) Calculate values for the following measures of investment risk:

(a) downside semi-variance

(b) shortfall probability based on the risk-free rate of return of 6%

(c) the expected shortfall below the risk-free return conditional on a shortfall occurring. [5]

(ii) Discuss the usefulness of downside semi-variance as a measure of investment risk for an investor. [3]
[Total 8]

4.5 An investor is contemplating an investment with a return of £ R, where:

Exam style

$$R = 250{,}000 - 100{,}000N$$

and N is a Normal $[1,1]$ random variable.

Calculate each of the following measures of risk:

(a) variance of return

(b) downside semi-variance of return

(c) shortfall probability, where the shortfall level is £50,000

(d) Value at Risk at the 95% confidence level

(e) Tail Value at Risk at the 95% confidence level, conditional on the VaR being exceeded.

[13]

Hint: *For part (e), you may wish to use the formula for the truncated first moment of a normal distribution given on page 18 of the Tables.*

4.6 (i) Explain the problem of adverse selection and how it might be dealt with by insurance companies. [2]

Exam style

(ii) Explain the problem of moral hazard and how it affects the price of insurance. [2]

[Total 4]

Chapter 4 Solutions

4.1 In the following we assume the investment return is given by a continuous random variable X with density function $f(x)$. This is the return over a chosen time period. Analogous formulae could be given for discrete or mixed cases.

(i) ***Variance***

$$\int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx$$

where $\mu = E[X]$ is the mean return for the chosen period. [1]

(ii) ***Downside semi-variance***

The downside semi-variance only takes into account returns below the mean return:

$$\int_{-\infty}^{\mu} (x-\mu)^2 f(x)dx$$ [1]

(iii) ***Shortfall probability***

A shortfall probability measures the probability of returns falling below a certain chosen benchmark level *L*:

$$P(X<L) = \int_{-\infty}^{L} f(x)dx$$ [1]

(iv) ***Value at Risk***

Value at Risk represents the maximum potential loss in value on a portfolio over a given future time period with a given degree of confidence. [1]

Alternatively, for a given confidence limit $(1-p)$:

$$VaR(X) = -t \quad \text{where} \quad P(X<t) = p$$

[Total 4]

4.2 The expected shortfall below a benchmark level *L* is defined as:

$$\int_{-\infty}^{L} (L-x)f(x)\,dx$$

For the risky asset, we have (working in percentage units):

$$f(x) = \begin{cases} \dfrac{1}{10-(-5)} = \dfrac{1}{15} & \text{if } -5<x<10 \\ 0 & \text{otherwise} \end{cases}$$

Adam

The expected shortfall of the *risky asset* is given by:

$$\int_{-5}^{-2} \frac{(-2-x)}{15}\, dx$$

$$= \frac{1}{15}\left[-2x - ½x^2\right]_{-5}^{-2}$$

$$= \frac{1}{15}\left[(4-2)-(10-12½)\right]$$

$$= \; 0.3\%$$

Alternatively, note that there is a chance of $1/5$ *that he will earn less than the benchmark, and in this case, the average shortfall will be 1½%. So the expected shortfall will be* $1/5 \times 1½\% = 0.3\%$.

The expected shortfall of the *risk-free* asset is 0%.

So Adam chooses the risk-free asset.

Barbara

The expected shortfall of the *risky asset* is given by:

$$\int_{-5}^{0} \frac{(-x)}{15}\, dx$$

$$= \frac{1}{15}\left[-½x^2\right]_{-5}^{-0}$$

$$= \frac{1}{15}\left[(0)-(-12½)\right]$$

$$= \; 0.833\%$$

The expected shortfall of the *risk-free* asset is again 0%.

So Barbara chooses the risk-free asset.

Charlie

The expected shortfall of the *risky asset* is given by:

$$\int_{-5}^{2} \frac{(2-x)}{15}\, dx$$

$$= \frac{1}{15}\left[2x - ½x^2\right]_{-5}^{2}$$

$$= \frac{1}{15}\left[(4-2)-(-10-12½)\right]$$

$$= \; 1.633\%$$

The expected shortfall of the *risk-free* asset is 2% × 1 = 2%.

So Charlie chooses the risky asset.

Thus, the expected shortfall increases with the benchmark return.

4.3 (i) ***Definition***

The shortfall probability for a continuous random variable, X, is:

$$P(X < L) = \int_{-\infty}^{L} f(x)\,dx$$

where L is the chosen benchmark level.

(ii)(a) ***Calculation based on lognormal distribution***

If $R \sim \log N(5,25)$, then the shortfall probability is:

$$P(R<1)=P\left(Z<\frac{\ln 1-5}{\sqrt{25}}\right)=P(Z<-1)$$

where $Z \sim N(0,1)$. Therefore:

$$P(R<1)=\Phi(-1)=1-\Phi(1)=1-0.84134=0.15866$$

(ii)(b) ***Calculation based on exponential distribution***

Since we know that $E(R)=5$, this means that $R \sim Exp(0.2)$.

So the shortfall probability is:

$$P(R<1)=\int_0^1 0.2e^{-0.2x}\,dx=\left[-e^{-0.2x}\right]_0^1=1-e^{-0.2}=0.18127$$

(iii) ***Main limitation***

The main limitation of the shortfall probability is that it ignores the *extent* of the shortfall below the benchmark L.

Thus, if L = 0, then the investor will prefer a gamble that offers +\$1 with probability 0.51 and –\$1,000,000 with probability 0.49 to one that that offers either \$1,000,000 or –\$2, each with probability of ½. This is somewhat unrealistic.

4.4 (i)(a) ***Downside semi-variance***

The expected return on the bond is given by:

$$0.90 \times 10\% + 0.05 \times 5\% + 0.05 \times 0\% = 9.25\% \qquad [1]$$

So the downside semi-variance is equal to:

$$(9.25-5)^2 \times 0.05 \; + \; (9.25-0)^2 \times 0.05 \; = \; 5.18\%\% \qquad [1]$$

(i)(b) ***Shortfall probability***

The probability of receiving less than 6% is equal to the sum of the probabilities of receiving 5% and 0%, *ie* 0.10. [1]

(i)(c) ***Expected conditional shortfall***

The expected shortfall below the risk-free rate of 6% is given by:

$$(6-5)\times 0.05 \; + \; (6-0)\times 0.05 \; = \; 0.35\% \qquad [1]$$

The expected shortfall below the risk-free return *conditional on a shortfall occurring* is equal to:

$$\frac{\text{expected shortfall}}{\text{shortfall probability}} = \frac{0.35\%}{0.10} = 3.5\% \qquad [1]$$

[Total 5]

We can see this directly by noting that, given that there is a shortfall, it is equally likely to be 1% or 6%. So the expected conditional shortfall is 3½%.

(ii) ***Usefulness of downside semi-variance***

- It gives more weight to downside risk, *ie* variability of investment returns below the mean, which is likely to be of greater concern to an investor than upside risk. [½]
- In fact, it completely ignores risk above the mean. [½]
- This is consistent with the investor being risk-neutral above the mean, which is unlikely to be the case in practice. [½]
- The mean is an arbitrary benchmark, which might not be appropriate for the particular investor. [½]
- If investment returns are symmetrically distributed about the mean (as they would be, for example, with a normal distribution) then it will give equivalent results to the variance. [½]
- However, it is less mathematically tractable than the variance. [½]

[Total 3]

4.5 (a) ***Variance of return***

N has a Normal $[1,1]$ distribution, so R has a Normal distribution with mean $150{,}000$ and variance $100{,}000^2$, *ie* $R \sim N(150{,}000, 100{,}000^2)$.

So, the variance of return is $100{,}000^2 = 10^{10}$. [2]

(b) ***Downside semi-variance of return***

Any normal distribution is symmetrical about its mean, so that the downside semi-variance of return is equal to half of the variance, *ie* 5×10^9. [2]

(c) ***Shortfall probability, where the shortfall level is £50,000***

The shortfall probability below £50,000 is:

$$\begin{aligned} P(R<50{,}000) &= P\left(\frac{R-150{,}000}{100{,}000}<\frac{50{,}000-150{,}000}{100{,}000}\right) \\ &= \Phi\left(\frac{50{,}000-150{,}000}{100{,}000}\right) \\ &= \Phi(-1) \\ &= 1-\Phi(1)=1-0.84134=0.15866 \end{aligned}$$ [2]

(d) ***Value at Risk***

From the *Tables*:

$$\Phi(-1.6449)=0.05$$

So, there is a 5% chance of the investment return R having a value less than:

$$\begin{aligned} R_{5\%} &= \mu_R-1.6449\sigma_R \\ &= 150{,}000-1.6449\times100{,}000 \\ &= -14{,}490 \end{aligned}$$

So, the Value at Risk at the 95% confidence level is £14,490. [2]

(e) ***Tail Value at Risk***

The VaR is £14,490. So, the formula for the conditional TVaR is:

$$\frac{1}{0.05}\int_{-\infty}^{-14{,}490}(-14{,}490-x)f(x)dx$$

where $f(x)$ is the *pdf* of a $N(150{,}000,100{,}000^2)$ distribution. [1]

Splitting this into two integrals:

$$\frac{-14{,}490}{0.05}\int_{-\infty}^{-14{,}490}f(x)dx-\frac{1}{0.05}\int_{-\infty}^{-14{,}490}x\,f(x)dx$$ [½]

Evaluating the first of these integrals:

$$\int_{-\infty}^{-14,490} f(x)dx = P\left(N(150{,}000, 100{,}000^2) < -14{,}490\right)$$

$$= P\left(Z < \frac{-14{,}490 - 150{,}000}{100{,}000}\right)$$

$$= P(Z < -1.6449)$$

$$= 0.05$$ [½]

This is as expected since $-14{,}490$ *is the VaR at the 95% confidence level.*

Evaluating the second integral, using the formula for the truncated first moment of a normal random variable on page 18 of the *Tables*:

$$\int_{-\infty}^{-14,490} x\,f(x)dx = 150{,}000\left[\Phi(U') - \Phi(L')\right] - 100{,}000\left[\phi(U') - \phi(L')\right]$$ [½]

where:

$$U' = \frac{-14{,}490 - 150{,}000}{100{,}000} = -1.6449$$ [¼]

and:

$$L' = \frac{-\infty - 150{,}000}{100{,}000} = -\infty$$ [¼]

Now $\Phi(t)$ is the cumulative distribution function of the standard normal distribution, so:

$$\Phi(U') = \Phi(-1.6449) = 0.05$$ [¼]

and:

$$\Phi(L') = \Phi(-\infty) = 0$$ [¼]

Also, $\phi(t)$ is the probability density function of the standard normal distribution, which is stated on page 160 of the *Tables*. So:

$$\phi(U') = \phi(-1.6449) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-1.6449)^2} = 0.10313$$ [¼]

and:

$$\phi(L') = \phi(-\infty) = 0$$ [¼]

Putting these values into the formula:

$$\int_{-\infty}^{-14,490} x\,f(x)dx = 150,000[0.05-0] - 100,000[0.10313-0]$$

$$= -2,813 \qquad [½]$$

Putting all this together, the conditional TVaR at the 95% confidence level is:

$$\frac{-14,490}{0.05} \times 0.05 - \frac{1}{0.05} \times (-2,813) = £41,770 \qquad [½]$$

So, the overall expected loss, given that the VaR at the 95% confidence level is exceeded, is:

$$14,490 + 41,770 = £56,260$$

[Total 13]

4.6 (i) ***Adverse selection***

Adverse selection refers to the fact that people who know that they are particularly bad risks are more inclined to take out insurance than those who know that they are good risks. [1]

To try to reduce the problems of adverse selection, insurance companies try to find out information about potential policyholders. Policyholders can then be put into small, reasonably homogeneous groups and charged appropriate premiums. [1]

[Total 2]

(ii) ***Moral hazard***

Moral hazard describes the fact that a policyholder may, because they have insurance, act in a way which makes the insured event more likely to occur. [1]

Moral hazard makes insurance more expensive. It may even push the price of insurance above the maximum premium that a person is prepared to pay. [1]

[Total 2]

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Loss distributions

Syllabus objectives

1.1 Loss distributions, with and without risk sharing

1.1.1 Describe the properties of the statistical distributions which are suitable for modelling individual and aggregate losses.

1.1.5 Estimate the parameters of a failure time or loss distribution when the data is complete, or when it is incomplete, using maximum likelihood and the method of moments.

1.1.6 Use R to fit a statistical distribution to a dataset and calculate appropriate goodness-of-fit measures.

0 Introduction

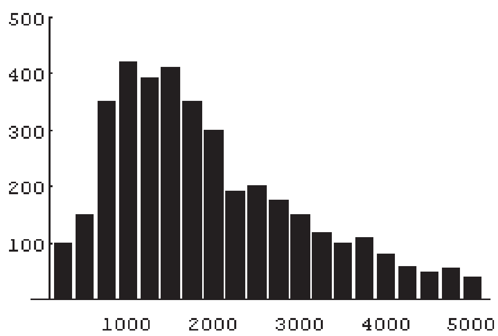
General insurance companies need to investigate claims experience and apply mathematical techniques for many purposes. These include:

- premium rating (*ie* deciding what premium rates to charge policyholders)
- reserving (*ie* assessing how much money should be set aside to cover the cost of claims)
- reviewing reinsurance arrangements
- testing for solvency (*ie* assessing the company's financial position).

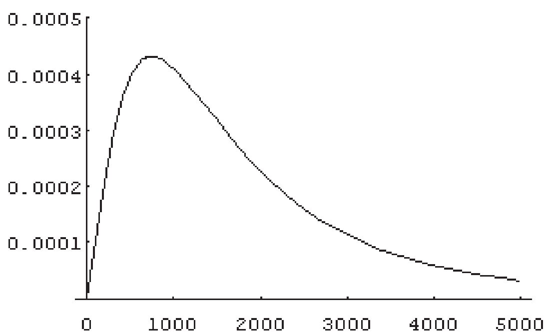
In this chapter we will look at loss distributions. These are statistical distributions that are used to model individual claim amounts. We will introduce some new distributions and we see how these can be fitted to observed claims data. We can then test for goodness of fit, and use the fitted loss distributions to estimate probabilities.

The total amount of claims in a particular time period is a quantity of fundamental importance to the proper management of an insurance company. The key assumption in all the models studied here is that the occurrence of a claim and the amount of a claim can be studied separately. Thus, a claim occurs according to some simple model for events occurring in time, then the amount of the claim is chosen from a distribution describing the claim amount.

Note carefully the distinction here. The frequency of claim amounts when plotted against size might look like this:



The statistical distributions in this chapter are used to approximate this distribution, which is called a *loss distribution*. For example, we might to decide to use a loss distribution like this as an approximation to the claims arising in the graph above:



A range of statistical techniques can be used to describe the distribution of random variables. The object is to describe the variation in claim amounts by finding a loss distribution that adequately describes the claims that actually occur. As usual this can be done at two levels.

At a first level, it can be assumed that the claims arise as realisations from a known distribution. For example, it may be possible to assume that the logarithm of the claim amount follows, to a reasonable approximation, a normal distribution with known mean and known standard deviation. Knowledge of the claim amount process would be complete, and interest would then centre on the consequences for insurance. For example, claims above a certain level might trigger some reinsurance arrangements or claims below a certain level might never be lodged if a policy excess was in force.

A policy excess means that the policyholder has to pay the first part of any claim. For example, with car insurance in the UK the policyholder often has to pay the first £200 of any claim. The insurer pays the rest.

In practice the exact claims distribution will hardly ever be known. At this second level a standard method of proceeding is to assume that the claims distribution is a member of a certain family. The parameters of the family must now be estimated using the claim amount records by an appropriate method such as maximum likelihood. Complications will arise if large claims have been limited (reinsurance) or some small claims have not been lodged (policy excess).

We will consider the effects of reinsurance and policy excesses in Chapter 18.

Many studies have been made of the kind of distribution that can be used to describe the variation in claim amounts.

The typical pattern is as shown in the histogram above, with a few small claims, rising to a peak, then tailing off gradually with a few very large claims.

The general conclusion is that claims distributions tend to be positively skewed and long tailed.

1 Simple loss distributions

In this section we will review some of the properties of the statistical distributions that are used to model claim amounts. In most cases we use a positively skewed, continuous distribution.

Recall that, for a continuous random variable, X :

- the cumulative distribution function (CDF) is:

$$F_X(x) = P(X \leq x)$$

- the probability density function (PDF) is:

$$f_X(x) = F'_X(x)$$

- probabilities can be expressed in terms of the PDF or CDF:

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx = F_X(b) - F_X(a)$$

- the moment generating function (MGF) is:

$$M_X(t) = E(e^{tX}) = \int_x e^{tx} f_X(x)\,dx$$

The CDF, PDF and MGF may also be denoted without the subscript as $F(x)$, $f(x)$ and $M(t)$, respectively, provided the meaning is clear.

The formulae for the densities, the moments and the moment generating functions (where they exist) for the distributions discussed in this chapter are given in the *Formulae and Tables for Actuarial Examinations*.

In the *Tables*, the abbreviation DF is used for cumulative distribution function.

1.1 The exponential distribution

A random variable X has the exponential distribution with parameter $\lambda > 0$ if it has CDF:

$$\boldsymbol{F(x) = 1 - e^{-\lambda x}, \quad x > 0}$$

In that case we write $X \sim Exp(\lambda)$.

The PDF is:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

The mean and variance are $\dfrac{1}{\lambda}$ and $\dfrac{1}{\lambda^2}$ respectively.

The PDF can also be written as:

$$f(x) = \frac{1}{\mu} e^{-x/\mu}$$

where μ is the mean.

The MGF is:

$$M(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}, \quad t < \lambda$$

All of these formulae are given on page 11 of the *Tables*.

Question

A portfolio of insurance policies contains two types of risk. Type I risks make up 70% of claims and give rise to loss amounts that are exponentially distributed with mean 500. Type II risks give rise to loss amounts that are exponentially distributed with mean 1,000.

Let X denote the amount of a randomly chosen loss. Determine $E(X)$, $\text{var}(X)$ and $M_X(t)$.

Solution

Since the amount of a loss depends on the type of risk from which it arises, we calculate $E(X)$ using the conditional expectation formula (from Subject CS1). This formula is given on page 16 of the *Tables*. In this case:

$$E(X) = E(E(X \mid \text{Type})) = E(X \mid \text{Type I})P(\text{Type I}) + E(X \mid \text{Type II})P(\text{Type II})$$

$$= 500 \times 0.7 + 1{,}000 \times 0.3 = 650$$

Similarly:

$$E(X^2) = E(E(X^2 \mid \text{Type})) = E(X^2 \mid \text{Type I})P(\text{Type I}) + E(X^2 \mid \text{Type II})P(\text{Type II})$$

Now:

$$E(X^2 \mid \text{Type I}) = \text{var}(X \mid \text{Type I}) + \left[E(X \mid \text{Type I})\right]^2 = 500^2 + 500^2 = 500{,}000$$

Here we are using the fact that the variance of an exponential random variable is the square of its mean. So the variance of losses from Type I risks is 500^2. We can use the same approach for Type II risks:

$$E(X^2 \mid \text{Type II}) = \text{var}(X \mid \text{Type II}) + \left[E(X \mid \text{Type II})\right]^2 = 1{,}000^2 + 1{,}000^2 = 2{,}000{,}000$$

So:

$$E(X^2) = 500{,}000 \times 0.7 + 2{,}000{,}000 \times 0.3 = 950{,}000$$

Hence:

$$\text{var}(X) = 950{,}000 - 650^2 = 527{,}500$$

Alternatively, we could calculate $\text{var}(X)$ using the conditional variance formula, which is also given on page 16 of the *Tables*:

$$\text{var}(X) = E[\text{var}(X \mid \text{Type})] + \text{var}[E(X \mid \text{Type})]$$

For notational convenience, let $V = \text{var}(X \mid \text{Type})$ and let $W = E(X \mid \text{Type})$. Then V has the following distribution:

v	500^2	$1{,}000^2$
$P(V = v)$	0.7	0.3

So:

$$E(V) = 500^2 \times 0.7 + 1{,}000^2 \times 0.3 = 475{,}000$$

In addition, we can calculate $\text{var}(W)$ from the distribution of W:

w	500	1,000
$P(W = w)$	0.7	0.3

We have:

$$E(W) = 500 \times 0.7 + 1{,}000 \times 0.3 = 650$$

$$E(W^2) = 500^2 \times 0.7 + 1{,}000^2 \times 0.3 = 475{,}000$$

and hence:

$$\text{var}(W) = 475{,}000 - 650^2 = 52{,}500$$

So:

$$\text{var}(X) = E[V] + \text{var}[W] = 475{,}000 + 52{,}500 = 527{,}500$$

Finally, we will consider the moment generating function of X. Again, we will use the conditional expectation formula:

$$M_X(t) = E(e^{tX}) = E\left[E(e^{tX} \mid \text{Type})\right]$$

$$= E(e^{tX} \mid \text{Type I})P(\text{Type I}) + E(e^{tX} \mid \text{Type II})P(\text{Type II})$$

$E(e^{tX} \mid \text{Type I})$ is the MGF of the exponential distribution with mean 500, *ie*:

$$E(e^{tX} \mid \text{Type I}) = (1-500t)^{-1}$$

Similarly, $E(e^{tX} \mid \text{Type II})$ is the MGF of the exponential distribution with mean 1,000, *ie*:

$$E(e^{tX} \mid \text{Type II}) = (1-1{,}000t)^{-1}$$

So:

$$M_X(t) = 0.7(1-500t)^{-1} + 0.3(1-1{,}000t)^{-1}$$

We can use R to simulate values from statistical distributions, plot their PDFs, and calculate probabilities and percentiles. An example involving the exponential distribution is given below.

Suppose we have an exponential distribution with parameter $\lambda = 0.5$. The R code for simulating 100 values is given by:

```
rexp(100,rate=0.5)
```

The PDF is obtained by `dexp(x, rate=0.5)` **and is useful for graphing. For example:**

```
plot(seq(0:5000),dexp(seq(0:5000), rate=0.5),type="l")
```

To calculate probabilities for a continuous distribution we use the CDF which is obtained by `pexp`**. For example, to calculate $P(X \le 2) = 0.6321206$ we use the R code:**

```
pexp(2,rate=0.5)
```

Similarly, the quantiles can be calculated with `qexp`**.**

This code can be adapted to deal with other statistical distributions.

1.2 The gamma distribution

The random variable X has a gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$ if it has PDF:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x), \quad x > 0$$

In that case we write $X \sim Ga(\alpha, \lambda)$.

This may also be written as $Gamma(\alpha, \lambda)$.

The gamma function, $\Gamma(\alpha)$, appears in the denominator of this PDF. The definition and properties of this function are given on page 5 of the *Tables*.

The mean and variance of X are:

$$E(X) = \frac{\alpha}{\lambda}$$

$$\text{var}(X) = \frac{\alpha}{\lambda^2}$$

Question

If $X \sim Gamma(\alpha, \lambda)$, show that the MGF of X is:

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}$$

Solution

Using the definition of the MGF, we have:

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x}\, dx = \int_0^\infty \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-(\lambda-t)x}\, dx$$

We can make the integrand look like the PDF of the $Gamma(\alpha, \lambda - t)$ distribution by writing:

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha \int_0^\infty \frac{1}{\Gamma(\alpha)} (\lambda - t)^\alpha x^{\alpha-1} e^{-(\lambda-t)x} dx$$

This integral is equal to 1 provided $\lambda - t > 0$, so:

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha = \left(\frac{\lambda - t}{\lambda}\right)^{-\alpha} = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}, \quad \text{for } t < \lambda$$

By differentiating the MGF, we can obtain the non-central moments, $E(X^k)$, $k = 1, 2, 3, \ldots$:

$$E(X) = M'_X(0), \quad E(X^2) = M''_X(0), \quad etc$$

The variance and skewness can be obtained more quickly using the cumulant generating function (CGF). Recall that:

$$C_X(t) = \ln M_X(t)$$

Question

Suppose that $X \sim Gamma(\alpha, \lambda)$.

Derive formulae for the skewness and coefficient of skewness of X.

Solution

The skewness of X is its third central moment, $E\left[(X-E(X))^3\right]$. It can be obtained by differentiating the CGF three times and evaluating the third derivative when $t=0$.

Since $X \sim Gamma(\alpha,\lambda)$:

$$C_X(t)=\ln\left(1-\frac{t}{\lambda}\right)^{-\alpha}=-\alpha\ln\left(1-\frac{t}{\lambda}\right)$$

Differentiating using the chain rule:

$$C'_X(t)=-\alpha\left(-\frac{1}{\lambda}\right)\left(1-\frac{t}{\lambda}\right)^{-1}=\frac{\alpha}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-1}$$

$$C''_X(t)=\frac{\alpha}{\lambda}\left(-\frac{1}{\lambda}\right)(-1)\left(1-\frac{t}{\lambda}\right)^{-2}=\frac{\alpha}{\lambda^2}\left(1-\frac{t}{\lambda}\right)^{-2}$$

$$C'''_X(t)=\frac{\alpha}{\lambda^2}\left(-\frac{1}{\lambda}\right)(-2)\left(1-\frac{t}{\lambda}\right)^{-3}=\frac{2\alpha}{\lambda^3}\left(1-\frac{t}{\lambda}\right)^{-3}$$

So:

$$\text{skew}(X)=C'''_X(0)=\frac{2\alpha}{\lambda^3}\left(1-\frac{0}{\lambda}\right)^{-3}=\frac{2\alpha}{\lambda^3}$$

The coefficient of skewness of X is:

$$\text{coeff of skew}(X)=\frac{\text{skew}(X)}{\left[\text{var}(X)\right]^{3/2}}$$

The variance can also be obtained from the CGF:

$$\text{var}(X)=C''_X(0)=\frac{\alpha}{\lambda^2}\left(1-\frac{0}{\lambda}\right)^{-2}=\frac{\alpha}{\lambda^2}$$

So:

$$\text{coeff of skew}(X)=\frac{\text{skew}(X)}{\left[\text{var}(X)\right]^{3/2}}=\frac{2\alpha/\lambda^3}{(\alpha/\lambda^2)^{3/2}}=\frac{2}{\alpha^{1/2}}=\frac{2}{\sqrt{\alpha}}$$

Formulae for the PDF, MGF, mean, variance, non-central moments and coefficient of skewness of the gamma distribution are all given on page 12 of the *Tables*.

There is no closed form (*ie* no simple formula) for the CDF of a gamma random variable, which means that it is not easy to find gamma probabilities directly without using a computer package such as R. However, these probabilities can be obtained using the relationship between the gamma and chi-squared distributions.

Relationship between gamma and chi-squared distributions

If $X \sim Gamma(\alpha, \lambda)$ and 2α is an integer, then:

$$2\lambda X \sim \chi^2_{2\alpha}$$

This result is also given on page 12 of the *Tables*.

As an illustration of how this relationship can be used, suppose that $X \sim Gamma(10,4)$ and we want to calculate $P(X > 4.375)$. Using the result above, we know that $8X \sim \chi^2_{20}$, so:

$$P(X > 4.375) = P(8X > 8 \times 4.375) = P(\chi^2_{20} > 35)$$

From page 166 of the *Tables*, we see that:

$$P(\chi^2_{20} \leq 35) = 0.9799$$

So:

$$P(X > 4.375) = 1 - 0.9799 = 0.0201$$

The R code for simulating a random sample of 100 values from the gamma distribution with $\alpha = 2$ and $\lambda = 0.25$ is:

```
rgamma(100, 2, 0.25)
```

Similarly, the PDF, CDF and quantiles can be obtained using the R functions `dgamma`**,** `pgamma` **and** `qgamma`**.**

1.3 The normal distribution

The normal distribution arises in a variety of contexts. It is of limited use for modelling loss distributions because of its symmetry (as loss distributions tend to be positively skewed).

Question

Derive the formula for the MGF of a standard normal random variable.

Solution

Suppose that $X \sim N(0,1)$. Then the PDF of X is:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-½x^2}$$

and the MGF is:

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-½x^2} \, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-½(x^2-2tx)} dx$$

Completing the square gives:

$$M_X(t) = e^{½t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-½(x-t)^2} dx$$

The integrand in the expression immediately above is the PDF of a $N(t,1)$ random variable. Integrating this over all possible values of x gives the total probability, which is 1. So:

$$M_X(t) = e^{½t^2}$$

2 Other loss distributions

The distributions given in Section 1 (exponential, gamma, normal) all have easily derivable MGFs. However, there is a wide variety of other distributions that may also be used to model losses. We consider some of these here. None of the distributions in this section have an MGF that is easy to derive or use.

2.1 The lognormal distribution

The definition of the lognormal distribution is very simple: X has a lognormal distribution if $\log X$ has a normal distribution. When $\log X \sim N(\mu, \sigma^2)$, $X \sim \log N(\mu, \sigma^2)$.

So the range of values taken by the lognormal distribution is 0 to ∞.

As usual, log here refers to the natural logarithm, *ie* log base e.

The mean and variance of a lognormal random variable can be obtained from the MGF of the corresponding normal distribution. If $X \sim \log N(\mu, \sigma^2)$, then:

$$E(X) = E(e^{\ln X}) = E(e^Y)$$

where $Y = \ln X \sim N(\mu, \sigma^2)$. However:

$$E(e^Y) = M_Y(1) = e^{\mu + \frac{1}{2}\sigma^2}$$

So $E(X) = e^{\mu + \frac{1}{2}\sigma^2}$.

Similarly:

$$E(X^2) = E(e^{2Y}) = M_Y(2) = e^{2\mu + 2\sigma^2}$$

and hence:

$$\text{var}(X) = e^{2\mu + 2\sigma^2} - \left(e^{\mu + ½\sigma^2}\right)^2 = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)$$

Alternatively, the mean and variance can be derived using integration.

Formulae for the PDF, mean, variance, non-central moments and coefficient of skewness of the lognormal distribution are given on page 14 of the *Tables*.

Lognormal probabilities can be evaluated by expressing them as standard normal probabilities and looking up the values given on pages 160 and 161 of the *Tables*.

The R code for simulating values and obtaining the PDF, CDF and quantiles from the lognormal distribution is similar to the R code used for other continuous distributions using the R functions `rlnorm`, `dlnorm`, `plnorm` and `qlnorm`.

2.2 The two-parameter Pareto distribution

A random variable X has the Pareto distribution with parameters $\alpha > 0$ and $\lambda > 0$ if it has CDF:

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha} \quad , \quad x > 0$$

In that case we write $X \sim Pa(\alpha, \lambda)$.

It is easily checked by differentiating $F(x)$ with respect to x that the Pareto distribution has PDF:

$$f(x) = \frac{\alpha\lambda^{\alpha}}{(\lambda + x)^{\alpha+1}} \quad , \quad x > 0$$

Question

Suppose that $X \sim Pa(\alpha, \lambda)$. Derive a formula for $E(X)$.

Solution

The expected value is:

$$E(X) = \int_x x\, f(x)\,dx = \int_0^{\infty} x \frac{\alpha\lambda^{\alpha}}{(\lambda + x)^{\alpha+1}}\,dx$$

One way to simplify this is to use the substitution $t = \lambda + x$. Using this substitution:

$$E(X) = \int_{\lambda}^{\infty} (t - \lambda)\frac{\alpha\lambda^{\alpha}}{t^{\alpha+1}}\,dt = \alpha\lambda^{\alpha}\int_{\lambda}^{\infty} t^{-\alpha}\,dt - \alpha\lambda^{\alpha+1}\int_{\lambda}^{\infty} t^{-\alpha-1}\,dt$$

Integrating gives:

$$E(X) = \alpha\lambda^{\alpha}\left[\frac{t^{-\alpha+1}}{-\alpha+1}\right]_{\lambda}^{\infty} - \alpha\lambda^{\alpha+1}\left[\frac{t^{-\alpha}}{-\alpha}\right]_{\lambda}^{\infty} = \frac{\alpha\lambda}{\alpha-1} - \lambda = \frac{\lambda}{\alpha-1}$$

This expression is valid only if the powers in the bracketed terms are both negative, *ie* if $\alpha > 1$.

Alternatively, this formula could be derived using integration by parts.

Formulae for the CDF, PDF, mean, variance, non-central moments and coefficient of skewness of the Pareto distribution are given on page 14 of the *Tables*.

Let's now consider the median of the Pareto distribution. By definition, the median m is the point where $F(m) = P(X \le m) = ½$. So, in this case, we have:

$$1-\left(\frac{\lambda}{\lambda+m}\right)^{\alpha}=\frac{1}{2}$$

and this can be rearranged to give:

$$m = \lambda(2^{1/\alpha} - 1)$$

We can compare the median and the mean by drawing a graph. We have just seen that the mean, μ, is equal to $\frac{\lambda}{\alpha - 1}$.

A sketch of the graphs of $\frac{m}{\lambda}$ (bottom curve) and $\frac{\mu}{\lambda}$ (top curve) for values of $\alpha > 1$ is shown below:

From this we see that the mean is always greater than the median, *ie* the Pareto distribution is always positively skewed.

There is no built in R code for the Pareto distribution so we would have to define the functions `rpareto`, `dpareto`, `ppareto` **and** `qpareto` **from first principles as follows:**

```
rpareto <- function(n,a,l){
rp <- l*((1-runif(n))^(-1/a)-1)
rp}

dpareto <- function(x,a,l){
a*l^(a)/((l+x)^(a+1))}

ppareto <- function(q,a,l){
1-(l/(l+q))^a}

qpareto <- function(p,a,l){
q <- l*((1-p)^(-1/a)-1)
q}
```

2.3 The Burr distribution

The CDF of the Pareto distribution $Pa(\alpha, \lambda)$ is:

$$F(x) = 1 - \frac{\lambda^\alpha}{(\lambda + x)^\alpha}, x > 0$$

A further parameter $\gamma > 0$ can be introduced by setting:

$$F(x) = 1 - \frac{\lambda^\alpha}{(\lambda + x^\gamma)^\alpha}, \quad x > 0$$

This is the CDF of the transformed Pareto or Burr distribution. The additional parameter gives extra flexibility when a fit to data is required.

Formulae for the CDF, PDF and non-central moments of the Burr distribution are given on page 15 of the *Tables*.

There is no built in R code for the Burr distribution so we would have to define the functions `rburr`, `dburr`, `pburr` **and** `qburr` **from first principles as follows:**

```
rburr <- function(n,a,l,g){
rp <- (l*((1-runif(n))^(-1/a)-1))^(1/g)
rp}

dburr <- function(x,a,l,g){
((a*g*l^a)*x^(g-1))/((l+x^g)^(a+1))}

pburr <- function(q,a,l,g){
1-(l/(l+q^g))^a}

qburr <- function(p,a,l,g){
q <- (l*((1-p)^(-1/a)-1))^(1/g)
q}
```

2.4 The three-parameter Pareto distribution

The PDF of the Pareto distribution $Pa(\alpha,\lambda)$ is:

$$f(x) = \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}}, \quad x > 0$$

Another generalisation of the Pareto distribution is to add a further parameter k so that the PDF becomes:

$$f(x) = \frac{\Gamma(\alpha + k)\,\delta^\alpha}{\Gamma(\alpha)\,\Gamma(k)} \frac{x^{k-1}}{(\delta + x)^{\alpha+k}}, \quad x > 0$$

Formulae for the PDF, mean, variance and non-central moments of the three-parameter Pareto distribution are given on page 15 of the *Tables*.

The three-parameter Pareto distribution is equivalent to the two-parameter Pareto distribution when $k=1$.

The moments of the generalised Pareto can be obtained either directly by evaluating $E(X^n) = \int_x x^n f(x)\,dx$ or by using a conditional expectation argument.

Here the Core Reading is using the phrase 'generalised Pareto distribution' to refer to the three-parameter Pareto distribution. However, this is not the same as the generalised Pareto distribution that we will meet in Chapter 16.

The easiest way to evaluate the integral expression:

$$\int_x x^n f(x)\,dx$$

is to make it look like the PDF of another three-parameter Pareto distribution.

Question

Suppose that X has a three-parameter Pareto distribution with parameters α, λ and k. Derive formulae for $E(X)$ and $\text{var}(X)$.

Solution

The mean is:

$$E(X) = \int_0^\infty x \frac{\Gamma(\alpha+k)\lambda^\alpha x^{k-1}}{\Gamma(\alpha)\Gamma(k)(\lambda+x)^{\alpha+k}}\,dx = \int_0^\infty \frac{\Gamma(\alpha+k)\lambda^\alpha x^k}{\Gamma(\alpha)\Gamma(k)(\lambda+x)^{\alpha+k}}\,dx$$

This expression can be simplified by making the integrand look like the PDF of the Pareto distribution with parameters $\alpha-1$, λ and $k+1$:

$$E(X) = \lambda \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} \frac{\Gamma(k+1)}{\Gamma(k)} \int_0^\infty \frac{\Gamma(\alpha+k)\lambda^{\alpha-1} x^k}{\Gamma(\alpha-1)\Gamma(k+1)(\lambda+x)^{\alpha+k}}\,dx$$

Since this integrand is a PDF, integrating it over all possible values of x gives us 1. So:

$$E(X) = \lambda \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} \frac{\Gamma(k+1)}{\Gamma(k)} = \frac{\lambda k}{\alpha-1}$$

Here we are using the result $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$, which is given on page 5 of the *Tables*.

We can use a similar method to calculate the second moment:

$$E(X^2) = \int_0^\infty x^2 \frac{\Gamma(\alpha+k)\lambda^\alpha x^{k-1}}{\Gamma(\alpha)\Gamma(k)(\lambda+x)^{\alpha+k}}\,dx = \int_0^\infty \frac{\Gamma(\alpha+k)\lambda^\alpha x^{k+1}}{\Gamma(\alpha)\Gamma(k)(\lambda+x)^{\alpha+k}}\,dx$$

Making the integrand look like the PDF of the Pareto distribution with parameters $\alpha-2$, λ and $k+2$, we have:

$$E(X^2)=\lambda^2\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}\frac{\Gamma(k+2)}{\Gamma(k)}\int_0^\infty \frac{\Gamma(\alpha+k)\lambda^{\alpha-2}x^{k+1}}{\Gamma(\alpha-2)\Gamma(k+2)(\lambda+x)^{\alpha+k}}\,dx=\frac{k(k+1)\lambda^2}{(\alpha-1)(\alpha-2)}$$

So the variance is:

$$\begin{aligned}\frac{\lambda^2k(k+1)}{(\alpha-1)(\alpha-2)}-\left(\frac{\lambda k}{\alpha-1}\right)^2&=\frac{\lambda^2k(k+1)(\alpha-1)-\lambda^2k^2(\alpha-2)}{(\alpha-1)^2(\alpha-2)}\\&=\frac{\lambda^2k\left[(k\alpha+\alpha-k-1)-(k\alpha-2k)\right]}{(\alpha-1)^2(\alpha-2)}\\&=\frac{\lambda^2k(k+\alpha-1)}{(\alpha-1)^2(\alpha-2)}\end{aligned}$$

There is no built in R code for the three-parameter Pareto distribution, so we would have to define the function `dgpareto` from first principles as we did for the Pareto. However, since the CDF does not exist in closed form it is not easy to create functions to obtain probabilities, percentage points or simulated values.

2.5 The Weibull distribution

The Pareto distribution is a distribution with an upper tail that tends to 0 as a power of x. This gives a distribution with a much heavier tail than the exponential. The expressions for the upper tails of the exponential and the Pareto distributions are:

exponential $\quad P(X>x)=\exp(-\lambda x)$

Pareto $\quad P(X>x)=(\lambda/(\lambda+x))^\alpha$

So, if we want to choose a model with a thick tail so as not to underestimate the probability of a large claim, we might well choose the Pareto distribution to model our claims (assuming that it is a suitable distribution in other respects).

However, these are not the only types of tail.

There is a further possibility. Set:

$$P(X>x)=\exp(-\lambda x^\gamma),\quad \gamma>0$$

There are now two cases. If $\gamma<1$, a distribution with a tail intermediate in weight between the exponential and the Pareto will be obtained, while if $\gamma>1$, the upper tail will be lighter than the exponential ($\gamma=1$ is the exponential distribution).

This distribution is called the Weibull distribution, a very flexible distribution, which can be used as a model for losses in insurance, usually with $\gamma < 1$. A random variable X has a Weibull distribution with parameters $c > 0$ and $\gamma > 0$ if it has CDF:

$$F(x) = 1 - \exp(-cx^{\gamma}), \quad x > 0$$

In that case we write $X \sim W(c,\gamma)$. (Note the change from λ to *c*; this is the notation used in the *Tables for Actuarial Examinations*).

The PDF of the $W(c,\gamma)$ distribution is:

$$f(x) = c\gamma x^{\gamma-1}\exp(-cx^{\gamma}), \quad x > 0$$

Formulae for the CDF, PDF and non-central moments of the Weibull distribution are given on page 15 of the *Tables*.

Question

Suppose that X has a Weibull distribution with parameters c and γ. Derive a formula for $E(X)$.

Solution

The mean is:

$$E(X) = \int_0^{\infty} x\, c\gamma x^{\gamma-1} e^{-cx^{\gamma}}\, dx$$

Making the substitution $u = cx^{\gamma}$, so that $\dfrac{du}{dx} = c\gamma\, x^{\gamma-1}$ and $x = \left(\dfrac{u}{c}\right)^{1/\gamma}$ gives:

$$E(X) = \int_0^{\infty} \left(\frac{u}{c}\right)^{1/\gamma} e^{-u} du$$

Now, manipulating the integrand so that it looks like the PDF of a $Gamma\left(1+\dfrac{1}{\gamma},1\right)$ random variable, we have:

$$E(X) = \Gamma\left(1+\frac{1}{\gamma}\right)\frac{1}{c^{1/\gamma}}\int_0^{\infty}\frac{1}{\Gamma(1+1/\gamma)}u^{1/\gamma}e^{-u}du$$

The integral is now equal to 1 (as we're integrating a PDF over all possible values of the random variable). So:

$$E(X) = \Gamma\left(1+\frac{1}{\gamma}\right)\frac{1}{c^{1/\gamma}}$$

The R code for simulating a random sample of 100 values from the Weibull distribution with $c = 2$ and $\gamma = 0.25$ is:

```
rweibull(100, 0.25, 2^(-1/0.25))
```

R uses a different parameterisation for the scale parameter, *c*.

Similarly, the PDF, CDF and quantiles can be obtained using the R functions `dweibull`, `pweibull` **and** `qweibull`**.**

Alternatively, we could redefine them from first principles as follows:

```
rweibull <- function(n,c,g){
rp <- (log(1-runif(n))/c)^(1/g)
rp}

dweibull <- function(x,c,g){
c*g*x^(g-1)*exp(-c*x^g)}

pweibull <- function(q,c,g){
1-exp(-c*x^g)}

qweibull <- function(p,c,g){
q <- (log(1-p)/c)^(1/g)
q}
```

2.6 Illustration of tail weights

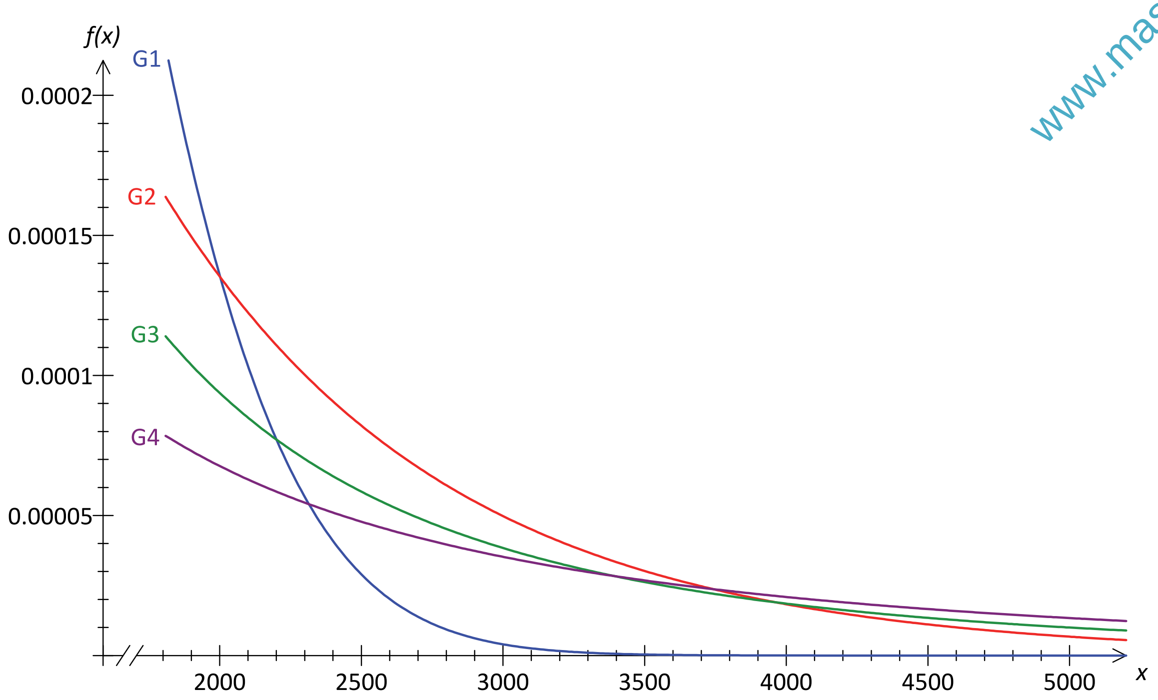
The PDFs shown in the diagram below illustrate the difference in the tails of the exponential, Pareto and Weibull distributions. All four of the distributions have a mean of 1,000.

G1 is a Weibull distribution with parameters $\gamma = 2$ and $c = \dfrac{\pi}{2{,}000^2} = 7.854 \times 10^{-7}$ (standard deviation = 522.7).

G2 is an exponential distribution with $\lambda = \dfrac{1}{1{,}000} = 0.001$ (standard deviation = 1,000).

G3 is a (two-parameter) Pareto distribution with $\alpha = 3$ and $\lambda = 2{,}000$ (standard deviation = 1,732).

G4 is a Weibull distribution with $\gamma = ½$ and $c = \dfrac{1}{\sqrt{500}} = 0.04472$ (standard deviation = 2,236).


f(x)
G1
G2
G3
G4
0.0002
0.00015
0.0001
0.00005
2000
2500
3000
3500
4000
4500
5000
x

3 Estimation

The methods of maximum likelihood, moments and percentiles can be used to fit distributions to sets of data.

We can check the fit in R by plotting a histogram of the data and superimposing the density function of the fitted distribution. Better yet, we can plot an empirical density function from the data using the function `density` and add the true density function of the fitted distribution.

A better way is to use the `qqplot` function to compare the sample data to simulated values from the fitted model distribution. A straight diagonal line indicates perfect fit:

```
qqplot(<simulated theoretical values>, <sample values>)
abline(0,1)
```

The fit of the distribution can also be tested formally by using a χ^2 test. The method of percentiles is outlined in Section 3.3; the other methods and the χ^2 test have been covered in Subject CS1, Actuarial Statistics 1.

We will now give a summary of the method of moments and maximum likelihood estimation. We will also introduce the method of percentiles and give a brief reminder of how the chi-squared test can be used to check the fit of a statistical distribution to a data set.

3.1 The method of moments

For a distribution with r parameters, the moments are as follows:

$$m_j = \frac{1}{n}\sum_{i=1}^{n} x_i^j \qquad j = 1, 2 \dots r$$

where:

$m_j = E(X^j \mid \theta)$, **a function of the unknown parameter,** θ, **being estimated**

$n =$ **the sample size**

$x_i =$ **the** i ***th*** **value in the sample**

The estimate for the parameter, θ, can be determined by solving the equation above. Where there is more than one parameter, they can be determined by solving the simultaneous equations for each m_j.

So, for example, if we are trying to estimate the value of a single parameter, and we have a sample of n claims whose sizes are $x_1, x_2, \dots, x_n$, we would solve the equation:

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

ie we would equate the first non-central moments for the population and the sample.

If we are trying to find estimates for two parameters (for example if we are fitting a gamma distribution and need to obtain estimates for both parameters), we would solve the simultaneous equations:

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad E(X^2) = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

In fact, in the two-parameter case, estimates are often obtained by equating sample and population means and variances. If we use the n-denominator sample variance:

$$s_n^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right]$$

this will give the same estimates as would be obtained by equating the first two non-central moments.

More generally, we use as many equations of the form $E(X^k) = \frac{1}{n}\sum_{i=1}^{n} x_i^k$, $k = 1, 2, \ldots$ as are needed to determine estimates of the relevant parameters.

3.2 Maximum likelihood estimation

The likelihood function of a random variable, X, is the probability (or PDF) of observing what was observed given a hypothetical value of the parameter, θ. The maximum likelihood estimate (MLE) is the one that yields the highest probability (or PDF), *ie* that maximises the likelihood function.

For the sample in Section 3.1 above, the likelihood function $L(\theta)$ can be expressed as:

$$L(\theta) = \prod_{i=1}^{n} P(X = x_i \mid \theta) \quad \text{for a discrete random variable, } X$$

or:

$$L(\theta) = \prod_{i=1}^{n} f(x_i \mid \theta) \quad \text{for a continuous random variable, } X$$

To determine the MLE the likelihood function needs to be maximised. Often it is practical to consider the log-likelihood function:

$$l(\theta) = \log L(\theta) = \sum_{i=1}^{n} \log P(X = x_i \mid \theta) \quad \text{for a discrete random variable, } X$$

or:

$$l(\theta) = \log L(\theta) = \sum_{i=1}^{n} \log f(x_i \mid \theta) \quad \text{for a continuous random variable, } X$$

If $l(\theta)$ can be differentiated with respect to θ, the MLE, expressed as $\hat{\theta}$, satisfies the expression:

$$\frac{d}{d\theta} l(\hat{\theta}) = 0$$

Where there is more than one parameter, the MLEs for each parameter can be determined by taking partial derivatives of the log-likelihood function and setting each to zero.

The determination of MLEs when the data are incomplete is covered in Chapter 18.

We will now look at the distributions described earlier in this chapter and consider how the parameters can be estimated in each case.

The exponential distribution

It is possible to use the method of maximum likelihood (ML) or the method of moments to estimate the parameter of the exponential distribution.

For example, suppose that an insurance company uses an exponential distribution to model the cost of repairing insured vehicles that are involved in accidents, and the average cost of repairing a random sample of 1,000 vehicles is £2,200.

We can calculate the maximum likelihood estimate of the exponential parameter as follows.

Let $x_1, x_2, \ldots, x_{1,000}$ denote the individual repair costs.

The likelihood of obtaining these values for the costs, if they come from an exponential distribution with parameter λ, is:

$$L = \prod_{i=1}^{1,000} \lambda e^{-\lambda x_i} = \lambda^{1,000} e^{-\lambda \Sigma x_i} = \lambda^{1,000} e^{-1,000 \lambda \bar{x}}$$

(where $\bar{x} = \frac{1}{1,000} \sum_{i=1}^{1,000} x_i$ denotes the average claim amount).

We want to determine the value of λ that maximises the likelihood, or equivalently the value that maximises the log-likelihood:

$$\ln L = 1,000 \ln \lambda - 1,000 \lambda \bar{x}$$

Differentiating with respect to λ:

$$\frac{\partial}{\partial \lambda} \ln L = \frac{1,000}{\lambda} - 1,000 \bar{x}$$

This is equal to 0 when:

$$\lambda = \frac{1}{\bar{x}}$$

The second derivative is:

$$\frac{\partial^2}{\partial\lambda^2}\log L = -\frac{1{,}000}{\lambda^2}$$

Since the second derivative is negative when $\lambda = \frac{1}{\bar{x}}$, the stationary point is a maximum. So, $\hat{\lambda}$, the maximum likelihood estimate of λ is $\frac{1}{\bar{x}}$, or $\frac{1}{2{,}200}$.

Alternatively, we could argue that the likelihood function is continuous and is always positive (by necessity) and that $\lambda^n e^{-\lambda n\bar{x}} \to 0$ as $\lambda \to 0$ or $\lambda \to \infty$. So any stationary point that we find must be a maximum.

To obtain ML estimates in R, we could use the `fitdistr` in the MASS package as follows:

```
fitdistr(<data vector>,"exponential")
```

Or we could define the log-likelihood function and use the function `nlm` on the negative value of the log-likelihood function.

```
nlm(-<log likelihood function>, <vector of parameters>)
```

So to fit an exponential distribution to a vector `<data>` with initial estimate of $\lambda = 0.5$ we would use:

```
params <- 0.5
n <- length(<data>)
sx <- sum(<data>)
fMLE <- function(params) {n*log(params[1])-params[1]*sx}
nlm(-fMLE,params)
```

The gamma distribution

The moments have a simple form and so the method of moments is very easy to apply. The MLEs for the gamma distribution cannot be obtained in closed form (*ie* in terms of elementary functions) but the moment estimators can be used as initial estimators in the search for the MLEs.

It is more convenient to obtain MLEs for the gamma distribution using a different parameterisation. Set $\mu = \alpha / \lambda$ and estimate the parameters α and μ. Then recover the MLE of λ by setting $\hat{\lambda} = \hat{\alpha} / \hat{\mu}$. This uses the invariance property of maximum likelihood estimators.

The invariance property says that if $\hat{\theta}$ is the maximum likelihood estimator of θ and $f(\theta)$ is a function of θ, then $f(\hat{\theta})$ is the maximum likelihood estimator of $f(\theta)$.

For the gamma distribution, λ is a function of both α and μ.

To obtain ML estimates in R, we could use the `fitdistr` **in the MASS package as follows:**

```
fitdistr(<data vector>,"gamma")
```

However, it is better to include the initial estimates obtained from the method of moments (and put a lower limit of say, 0.001 > 0, to prevent invalid answers). For example:

```
fitdistr(<data vector>,dgamma, list(shape = <alpha>,
rate = <lambda>), lower = 0.001)
```

Alternatively, we could define the log-likelihood function and use the function `nlm` **on the negative value of the log-likelihood function as before.**

The normal distribution

The method of moments and maximum likelihood estimation are both straightforward to apply in this case. Both give the following estimates:

$$\hat{\mu} = \bar{x} \qquad \text{and} \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

The estimate for the population variance is $\dfrac{n-1}{n} \times$ the usual sample variance. Of course, provided the sample size is large, there will be little difference between estimates calculated using the two different sample variance formulae.

The lognormal distribution

Estimation for the lognormal distribution is straightforward since μ and σ^2 may be estimated using the log-transformed data. Let $x_1, x_2, ..., x_n$ be the observed values and let $y_i = \log x_i$. The MLEs of μ and σ^2 are $\bar{y}$ and s_y^2, where the subscript y signifies a sample variance (n-denominator) computed on the y values.

In other words, the maximum likelihood estimate of σ^2 is:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

Alternatively the method of moments can be used to estimate the parameters.

As an example, suppose that based on an analysis of past claims, an insurance company believes that individual claims in a particular category for the coming year will be lognormally distributed with a mean size of £5,000 and a standard deviation of £7,500. The company wants to estimate the proportion of claims that will exceed £25,000.

To do this, it needs to estimate the parameters, μ and σ^2, of the lognormal distribution. Equating the formulae for the mean and standard deviation of the lognormal distribution to the values given gives:

$$e^{\mu+\frac{1}{2}\sigma^2}=5,000 \quad \text{and} \quad e^{\mu+\frac{1}{2}\sigma^2}\sqrt{e^{\sigma^2}-1}=7,500$$

Dividing the second equation by the first gives:

$$\sqrt{e^{\sigma^2}-1}=\frac{7,500}{5,000}=1.5$$

$$\Rightarrow \quad \sigma^2=1.179$$

We can now solve for μ:

$$\mu=\log 5,000-½(1.179)=7.928$$

So the proportion of claims expected to exceed £25,000 is:

$$\begin{aligned}P(X>25,000)&=P(\ln X>\ln 25,000)\\&=P(N(7.928,1.179)>\ln 25,000)\\&=P\left(N(0,1)>\frac{\ln 25,000-7.928}{\sqrt{1.179}}\right)\\&=1-\Phi(2.025)=0.021\end{aligned}$$

ie 2.1% of claims are expected to exceed £25,000.

To obtain ML estimates in R, we could use the `fitdistr` in the MASS package as follows:

```
fitdistr(<data vector>,"log-normal")
```

Alternatively, we could define the log-likelihood function and use the function `nlm` on the negative value of the log-likelihood function as before.

The two-parameter Pareto distribution

The method of moments is very easy to apply in the case of the two-parameter Pareto distribution, but the estimators obtained in this way will tend to have rather large standard errors, mainly because S^2, the sample variance, has a very large variance. However, the method does provide initial estimates for more efficient methods of estimation that may not be so simple to apply, like maximum likelihood, where numerical methods may need to be used.

Question

Claims arising from a particular group of policies are believed to follow a Pareto distribution with parameters α and λ. A random sample of 20 claims gives values such that $\sum x = 1,508$ and $\sum x^2 = 257,212$. Estimate α and λ using the method of moments.

Solution

Suppose that X is the claim amount random variable. Then:

$$E(X) = \frac{\lambda}{\alpha - 1}$$

Rearranging the variance formula to find $E(X^2)$, we have:

$$E(X^2) = \text{var}(X) + [E(X)^2] = \frac{2\lambda^2}{(\alpha-1)(\alpha-2)}$$

So we set:

$$\frac{\lambda}{\alpha-1} = \frac{1,508}{20} = 75.4 \quad \text{and} \quad \frac{2\lambda^2}{(\alpha-1)(\alpha-2)} = \frac{257,212}{20} = 12,860.6$$

Squaring the first of these equations and substituting into the second, we see that:

$$\frac{2 \times 75.4^2(\alpha-1)}{\alpha-2} = 12,860.6$$

Solving this equation, we find that the method of moments estimates of α and λ are 9.630 and 650.7, respectively.

The three-parameter Pareto distribution

Things are not quite so easy for the three-parameter Pareto distribution.

As for estimation, the CDF does not exist in closed form, so the method of percentiles is not available.

The method of percentiles is described in Section 3.3.

ML can be used, but again suitable computer software is required; the method of moments can provide initial estimates for any iterative scheme.

We will need to define the log-likelihood function and use the function `nlm` on the negative value of the log-likelihood function as before.

The Weibull and Burr distributions

Neither the method of moments nor maximum likelihood is elementary to apply if both c and γ are unknown (although if a computer is available, as would be the case in practice, the equations are simple enough).

To obtain ML estimates in R, we could use the `fitdistr` in the MASS package as follows:

```
fitdistr(<data vector>,"weibull")
```

However, it is better to include the initial estimates obtained from the method of percentiles (and put a lower limit of say, 0.001 > 0, to prevent invalid answers). For example:

```
fitdistr(<data vector>,dweibull,
list(shape = <gamma>, scale = <c^(-1/gamma)>, lower = 0.001)
```

Alternatively, we could define the log-likelihood function and use the function `nlm` on the negative value of the log-likelihood function as before.

In the case where γ has the known value γ^*, maximum likelihood is easy enough.

To do this, we let $y_i = x_i^{\gamma}$. If the original distribution is Weibull, the y values now have an exponential distribution. If the original distribution is Burr, the y values now come from a Pareto distribution. This can be seen by comparing the CDFs.

In the case of the Weibull, the MLE of c can now be determined in the usual way. In the case of the Burr distribution, the estimates have to be calculated numerically (since the MLEs of the parameters of the Pareto distribution cannot be calculated algebraically).

3.3 The method of percentiles

The distribution function of the $W(c,\gamma)$ distribution is an elementary function, and a simple method of estimation of both c and γ is based on this. The method involves equating selected sample percentiles to the distribution function; for example, equate the sample quartiles, the 25th and 75th sample percentiles, to the population quartiles. This corresponds to the way in which sample moments are equated to population moments in the method of moments. This method will be referred to as the method of percentiles.

In the method of moments, the first two moments are used if there are two unknown parameters, and this seems intuitively reasonable (although the theoretical basis for this is not so clear). In a similar fashion, when using the method of percentiles, the median would be used if there were one parameter to estimate. With two parameters, the best procedure is less clear, but the lower and upper quartiles seem a sensible choice.

Example

Estimate c and γ in the Weibull distribution using the method of percentiles, where the first sample quartile is 401 and the third sample quartile is 2,836.75.

Solution

The two equations for c and γ are:

$$F(401) = 1 - \exp(-c \times 401^{\gamma}) = 0.25$$

$$F(2{,}836.75) = 1 - \exp(-c \times 2{,}836.75^{\gamma}) = 0.75$$

which can be rewritten as:

$$-c \times 401^{\gamma} = \ln 0.75$$

and: $$-c \times 2{,}836.75^{\gamma} = \ln 0.25$$

Dividing, it is found that $\tilde{\gamma}$ = 0.8038, and hence $\tilde{c}$ = 0.002326, where ~ denotes the percentile estimate. Note that $\tilde{\gamma}$ is less than 1, indicating a fatter tail than the exponential distribution gives.

We can apply the method of percentiles to any distribution for which it is possible to calculate a closed form for the cumulative distribution function, although the resulting algebra can be messy.

Question

Claims arising from a particular group of policies are believed to follow a Pareto distribution with parameters α and λ. A random sample of 20 claims has a lower quartile of 11 and an upper quartile of 85. Estimate the values of α and λ using the method of percentiles.

Solution

The cumulative distribution function of the Pareto distribution is $F(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}$.

Q_1, the lower quartile of the distribution, satisfies the equation:

$$F(Q_1) = 1 - \left(\frac{\lambda}{\lambda + Q_1}\right)^{\alpha} = 0.25$$

and Q_3, the upper quartile of the distribution, satisfies the equation:

$$F(Q_3) = 1 - \left(\frac{\lambda}{\lambda + Q_3}\right)^{\alpha} = 0.75$$

So:

$$Q_1 = \lambda\left[(3/4)^{-1/\alpha} - 1\right] \qquad \text{and} \qquad Q_3 = \lambda\left[(1/4)^{-1/\alpha} - 1\right]$$

The method of percentiles estimates $\tilde{\alpha}$ and $\tilde{\lambda}$ are obtained by setting $Q_1 = 11$ and $Q_3 = 85$:

$$11 = \tilde{\lambda}\left[(3/4)^{-1/\tilde{\alpha}} - 1\right]$$

$$85 = \tilde{\lambda}\left[(1/4)^{-1/\tilde{\alpha}} - 1\right]$$

We can eliminate $\tilde{\lambda}$ by dividing these equations. This gives:

$$\frac{11}{85} = \frac{(3/4)^{-1/\tilde{\alpha}} - 1}{(1/4)^{-1/\tilde{\alpha}} - 1}$$

We cannot solve this algebraically but it can easily be done on a computer, *eg* using the goalseek function in Excel. Doing this, we find that $\tilde{\alpha} = 1.284$ and hence $\tilde{\lambda} = 43.790$.

These estimates are very different from those obtained using the method of moments. (In an earlier question, we calculated the method of moments estimates of α and λ to be 9.630 and 650.7, respectively.)

The method of percentiles is very unreliable for estimating the parameters of a Pareto distribution unless we use extremely large samples. In this particular case, the method of percentiles is unlikely to give us reasonable estimates unless we use samples of, say, 1,000 or more.

We now turn to the Burr distribution.

Since the CDF exists in closed form, it may be possible to fit the Burr distribution to data by using the method of percentiles; ML will certainly require the use of computer software that allows non-linear optimisation.

We will need to define the log-likelihood function and use the function `nlm` **on the negative value of the log-likelihood function as before.**

4 Goodness-of-fit tests

As mentioned earlier, one way of testing whether a given loss distribution provides a good model for the observed claim amounts is to apply a chi-squared goodness-of-fit test.

Recall that the formula for the test statistic is $\sum \frac{(O-E)^2}{E}$, where:

- O is the observed number in a particular category
- E is the corresponding expected number predicted by the assumed probabilities
- the sum is over all possible categories.

A high value for the total indicates that the overall discrepancy is quite large and would lead us to reject the model.

As an example, suppose that an insurance company uses an exponential distribution to model the cost of repairing insured vehicles that are involved in accidents, and the average cost of repairing a random sample of 1,000 vehicles is £2,200. A breakdown of the repair costs revealed the following numbers in different bands:

Repair cost, £	Observed number
0 – 1,000	200
1,000 – 2,000	300
2,000 – 3,000	250
3,000 – 4,000	150
4,000 – 5,000	100
5,000+	0

We can use this information to test whether the exponential distribution provides a good model for the individual repair costs.

Here we are testing:

H_0: repair costs are exponentially distributed

against:

H_1: repair costs are not exponentially distributed

In order to apply the chi-squared test, we need to calculate the expected number of repair costs in each interval based on the assumption that the null hypothesis is true.

Using the maximum likelihood estimate of the value of λ (*ie* $1/2,200$), the probability that an individual repair cost will fall in the interval £2,000 - £3,000 is:

$$\int_{2,000}^{3,000} \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_{2,000}^{3,000} = e^{-2,000\lambda} - e^{-3,000\lambda} = 0.1472$$

and the expected number for this band is: $1,000 \times 0.1472 = 147.2$

The expected numbers for all the intervals can be calculated in a similar way, giving the following results:

365.3, 231.8, 147.2, 93.4, 59.3, 103.0

The value of the test statistic is:

$$\sum \frac{(O-E)^2}{E} = \frac{(200-365.3)^2}{365.3} + \frac{(300-231.8)^2}{231.8} + \cdots + \frac{(0-103.0)^2}{103.0} = 331.89$$

We have 6 intervals, but we have equated the totals and estimated one parameter. So there are $6-1-1=4$ degrees of freedom.

The observed value of the chi-squared statistic far exceeds 14.86, the upper 99.95% point of the chi-squared distribution with 4 degrees of freedom (given on p169 of the *Tables*). So we can reject H_0 with almost total confidence and conclude that the repair costs do not conform to an exponential distribution.

In fact we need only work out the value of the first term in the chi-squared statistic to see that we will reject the null hypothesis.

This conclusion is supported by the observation that, if the values did come from an exponential distribution, we would expect the numbers in each band to decline steadily. However, we recorded 100 fewer values in the first band than in the second.

Chapter 15 Summary

Loss distributions

Individual claim amounts can be modelled using a loss distribution. Loss distributions are often positively skewed and long-tailed.

The (cumulative) distribution function of X is denoted by $F_X(x)$. It is defined by the equation: $F_X(x) = P(X \le x)$.

The (probability) density function of X is denoted by $f_X(x)$. It is defined by the equation: $f_X(x) = F'_X(x)$, wherever this derivative exists.

Distributions such as the exponential, normal, lognormal, gamma, Pareto, Burr and Weibull distributions are commonly used to model individual claim amounts.

Once the form of the loss distribution has been decided upon, the values of the parameters must be estimated. This might be done using the method of maximum likelihood, the method of moments, or the method of percentiles. Goodness of fit can then be checked using a chi-squared test.

Method of moments

The method of moments involves equating population and sample moments to solve for the unknown parameter values. If there is one parameter to estimate, we equate the population mean with the sample mean. If there are two parameters to estimate, we could equate the first two non-central population moments with the equivalent non-central sample moments. Equivalently, we could equate the first two central population moments with the equivalent central sample moments, noting that (for equivalence) we would need to use the n-denominator sample variance.

Method of maximum likelihood

The steps involved in finding a maximum likelihood estimate (MLE) are as follows:

- write down the likelihood function L – this the probability/PDF of obtaining the values we have observed
- take logs and simplify the resulting expression
- differentiate the log-likelihood with respect to each parameter to be estimated – this will involve partial differentiation if there is more than one parameter to be estimated
- set the derivatives equal to 0 and solve the equations simultaneously
- check that the resulting values are maxima.

Method of percentiles

The method of percentiles involves equating population and sample percentiles to solve for the unknown parameter values. If there is just one parameter to estimate, we equate the population median with the sample median. If there are two parameters to estimate, we equate the population lower and upper quartiles with the sample lower and upper quartiles.

Testing goodness of fit

We can test whether a given loss distribution provides a good model for the observed claim amounts by applying a chi-squared goodness-of-fit test.

The formula for the test statistic is $\sum \frac{(O-E)^2}{E}$, where:

- O is the observed number in a particular category
- E is the corresponding expected number predicted by the assumed probabilities
- the sum is over all possible categories.

Under the null hypothesis (that the model is correct), the test statistic has a chi-squared distribution.

Chapter 15 Practice Questions

15.1 Losses arising from a portfolio follow a Pareto distribution with parameters $\alpha = 3$ and $\lambda = 2{,}000$.

Calculate the probability that a randomly chosen loss amount exceeds the mean loss amount.

15.2 Suppose that X has a Weibull distribution with parameters c and γ.

(i) Using the formula for $E(X^r)$ given in the *Tables,* write down an expression for $\text{var}(X)$.

(ii) Show that, when $\gamma = 1$, this reduces to the formula for the variance of an exponential random variable.

15.3 Show that if $\gamma = ½$, the standard deviation of the Weibull distribution is greater than the mean, whereas if $\gamma = 2$ the opposite is true.

15.4 The random variable X follows a gamma distribution with parameters $\alpha = 20$ and $\lambda = 0.1$. Determine the value of a such that:

$$P(X > a) = 0.05$$

15.5 The random variable X has a Burr distribution with parameters $\gamma = 2$ and $\lambda = 500$.

(i) Show that the maximum likelihood estimate of the parameter α, based on a random sample $x_1, x_2, \ldots, x_n$ is:

$$\hat{\alpha} = \frac{n}{\Sigma \log(500 + x_i^2) - n \log 500}$$

You may assume that this is a maximum.

(ii) Evaluate this based on a sample consisting of the five values 52, 109, 114, 163 and 181.

15.6 Claim amounts from a particular group of policies have the following distribution:

Amount	£200	£500
Probability	p	$1-p$

In a random sample of 40 claims, 25 were for £200 and the other 15 were for £500.

Calculate the maximum likelihood estimate of p.

15.7 A loss amount random variable has MGF:

$$M(t) = 0.4(1-20t)^{-2} + 0.6(1-30t)^{-3}$$

Calculate the expected loss amount.

15.8 Exam style

Individual claim amounts on a portfolio of motor insurance policies follow a gamma distribution with parameters α and λ. It is known that $\lambda = 0.8$ for all drivers, but the value of α varies across the population.

Given that $\alpha \sim Gamma(200, 0.5)$, calculate the mean and variance of a randomly chosen claim amount. [5]

15.9 Exam style

(i) The distribution of claims on a portfolio of general insurance policies is a Weibull distribution, with density function $f_1(x)$ where:

$$f_1(x) = 2cxe^{-cx^2} \quad (x > 0)$$

It is expected that one claim out of every 100 will exceed £1,000. Use this information to estimate c. [2]

(ii) An alternative suggestion is that the density function is $f_2(x)$, where:

$$f_2(x) = \lambda e^{-\lambda x} \qquad (x > 0)$$

Use the same information as in part (i) to estimate λ. [2]

(iii) (a) For each of $f_1(x)$ and $f_2(x)$ calculate the value of M such that:

$$P(X > M) = 0.001$$

(b) Comment on these results. [3]

[Total 7]

15.10 Exam style

A random sample of 100 claim amounts $x_1, x_2, \ldots, x_{100}$ is observed from a Weibull distribution with parameter $\gamma = 2$, where c is unknown. For these data:

$$\sum x_i = 487{,}926 \qquad \sum x_i^2 = 976{,}444{,}000 \qquad \text{sample median} = 4{,}500$$

(i) Show that the maximum likelihood estimate for c based on a sample of size n is given by:

$$\hat{c} = n \Big/ \sum x_i^2$$

and hence estimate the value of c. [4]

(ii) Estimate the value of c using the method of moments. [2]

(iii) Calculate the method of percentiles estimate of c. [2]

[Total 8]

15.11 Exam style

Claims arising from a certain type of insurance policy are believed to follow an exponential distribution. The lower quartile claim is 200.

Calculate the mean claim size. [3]

Chapter 15 Solutions

15.1 Let X denote the loss amount random variable. Then $X \sim Pa(3,2000)$ and $E(X) = \frac{2,000}{3-1} = 1,000$.

So the required probability is:

$$P(X > 1,000) = 1 - F_X(1,000) = 1 - \left[1 - \left(\frac{2,000}{2,000+1,000}\right)^3\right] = \left(\frac{2,000}{3,000}\right)^3 = 0.29630$$

15.2 (i) ***Variance***

We have:

$$E(X) = \Gamma\left(1+\frac{1}{\gamma}\right)\frac{1}{c^{1/\gamma}} \quad \text{and} \quad E(X^2) = \Gamma\left(1+\frac{2}{\gamma}\right)\frac{1}{c^{2/\gamma}}$$

So:

$$\text{var}(X) = E(X^2) - [E(X)]^2 = \Gamma\left(1+\frac{2}{\gamma}\right)\frac{1}{c^{2/\gamma}} - \left[\Gamma\left(1+\frac{1}{\gamma}\right)\frac{1}{c^{1/\gamma}}\right]^2$$

(ii) ***Simplification when $\gamma = 1$***

When $\gamma = 1$, this becomes:

$$\Gamma(3) \times \frac{1}{c^2} - \left[\Gamma(2) \times \frac{1}{c}\right]^2 = \frac{2}{c^2} - \left[\frac{1}{c}\right]^2 = \frac{1}{c^2}$$

which is the formula for the variance of an $Exp(c)$ random variable.

15.3 When $\gamma = ½$:

$$E(X) = \frac{\Gamma(1+2)}{c^2} = \frac{2!}{c^2} = \frac{2}{c^2}$$

$$E(X^2) = \frac{\Gamma(1+4)}{c^4} = \frac{4!}{c^4} = \frac{24}{c^4}$$

and:

$$\text{var}(X) = E(X^2) - [E(X)]^2 = \frac{24}{c^4} - \left(\frac{2}{c^2}\right)^2 = \frac{20}{c^4}$$

So the standard deviation is $\frac{\sqrt{20}}{c^2}$, which is greater than $E(X)$.

When $\gamma = 2$:

$$E(X) = \frac{\Gamma(1+½)}{c^{½}} = \frac{\Gamma(1.5)}{c^{½}}$$

Using the properties of the gamma function given on page 5 of the *Tables*:

$$\Gamma(1.5) = 0.5\Gamma(0.5) = 0.5\sqrt{\pi}$$

So:

$$E(X) = \frac{0.5\sqrt{\pi}}{c^{½}} = \frac{0.886227}{c^{½}}$$

Also:

$$E(X^2) = \frac{\Gamma(1+2\times½)}{c^{2\times½}} = \frac{\Gamma(2)}{c} = \frac{1}{c}$$

and:

$$\text{var}(X) = E(X^2) - [E(X)]^2 = \frac{1}{c} - \left(\frac{0.5\sqrt{\pi}}{c^{½}}\right)^2 = \frac{1-0.25\pi}{c}$$

So the standard deviation is $\sqrt{\dfrac{1-0.25\pi}{c}} = \dfrac{0.463251}{c^{½}}$, which is less than $E(X)$.

In fact the mean and standard deviation are equal when $\gamma = 1$.

15.4 We can calculate the value of a using the relationship between the gamma distribution and the chi-squared distribution:

$$X \sim Gamma(20, 0.1) \Leftrightarrow 2\times 0.1X \sim \chi^2_{2\times 20}$$

So:

$$P(X > a) = P(0.2X > 0.2a) = P(\chi^2_{40} > 0.2a) = 0.05$$

ie $0.2a$ is the upper 5% point of χ^2_{40}. From page 169 of the *Tables*, we see that the upper 5% point of this chi-squared distribution is 55.76. So:

$$a = \frac{55.76}{0.2} = 278.8$$

The value of a *can also be determined in R using the command* $qgamma(0.95, 20, 0.1)$. *The R command* q *gives us the percentiles of a distribution. We follow the letter* q *with the name of the distribution. Here we want the upper 5% point, ie the 95th percentile, so the first argument is 0.95. The second and third arguments are the parameters of the gamma distribution.*

15.5 (i) ***Maximum likelihood estimate***

The likelihood function is:

$$L(\alpha,\lambda,\gamma)=\prod_{i=1}^{n} f(x_i)$$

$$=\prod_{i=1}^{n}\frac{\alpha\gamma\lambda^{\alpha}x_i^{\gamma-1}}{(\lambda+x_i^{\gamma})^{\alpha+1}}$$

$$=\alpha^n\gamma^n\lambda^{n\alpha}\prod_{i=1}^{n}\frac{x_i^{\gamma-1}}{(\lambda+x_i^{\gamma})^{\alpha+1}}$$

Taking logs:

$$\ln L=n\ln\alpha+n\ln\gamma+n\alpha\ln\lambda+(\gamma-1)\sum_{i=1}^{n}\ln x_i-(\alpha+1)\sum_{i=1}^{n}\ln(\lambda+x_i^{\gamma})$$

Differentiating with respect to α:

$$\frac{\partial}{\partial\alpha}\ln L=\frac{n}{\alpha}+n\ln\lambda-\sum_{i=1}^{n}\ln(\lambda+x_i^{\gamma})$$

This is equal to 0 when:

$$\alpha=\frac{n}{\sum_{i=1}^{n}\ln(\lambda+x_i^{\gamma})-n\ln\lambda}$$

Since we can assume that this is a maximum, we can say that the maximum likelihood estimate of α is:

$$\hat{\alpha}=\frac{n}{\sum_{i=1}^{n}\ln(\lambda+x_i^{\gamma})-n\ln\lambda}$$

(ii) ***Numerical value***

For the sample given, we have $\sum_{i=1}^{5}\ln(500+x_i^2)=47.6245$. So:

$$\hat{\alpha}=\frac{5}{47.6245-5\ln 500}=0.3021$$

α is the easy parameter to estimate. λ and γ are much more difficult to estimate using MLE because of the form of the last term in the log-likelihood function.

15.6 Let X denote the claim amount random variable. Then the likelihood function is:

$$L = C\left[P(X=200)\right]^{25}\left[P(X=500)\right]^{15} = C\,p^{25}(1-p)^{15}$$

where C is a constant.

The log-likelihood function is:

$$\ln L = \ln C + 25\ln p + 15\ln(1-p)$$

Differentiating this with respect to p gives:

$$\frac{d\ln L}{dp} = \frac{25}{p} - \frac{15}{1-p}$$

Now:

$$\frac{d\ln L}{dp} = 0 \Leftrightarrow \frac{25}{p} = \frac{15}{1-p}$$

$$\Leftrightarrow 25 - 25p = 15p$$

$$\Leftrightarrow 25 = 40p$$

$$\Leftrightarrow p = \frac{25}{40} = 0.625$$

So we have a stationary point when $p = 0.625$. To determine the nature of the stationary point, we check the sign of the second derivative:

$$\frac{d^2\ln L}{dp^2} = -\frac{25}{p^2} - \frac{15}{(1-p)^2}$$

This is negative when $p = 0.625$. (In fact, this second derivative is always negative.) So the maximum likelihood estimate of p is 0.625.

15.7 Differentiating the MGF:

$$M'(t) = 0.4(-2)(-20)(1-20t)^{-3} + 0.6(-3)(-30)(1-30t)^{-4}$$

$$= 16(1-20t)^{-3} + 54(1-30t)^{-4}$$

The expected loss amount is:

$$M'(0) = 16 + 54 = 70$$

15.8 Let X denote the amount of a randomly chosen claim. We know that $X \mid \alpha \sim Gamma(\alpha, 0.8)$. So, using the conditional expectation formula:

$$E(X) = E(E(X \mid \alpha)) = E\left(\frac{\alpha}{0.8}\right) = \frac{1}{0.8}E(\alpha) \qquad [1]$$

Then using the fact that $\alpha \sim Gamma(200, 0.5)$:

$$E(X) = \frac{1}{0.8} \times \frac{200}{0.5} = 500 \qquad [1]$$

Using the conditional variance formula:

$$\text{var}(X) = E(\text{var}(X \mid \alpha)) + \text{var}(E(X \mid \alpha)) = E\left(\frac{\alpha}{0.8^2}\right) + \text{var}\left(\frac{\alpha}{0.8}\right) = \frac{1}{0.8^2}E(\alpha) + \frac{1}{0.8^2}\text{var}(\alpha) \qquad [2]$$

Then using the fact that $\alpha \sim Gamma(200, 0.5)$:

$$\text{var}(X) = \frac{1}{0.8^2} \times \frac{200}{0.5} + \frac{1}{0.8^2} \times \frac{200}{0.5^2} = 1{,}875 \qquad [1]$$

15.9 *This is Subject 106, September 2003, Question 5.*

(i) ***Estimate c***

The random variable X has a Weibull distribution. Comparing the given PDF with the Weibull PDF from page 15 of the *Tables*:

$$f_1(x) = 2cxe^{-cx^2} = c\gamma x^{\gamma-1}e^{-cx^{\gamma}} \quad \Rightarrow \quad \gamma = 2$$

We are told that $P(X > 1{,}000)$ is expected to be 0.01 and we know that:

$$P(X > 1{,}000) = 1 - F(1{,}000) = e^{-c \times 1{,}000^2} \qquad [1]$$

Setting this equal to 0.01 gives the estimated value of c to be:

$$\hat{c} = -\frac{\ln 0.01}{1{,}000^2} = 4.605 \times 10^{-6} \qquad [1]$$

(ii) ***Estimate λ***

Here, X has an exponential distribution and:

$$P(X > 1{,}000) = 1 - F(1{,}000) = e^{-1{,}000\lambda} \qquad [1]$$

Setting this equal to 0.01 gives the estimated value of λ to be:

$$\hat{\lambda} = -\frac{\ln 0.01}{1{,}000} = 4.605 \times 10^{-3} \qquad [1]$$

(iii)(a) ***Calculate M***

We require M such that:

$$P(X>M)=1-F(M)=0.001$$

For the Weibull distribution with $c=4.605\times 10^{-6}$:

$$e^{-4.605\times 10^{-6}\times M^2}=0.001$$

$$\Rightarrow M^2=-\frac{\ln 0.001}{4.605\times 10^{-6}}=1,500,000$$

$$\Rightarrow M=1,225$$ [1]

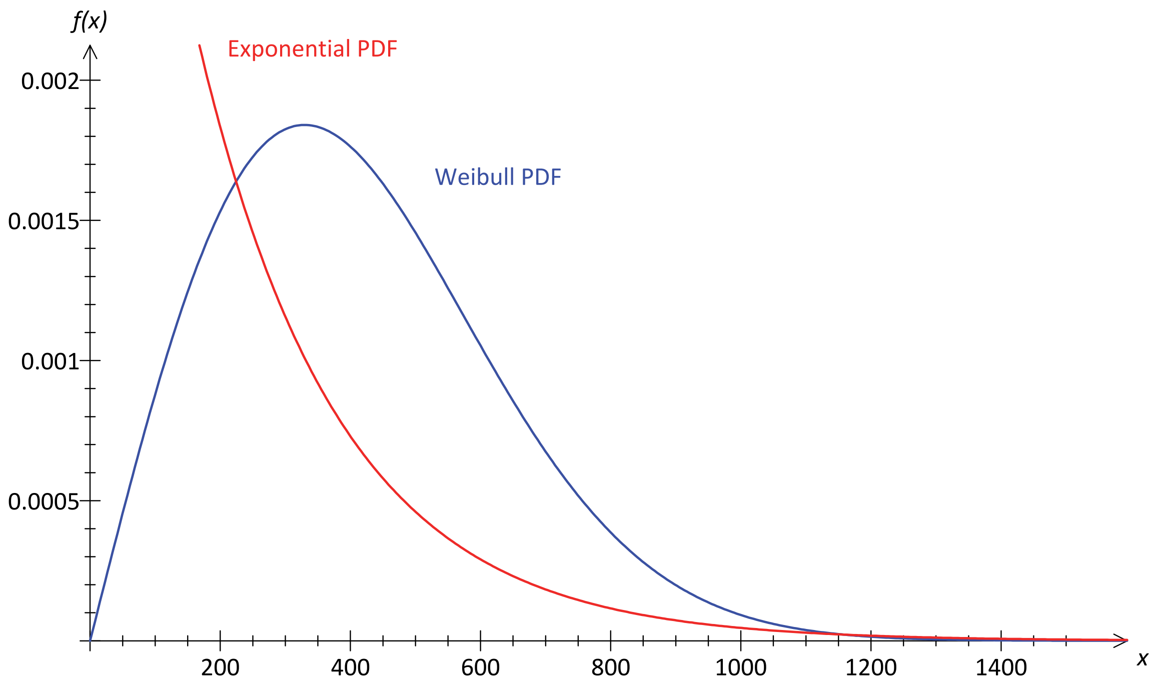
For the exponential distribution with $\lambda=4.605\times 10^{-3}$:

$$e^{-4.605\times 10^{-3}\times M}=0.001 \quad\Rightarrow\quad M=-\frac{\ln 0.001}{4.605\times 10^{-3}}=1,500$$ [1]

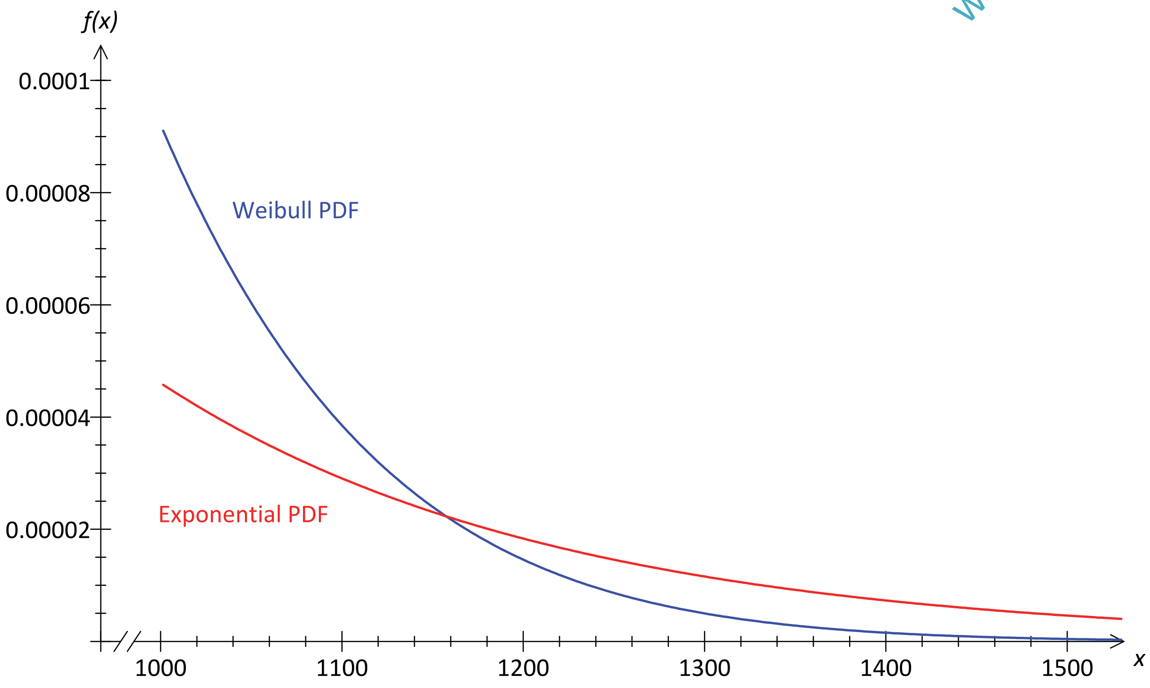
(iii)(b) ***Comment***

The probability that the Weibull random variable exceeds 1,225 is 0.001 but the probability that the exponential random variable exceeds 1,225 is more than 0.001. This is because the exponential distribution has a heavier tail than the Weibull. [1]

A graph of the distributions is shown below:



Looking more closely at the tails, it is clear that the exponential distribution has a heavier tail than the Weibull distribution:



15.10 (i)(a) ***Maximum likelihood estimate***

The PDF of the Weibull distribution is:

$$f(x) = c\gamma x^{\gamma-1}e^{-cx^{\gamma}} \ , \ x > 0$$

So the likelihood function in this case is:

$$L(c) = 2cx_1e^{-cx_1^2} \times \cdots \times 2cx_ne^{-cx_n^2} = \text{constant} \times c^n e^{-c\sum x_i^2}$$ [½]

Taking logs, we obtain:

$$\ln L = \text{constant} + n\ln c - c\sum_{i=1}^{n} x_i^2$$ [½]

Differentiating this with respect to c, we obtain:

$$\frac{d}{dc}\ln L = \frac{n}{c} - \sum x_i^2$$ [½]

This is equal to 0 when:

$$c = \frac{n}{\sum x_i^2}$$ [½]

We can check that this does give us a maximum by examining the second derivative of the log-likelihood:

$$\frac{d^2}{dc^2}\ln L = -\frac{n}{c^2}$$ [½]

This is negative when $c=\dfrac{n}{\sum x_i^2}$. So, $\hat{c}$, the maximum likelihood estimate of c is $\dfrac{n}{\sum x_i^2}$. [½]

Substituting the sample data results into the formula from part (i)(a) gives:

$$\hat{c}=\frac{100}{976{,}444{,}000}=1.0241\times 10^{-7}$$ [1]

(ii) ***Method of moments estimate***

We obtain the corresponding method of moments estimate for c by equating the sample and population means.

Using the formula for the mean of the Weibull distribution with $\gamma=2$ (given on page 15 of the *Tables*) and the properties of the gamma function (given on page 5 of the *Tables*), we have:

$$E(X)=\frac{\Gamma(1+½)}{c^{½}}=\frac{0.5\Gamma(0.5)}{c^{½}}=\frac{0.5\sqrt{\pi}}{c^{½}}$$ [1]

From the data we have:

$$\bar{x}=\frac{487{,}926}{100}=4{,}879.26$$ [½]

Equating $E(X)$ and $\bar{x}$ gives:

$$\hat{c}=\left(\frac{0.5\sqrt{\pi}}{\bar{x}}\right)^2=\left(\frac{0.5\sqrt{\pi}}{4{,}879.26}\right)^2=3.299\times 10^{-8}$$ [½]

(iii) ***Method of percentiles estimate***

The median of the distribution is the value of M such that $F(M)=½$.

Equating this to the sample median of 4,500 gives the method of percentiles estimate, $\tilde{c}$:

$$F(4{,}500)=1-e^{-\tilde{c}\times 4{,}500^2}=½ \quad\Rightarrow\quad \tilde{c}=-\frac{\ln ½}{4{,}500^2}=3.423\times 10^{-8}$$ [2]

15.11 Since the lower quartile is 200, we have:

$$F(200) = 0.25$$

Also, using the fact that claim amounts follow an exponential distribution:

$$F(200) = 1 - e^{-200\lambda}$$ [½]

So:

$$e^{-200\lambda} = 1 - 0.25 = 0.75$$ [½]

Taking logs:

$$\lambda = -\frac{1}{200}\ln 0.75 = 0.0014384$$ [1]

So the mean claim amount is:

$$\frac{1}{\lambda} = 695.21$$ [1]

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Reinsurance

Syllabus objectives

1.1 Loss distributions, with and without risk sharing

- 1.1.2 Explain the concepts of excesses (deductibles), and retention limits.
- 1.1.3 Describe the operation of simple forms of proportional and excess of loss reinsurance.
- 1.1.4 Derive the distribution and corresponding moments of the claim amounts paid by the insurer and the reinsurer in the presence of excesses (deductibles) and reinsurance.
- 1.1.5 Estimate the parameters of a failure time or loss distribution when the data is complete, or when it is incomplete, using maximum likelihood and the method of moments.

0 Introduction

The claims on an insurance company must be met in full, but, to protect itself from large claims, the company itself may take out an insurance policy; such a policy is called a reinsurance policy. For the purposes of this chapter, it will be assumed that the reinsurance contract is one of two very simple types: individual excess of loss reinsurance or proportional reinsurance.

0.1 Proportional reinsurance

Under a proportional reinsurance arrangement, the *direct writer* (*ie* the original insurance company) and the reinsurer share the cost of all claims for each risk. For example, for a particular building insured against fire, the direct writer might retain 75% of the premium and will be liable to pay 75% of all claims, large or small. The direct writer must pay a premium to effect this reinsurance. The direct writer is sometimes referred to as the *direct insurer* or even just the *insurer*.

Proportional reinsurance operates in two forms:

1. With *quota share* reinsurance, the proportions are the same for all risks.

2. With *surplus* reinsurance, the proportions can vary from one risk to the next.

In this course we will focus on quota share reinsurance.

0.2 Non-proportional reinsurance

Under a non-proportional reinsurance arrangement, the direct writer pays a fixed premium to the reinsurer. The reinsurer will only be required to make payments where part of the claim amount falls in a particular reinsurance *layer* (*eg* between £1m and £5m). The layer will be defined by a lower limit, the *retention limit* (*eg* £1m), and an upper limit (*eg* £5m or infinity if the cover is *unlimited*). Usually, most claims are paid in full by the direct writer.

We will mention two forms of non-proportional reinsurance here:

1. With *individual excess of loss* (XOL) reinsurance, the reinsurer will be required to make a payment when the claim amount for any individual claim exceeds a specified *excess point* or *retention*. For example, the reinsurer might agree to pay the excess when any claim from a motor policy exceeds £50,000, but with an upper limit of £2 million.

2. With *stop loss* reinsurance, the reinsurer will be required to make payments if the total claim amount for a specified group of policies exceeds a specified amount (which may be expressed as a percentage of the gross premium). We will look at this in Chapter 20.

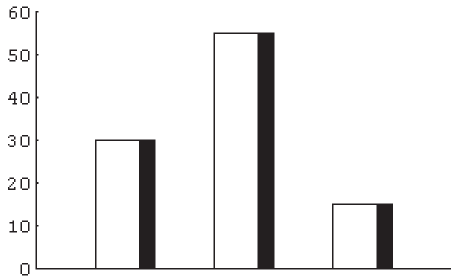
The diagram below shows how much the direct writer and the reinsurer would pay when there are claims for £30,000, £55,000 and £15,000:

(a) under a 25% quota share arrangement, and

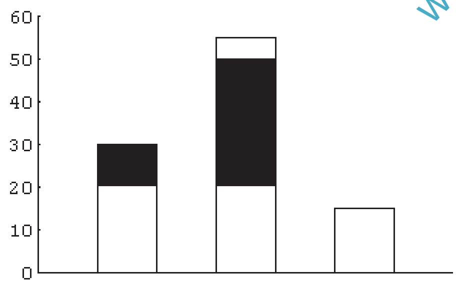
(b) under an individual XOL arrangement with a reinsurance layer of £30,000 in excess of £20,000.

The parts of each claim paid by the reinsurer are shown in black.

25% Quota Share



XOL (£30k in excess of £20k)



1 Reinsurance arrangements

The actual amount that the direct insurer ends up paying after allowing for payments under the reinsurance arrangements is called the *net claim amount*. The actual premium that the direct writer gets to keep after making any payments for reinsurance is the *insurer's net premium income*. The original amounts without adjustment for reinsurance are referred to as the *gross claim amount* and the *insurer's gross premium income*.

In this chapter we will use the following notation.

Notation

X is the gross claim amount random variable

Y is the net claim amount, *ie* the amount of the claim paid by the insurer in respect of a single claim (after receiving the reinsurance recovery)

Z is the amount paid by the reinsurer in respect of a single claim.

For a given reinsurance arrangement, we can express the random variables Y and Z in terms of X.

For example, suppose that a reinsurer has agreed to make the following payments in respect of individual claims incurred by a direct insurer:

- nothing, if the claim is less than £5,000
- the full amount reduced by £5,000, if the claim is between £5,000 and £10,000
- half the full amount, if the claim is between £10,000 and £20,000
- £10,000, if the claim exceeds £20,000.

Then:

$$Z = \begin{cases} 0 & \text{if } X \le £5{,}000 \\ X - 5{,}000 & \text{if } £5{,}000 < X \le £10{,}000 \\ X/2 & \text{if } £10{,}000 < X \le £20{,}000 \\ 10{,}000 & \text{if } X > £20{,}000 \end{cases}$$

and:

$$Y = \begin{cases} X & \text{if } X \le £5{,}000 \\ 5{,}000 & \text{if } £5{,}000 < X \le £10{,}000 \\ X/2 & \text{if } £10{,}000 < X \le £20{,}000 \\ X - 10{,}000 & \text{if } X > £20{,}000 \end{cases}$$

Note that $Y + Z = X$.

We are now in a position to consider the statistical calculations relating to reinsurance arrangements.

1.1 Excess of loss reinsurance

In excess of loss reinsurance, the insurer will pay any claim in full up to an amount M, the retention level; any amount above M will be borne by the reinsurer.

The excess of loss reinsurance arrangement can be written in the following way: if the claim is for amount X, then the insurer will pay Y where:

$$Y = X \quad \text{if } X \le M$$

$$Y = M \quad \text{if } X > M$$

The reinsurer pays the amount $Z = X - Y$.

Question

Write down an expression for Y if only a layer between M and $2M$ is reinsured.

Solution

$$Y = \begin{cases} X & \text{if} \quad X \le M \\ M & \text{if} \quad M < X \le 2M \\ X - M & \text{if} \quad X > 2M \end{cases}$$

The insurer's liability is affected in two obvious ways by reinsurance:

(i) the mean amount paid is reduced;

(ii) the variance of the amount paid is reduced.

Both these conclusions are simple consequences of the fact that excess of loss reinsurance puts an upper limit on large claims.

The mean amounts paid by the insurer and the reinsurer under excess of loss reinsurance can now be obtained. Observe that the mean amount paid by the insurer without reinsurance is:

$$E(X) = \int_0^\infty x\, f(x)\, dx \tag{18.1}$$

where $f(x)$ is the PDF of the claim amount X. With a retention level of M the mean amount paid by the insurer becomes:

$$E(Y) = \int_0^M xf(x)dx + M\,P(X > M) \tag{18.2}$$

This is because:

$$E(Y)=\int_0^M x\,f(x)dx+\int_M^\infty M\,f(x)dx=\int_0^M x\,f(x)dx+M\int_M^\infty f(x)dx$$

and:

$$\int_M^\infty f(x)dx=P(X>M)$$

We can calculate $E(Y^2)$ in a similar way:

$$E(Y^2)=\int_0^M x^2\,f(x)dx+\int_M^\infty M^2\,f(x)dx=\int_0^M x^2\,f(x)dx+M^2\,P\left(X>M\right)$$

Then $\text{var}(Y)=E(Y^2)-\left[E(Y)\right]^2$.

More generally, the moment generating function of Y, the amount paid by the insurer, is:

$$\boldsymbol{M_Y(t)=E(e^{tY})=\int_0^M e^{tx}f(x)\,dx+e^{tM}\;P(X>M)}$$

Here we are using the formula for the expected value of a function of a continuous random variable:

$$E(h(X))=\int_x h(x)\,f(x)\,dx$$

with:

$$h(X)=\begin{cases}e^{tX} & \text{if } X\le M\\ e^{tM} & \text{if } X>M\end{cases}$$

Question

Suppose that claim amounts are uniformly distributed over the interval $(0,500)$. The insurer effects individual excess of loss reinsurance with a retention limit of 375.

Calculate the expected amounts paid by the insurer and the reinsurer in respect of a single claim.

Solution

Since $X\sim U(0,500)$, the expected gross claim amount is:

$$E(X)=\frac{500}{2}=250$$

The expected amount paid by the insurer is:

$$E(Y) = \int_0^{375} x\, f(x)\, dx + 375 P(X > 375)$$

$$= \int_0^{375} \frac{x}{500}\, dx + 375\left[1 - F(375)\right]$$

$$= \left[\frac{x^2}{1,000}\right]_0^{375} + 375\left[1 - \frac{375}{500}\right]$$

$$= 140.625 + 93.75$$

$$= 234.375$$

Also, since $Y + Z = X$:

$$E(Z) = E(X) - E(Y) = 250 - 234.375 = 15.625$$

Under excess of loss reinsurance, the reinsurer will pay Z where:

$$Z = \begin{cases} 0 & \text{if } X \le M \\ X - M & \text{if } X > M \end{cases}$$

The mean amount paid by the reinsurer is:

$$\boldsymbol{E(Z) = \int_M^{\infty} (x - M) f(x)\, dx} \qquad \textbf{(18.3)}$$

Similarly, we can calculate $E(Z^2)$ using:

$$E(Z^2) = \int_M^{\infty} (x - M)^2 f(x)\, dx$$

Then $\text{var}(Z) = E(Z^2) - \left[E(Z)\right]^2$.

More generally, the moment generating function of Z is:

$$M_Z(t) = E(e^{tZ})$$

$$= \int_0^M e^{t0} f(x)\, dx + \int_M^{\infty} e^{t(x-M)} f(x)\, dx$$

$$= \int_0^M f(x)\, dx + \int_M^{\infty} e^{t(x-M)} f(x)\, dx$$

$$= P(X \le M) + \int_M^{\infty} e^{t(x-M)} f(x)\, dx$$

We can use R to simulate gross claim amounts and hence the amounts paid by the insurer and reinsurer for any given retention limit.

Suppose claims (in £'s) have an exponential distribution with parameter $\lambda = 0.0005$. The R code for simulating 10,000 claims, *x*, is given by:

```
x <- rexp(10000,rate=0.0005)
```

We can then obtain the claims paid by the insurer, *y*, and the reinsurer, *z* with retention *M* using:

```
y <- pmin(x,M)
z <- pmax(0,x-M)
```

We can then obtain their means and variances using the R functions `mean` **and** `var`**.**

We can use these vectors to estimate probabilities. For example, to estimate the probability that the insurer pays less than £1,000 we would use:

```
length(y[y<1000])/length(y)
```

Similarly we could estimate the claim size for a given percentile. For example, to estimate the claim size corresponding to the 90th percentile of the insurer's claims we would use:

```
quantile(y,0.9)
```

1.2 The reinsurer's conditional claims distribution

Now consider reinsurance (once again) **from the point of view of the reinsurer. The reinsurer may have a record only of claims that are greater than M. If a claim is for less than M the reinsurer may not even know a claim has occurred. The reinsurer thus has the problem of estimating the underlying claims distribution when only those claims greater than M are observed. The statistical terminology is to say that the reinsurer observes claims from a truncated distribution.**

In this case the values observed by the reinsurer relate to a conditional distribution, since the numbers are conditional on the original claim amount exceeding the retention limit.

Let W be the random variable with this truncated distribution. Then:

$$W = X - M \mid X > M$$

This can also be expressed as follows:

$$W = Z \mid Z > 0$$

Suppose that the underlying claim amounts have PDF $f(x)$ and CDF $F(x)$. Suppose that the reinsurer is only informed of claims greater than the retention *M* and has a record of $w = x - M$. What is the PDF $g(w)$ of the amount, w, paid by the reinsurer?

The argument goes as follows:

$$P(W < w) = P(X < w + M \mid X > M)$$

$$= \frac{P(X < w + M \text{ and } X > M)}{P(X > M)}$$

$$= \frac{P(M < X < w + M)}{P(X > M)}$$

$$= \int_M^{w+M} \frac{f(x)}{1 - F(M)}\, dx$$

$$= \frac{F(w + M) - F(M)}{1 - F(M)}$$

This derivation also uses the result:

$$P(a < X < b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

Differentiating with respect to w, the PDF of the reinsurer's claims is:

$$g(w) = \frac{f(w + M)}{1 - F(M)}, \quad w > 0 \qquad \textbf{(18.4)}$$

This is just the original PDF evaluated at the gross amount $w + M$, divided by the probability that the claim exceeds M.

The PDF of W may be denoted by $f_W(w)$ rather than $g(w)$. With this notation, the result can be stated as follows:

PDF of the reinsurer's conditional claim amount random variable

If $W = X - M \mid X > M$, then:

$$f_W(w) = \frac{f_X(w + M)}{1 - F_X(M)} = \frac{f_X(w + M)}{P(X > M)}$$

Question

Using the notation above, determine the distribution of W if:

(a) $X \sim Exp(\lambda)$

(b) $X \sim Pa(\alpha, \lambda)$

Solution

(a) ***Exponential***

If $X \sim Exp(\lambda)$, then $f_X(x) = \lambda e^{-\lambda x}$ and $F_X(x) = 1 - e^{-\lambda x}$. So:

$$f_W(w) = \frac{f_X(w+M)}{1-F_X(M)} = \frac{\lambda e^{-\lambda(w+M)}}{e^{-\lambda M}} = \lambda e^{-\lambda w}, \quad w > 0$$

This is the PDF of $Exp(\lambda)$. So $W \sim Exp(\lambda)$, the same as the original claims distribution. This illustrates the memoryless property of the exponential distribution.

(b) ***Pareto***

If $X \sim Pa(\alpha, \lambda)$, then $f_X(x) = \dfrac{\alpha \lambda^\alpha}{(\lambda+x)^{\alpha+1}}$ and $F_X(x) = 1 - \left(\dfrac{\lambda}{\lambda+x}\right)^\alpha$. So:

$$f_W(w) = \frac{\alpha\lambda^\alpha \big/ (\lambda+w+M)^{\alpha+1}}{\lambda^\alpha \big/ (\lambda+M)^\alpha} = \frac{\alpha(\lambda+M)^\alpha}{(\lambda+M+w)^{\alpha+1}}, \quad w > 0$$

This is the PDF of $Pa(\alpha, \lambda+M)$. So $W \sim Pa(\alpha, \lambda+M)$.

We can now calculate the expected value of W. Using the PDF of $W = X - M \mid X > M$, we have:

$$E(W) = \int_0^\infty w\, f_W(w)\, dw = \frac{\int_0^\infty w\, f_X(w+M)\, dw}{1-F_X(M)} = \frac{\int_M^\infty (x-M)\, f_X(x)\, dx}{1-F_X(M)} = \frac{E(Z)}{P(X>M)} = \frac{E(Z)}{P(Z>0)}$$

So the reinsurer's expected claim payment on a claim in which it is involved is just the reinsurer's expected claim payment (on all claims), $E(Z)$, divided by the probability that the claim involves the reinsurer.

If *z* is the vector of the reinsurer's claims:

```
z <- pmax(0,x-M)
```

Then we can obtain the truncated distribution, *w*, using:

```
w <- z[z>0]
```

We can then calculate moments, probabilities and quantiles as before.

1.3 Proportional reinsurance

In proportional reinsurance the insurer pays a fixed proportion of the claim, whatever the size of the claim. Using the same notation as above, the proportional reinsurance arrangement can be written as follows: if the claim is for an amount X then the company will pay Y where:

$$\boldsymbol{Y = \alpha X \qquad 0 < \alpha < 1}$$

The parameter α is known as the retained proportion or retention level; note that the term retention level is used in both excess of loss and proportional reinsurance though it means different things.

Since $Y + Z = X$, we must have $Z = (1-\alpha)X$. The mean and variance of Y and Z are calculated as follows:

$$E(Y) = \alpha E(X) \qquad E(Z) = (1-\alpha)E(X)$$

$$\text{var}(Y) = \alpha^2 \text{var}(X) \qquad \text{var}(Z) = (1-\alpha)^2 \text{var}(X)$$

Question

Claims from a particular portfolio have a generalised Pareto distribution with parameters $\alpha = 6$, $\lambda = 200$ and $k = 4$. A proportional reinsurance arrangement is in force with a retained proportion of 80%.

Calculate the mean and variance of the amount paid by the insurer and the reinsurer in respect of a single claim.

Solution

Using X to represent the individual claim amount random variable and the formulae for the mean and variance of a three-parameter Pareto random variable (from page 15 of the *Tables*), we have:

$$E(X) = \frac{k\lambda}{\alpha - 1} = \frac{4 \times 200}{6-1} = \frac{800}{5} = 160$$

and:

$$\text{var}(X) = \frac{k(k+\alpha-1)\lambda^2}{(\alpha-1)^2(\alpha-2)} = \frac{4(4+6-1)\times 200^2}{(6-1)^2(6-2)} = \frac{1,440,000}{100} = 14,400$$

The amount paid by the insurer is $Y = 0.8X$. So:

$$E(Y) = 0.8 \times 160 = 128$$

and:

$$\text{var}(Y) = 0.8^2 \times 14,400 = 9,216$$

The amount paid by the reinsurer is $Z = 0.2X$. So:

$$E(Z) = 0.2 \times 160 = 32$$

and:

$$\text{var}(Z) = 0.2^2 \times 14{,}400 = 576$$

As the amount paid by the insurer on a claim X is $Y = \alpha X$ and the amount paid by the reinsurer is $Z = (1-\alpha)X$, the distribution of both of these amounts can be found by a simple change of variable.

Question

Claims from a particular portfolio have an exponential distribution with mean 1,000. The insurer takes out proportional reinsurance with a retained proportion of 0.9.

Determine the distribution of the insurer's net claim amount random variable.

Solution

We know that X is exponential with mean 1,000, so the exponential parameter is $\dfrac{1}{1{,}000}$.

From page 11 of the *Tables*, the MGF of X is:

$$M_X(t) = (1 - 1{,}000t)^{-1}, \quad t < \frac{1}{1{,}000}$$

Since $Y = 0.9X$, the MGF of Y is:

$$M_Y(t) = E(e^{tY}) = E(e^{0.9tX}) = M_X(0.9t) = (1 - 1{,}000 \times 0.9t)^{-1} = (1 - 900t)^{-1}, \quad t < \frac{1}{900}$$

This is the MGF of the exponential distribution with mean 900. By the uniqueness property of MGFs, it follows that the distribution of the insurer's net claim amount random variable is exponential with mean 900.

The payments of the insurer, y, and the reinsurer, z, with retained proportion a would be:

```
y <- a*x
z <- (1-a)*x
```

We can then calculate moments, probabilities and quantiles as before.

2 Normal and lognormal distributions

There are useful integral formulae that simplify reinsurance calculations when working with normal and lognormal distributions.

2.1 Normal distribution

Truncated mean of the normal distribution

If $X \sim N(\mu, \sigma^2)$, then:

$$\int_L^U x\, f_X(x)\, dx = \mu\, [\Phi(U') - \Phi(L')] - \sigma\, [\phi(U') - \phi(L')]$$

where:

$$L' = \frac{L-\mu}{\sigma}$$

$$U' = \frac{U-\mu}{\sigma}$$

$\phi(z)$ is the PDF of the standard normal distribution

$\Phi(z)$ is the CDF of the standard normal distribution.

This result is given on page 18 of the *Tables*. It is proved as follows.

Using the formula for $f_X(x)$ and the substitution $z = \dfrac{x-\mu}{\sigma}$:

$$\begin{aligned}
\int_L^U x\, f_X(x)\, dx &= \int_L^U x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= \int_{L'}^{U'} (\mu + \sigma z) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \int_{L'}^{U'} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \sigma \int_{L'}^{U'} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz
\end{aligned}$$

Now, since $\dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ is the PDF of $N(0,1)$:

$$\int_{L'}^{U'} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = P\left(L' < N(0,1) < U'\right)$$

So:

$$\int_L^U x\, f_X(x)\, dx = \mu P\left(L' < N(0,1) < U'\right) + \sigma \int_{L'}^{U'} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

$$= \mu P\left(L' < N(0,1) < U'\right) + \sigma \left[-\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \right]_{L'}^{U'}$$

$$= \mu\left[\Phi(U') - \Phi(L')\right] - \sigma\left[\phi(U') - \phi(L')\right]$$

$$= \mu\left[\Phi\left(\frac{U-\mu}{\sigma}\right) - \Phi\left(\frac{L-\mu}{\sigma}\right)\right] - \sigma\left[\phi\left(\frac{U-\mu}{\sigma}\right) - \phi\left(\frac{L-\mu}{\sigma}\right)\right]$$

When $L = -\infty$ or $U = \infty$, these formulae can be simplified because:

$$\phi(-\infty) = \phi(\infty) = 0, \quad \Phi(-\infty) = 0, \quad \Phi(\infty) = 1$$

Question

Claims from a particular portfolio are normally distributed with mean 800 and standard deviation 100. An individual excess of loss arrangement with retention limit is 860 is in place.

Calculate the insurer's mean claim payment net of reinsurance.

Solution

The insurer's mean claim payment is:

$$E(Y) = \int_0^{860} x\, f_X(x)\, dx + 860\, P(X > 860)$$

where $X \sim N(800, 100^2)$.

Using the formula for the truncated mean of a normal random variable:

$$\int_0^{860} x\, f_X(x)\, dx = 800\left[\Phi\left(\frac{860-800}{100}\right) - \Phi\left(\frac{0-800}{100}\right)\right]$$

$$-100\left[\phi\left(\frac{860-800}{100}\right) - \phi\left(\frac{0-800}{100}\right)\right]$$

$$= 800\left[\Phi(0.6) - \Phi(-8)\right] - 100\left[\phi(0.6) - \phi(-8)\right]$$

From pages 160 and 161 of the *Tables*:

$$\Phi(0.6) = 0.72575$$

$$\Phi(-8) = 1 - \Phi(8) \approx 0$$

Also, using the formula for the PDF of the standard normal distribution from page 10 of the *Tables*:

$$\phi(0.6) = \frac{1}{\sqrt{2\pi}} e^{-½\times 0.6^2} = 0.33322$$

$$\phi(-8) = \frac{1}{\sqrt{2\pi}} e^{-½\times(-8)^2} \approx 0$$

So:

$$\int_0^{860} x\, f_X(x)\, dx \approx 800\left[0.72575 - 0\right] - 100\left[0.33322 - 0\right] = 547.278$$

The second term in the expression for $E(Y)$ is:

$$\begin{aligned} 860\, P(X > 860) &= 860\left[1 - \Phi\left(\frac{860-800}{100}\right)\right] \\ &= 860\left[1 - \Phi(0.6)\right] \\ &= 860\left[1 - 0.72575\right] \\ &= 235.855 \end{aligned}$$

Hence:

$$E(Y) \approx 547.278 + 235.855 = 783.13$$

2.2 Lognormal distribution

Truncated moments of the lognormal distribution

If $X \sim \log N(\mu, \sigma^2)$, then:

$$\int_L^U x^k f_X(x)\, dx = e^{k\mu + ½k^2\sigma^2}\left[\Phi(U_k) - \Phi(L_k)\right]$$

where:

$$L_k = \frac{\ln L - \mu}{\sigma} - k\sigma$$

$$U_k = \frac{\ln U - \mu}{\sigma} - k\sigma$$

$\Phi(z)$ is the CDF of the standard normal distribution.

This result is also given on page 18 of the *Tables*. It is proved as follows.

Using the formula for the PDF of the lognormal distribution from page 14 of the *Tables*, we have:

$$\int_L^U x^k f_X(x)\,dx = \int_L^U x^k \frac{1}{\sigma\sqrt{2\pi}}\frac{1}{x} e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2} dx$$

Making the substitution $t = \dfrac{\ln x - \mu}{\sigma} - k\sigma$ gives:

$$\int_L^U x^k f_X(x)\,dx = \int_{L_k}^{U_k} e^{k(\mu+\sigma t+k\sigma^2)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t+k\sigma)^2} dt$$

$$= \int_{L_k}^{U_k} \frac{1}{\sqrt{2\pi}} e^{k\mu+k\sigma t+k^2\sigma^2} e^{-½t^2-k\sigma t-½k^2\sigma^2} dt$$

$$= e^{k\mu+½k^2\sigma^2} \int_{L_k}^{U_k} \frac{1}{\sqrt{2\pi}} e^{-½t^2} dt$$

$$= e^{k\mu+½k^2\sigma^2} [\Phi(U_k) - \Phi(L_k)]$$

When $L=0$ or $U=\infty$, these formulae can be simplified using the facts that:

$$\Phi(-\infty)=0, \Phi(0)=½, \Phi(\infty)=1$$

By setting $k=1$ in the truncated moments formula, we can calculate the insurer's expected claim payment under excess of loss reinsurance when the original claims follow a lognormal distribution.

Question

An insurer is considering taking out one of the following reinsurance treaties:

Treaty 1: Proportional reinsurance with a retained proportion of 0.75

Treaty 2: Individual excess of loss cover with a retention limit of £25,000

The claims distribution is lognormal with parameters $\mu = 8.5$ and $\sigma^2 = 0.64$.

Calculate the insurer's expected net claim payments in the following cases:

(a) without either treaty

(b) with Treaty 1 only

(c) with Treaty 2 only.

Solution

(a) ***No reinsurance***

Without either treaty, the insurer pays the full amount of each loss. So:

$$E(Y) = E(X) = e^{\mu + ½\sigma^2} = e^{8.5 + ½\times 0.64} = £6,768$$

(b) ***Treaty 1***

Under Treaty 1, the insurer pays 75% of each loss. So $Y = 0.75X$ and:

$$E(Y) = 0.75E(X) = 0.75 \times 6,768 = £5,076$$

(c) ***Treaty 2***

Under Treaty 2, the insurer pays the first £25,000 of each loss. So:

$$Y = \begin{cases} X & \text{if } X \le 25,000 \\ 25,000 & \text{if } X > 25,000 \end{cases}$$

and:

$$E(Y) = \int_0^{25,000} x\, f_X(x)\, dx + 25,000 P(X > 25,000)$$

Using the truncated moments formula with $k = 1$ gives:

$$\int_0^{25,000} x\, f_X(x)\, dx = e^{\mu + ½\sigma^2} \left[\Phi(U_1) - \Phi(L_1)\right]$$

where:

$$U_1 = \frac{\ln U - \mu}{\sigma} - \sigma = \frac{\ln 25,000 - 8.5}{\sqrt{0.64}} - \sqrt{0.64} = 1.23329$$

$$\begin{aligned} \Phi(U_1) &\approx (1 - 0.329)\Phi(1.23) + 0.329\,\Phi(1.24) \\ &= 0.671 \times 0.89065 + 0.329 \times 0.89251 \\ &= 0.89126 \end{aligned}$$

and:

$$\Phi(L_1) = \Phi\left(\frac{\ln L - \mu}{\sigma} - \sigma\right) = 0 \quad \text{since } \ln L \to -\infty \text{ as } L \to 0$$

So:

$$\int_0^{25,000} x\, f_X(x)\, dx = e^{8.5+½\times 0.64} \times 0.89126 = 6,032.30$$

The second term in the expression for $E(Y)$ is:

$$25,000P(X > 25,000) = 25,000\left[1 - \Phi\left(\frac{\ln 25,000 - 8.5}{\sqrt{0.64}}\right)\right]$$

$$= 25,000\left[1 - \Phi(2.03329)\right]$$

Interpolating gives:

$$\Phi(2.03329) \approx (1-0.329)\Phi(2.03) + 0.329\,\Phi(2.04)$$

$$= 0.671 \times 0.97882 + 0.329 \times 0.97932$$

$$= 0.97898$$

So:

$$E(Y) \approx 6,032.30 + 25,000 \times (1 - 0.97898) = £6,558$$

Setting $k = 2$ in the truncated moments formula, we can calculate the second non-central moment of the insurer's claim payment. We now extend the previous question to calculate the standard deviation of the insurer's net claim amount.

Question

An insurer is considering taking out one of the following reinsurance treaties:

Treaty 1: Proportional reinsurance with a retained proportion of 0.75

Treaty 2: Individual excess of loss cover with a retention limit of £25,000

The claims distribution is lognormal with parameters $\mu = 8.5$ and $\sigma^2 = 0.64$.

Calculate the standard deviation of the insurer's net claim payments in the following cases:

(a) without either treaty

(b) with Treaty 1 only

(c) with Treaty 2 only.

Solution

(a) ***No reinsurance***

Without either treaty, the variance is:

$$\text{var}(Y) = \text{var}(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) = e^{2(8.5)+0.64}(e^{0.64} - 1) = 41{,}067{,}256.6$$

So the standard deviation is £6,408.

(b) ***Treaty 1***

Under Treaty 1:

$$\text{var}(Y) = 0.75^2 \text{var}(X)$$

So the standard deviation is $0.75 \times 6{,}408 = £4{,}806$.

(c) ***Treaty 2***

Under Treaty 2:

$$E(Y^2) = \int_0^M x^2 f_X(x)\,dx + \int_M^\infty M^2 f_X(x)\,dx$$

Using the truncated moments formula with $k = 2$ gives:

$$\int_0^{25{,}000} x^2\, f_X(x)\,dx = e^{2\mu+2\sigma^2}\left[\Phi(U_2) - \Phi(L_2)\right]$$

where:

$$U_2 = \frac{\ln U - \mu}{\sigma} - 2\sigma = \frac{\ln 25{,}000 - 8.5}{\sqrt{0.64}} - 2\sqrt{0.64} = 0.43329$$

$$\begin{aligned}\Phi(U_2) &\approx (1-0.329)\Phi(0.43) + 0.329\,\Phi(0.44)\\ &= 0.671 \times 0.66640 + 0.329 \times 0.67003\\ &= 0.66759\end{aligned}$$

and:

$$\Phi(L_2) = \Phi\left(\frac{\ln L - \mu}{\sigma} - 2\sigma\right) = 0 \quad \text{since } \ln L \to -\infty \text{ as } L \to 0$$

So:

$$\int_0^{25,000} x^2\, f_X(x)\, dx = e^{2\times 8.5+2\times 0.64} \times 0.66759 = 57,998,646$$

$$E(Y^2) = 57,998,646 + 25,000^2 \times (1-0.97898) = 71,130,942$$

$$\text{var}(Y) = 71,130,942 - 6,558^2 = 5,304^2$$

and the standard deviation is £5,304. (The calculations are very sensitive to rounding, so we have used Excel to obtain accurate values.)

Our calculations have shown that reinsurance reduces both the mean and variance of the insurer's claim payments, as expected.

3 Inflation

The examples we have considered so far have assumed that claim distributions don't change over time (or at least that we are looking at a sufficiently short time period for us to be able to make this assumption).

In practice claims are likely to increase because of inflation, at least in the longer term. A claim distribution that is suitable for modelling claim amounts in one year may well not be suitable a year or two later. We need to adjust our claim distributions to allow for inflation.

In this section we will look at how claims inflation affects reinsurance arrangements. It is easy to deal with claims inflation in the proportional reinsurance situation.

Question

Claims from a portfolio of policies are believed to follow an $Exp(\lambda)$ distribution. A proportional reinsurance arrangement with a retained proportion α is in force.

(i) Give an expression for the insurer's expected claim payment.

(ii) Next year, claim amounts are expected to increase by an inflationary factor of k. Derive an expression for the insurer's expected claim payment next year.

Solution

(i) ***Expected claim payment this year***

We know that $X \sim Exp(\lambda)$ and $Y = \alpha X$. So:

$$E(Y) = E(\alpha X) = \alpha E(X) = \frac{\alpha}{\lambda}$$

(ii) ***Expected claim payment next year***

Next year, the gross claim amount random variable is kX and the insurer's net claim payment is αkX. So the insurer's expected claim payment is:

$$E(\alpha kX) = \alpha k E(X) = \frac{\alpha k}{\lambda}$$

With excess of loss reinsurance, inflation can cause a problem. Suppose that the claims X are inflated by a factor of k but the retention M remains fixed. What effect does this have on the arrangement?

The amount claimed is kX, and the amount paid by the insurer, Y, is:

$$Y = kX \qquad \text{if } kX \le M$$

$$Y = M \qquad \text{if } kX > M$$

In other words:

$$Y = \begin{cases} kX & \text{if } X \le \frac{M}{k} \\ M & \text{if } X > \frac{M}{k} \end{cases}$$

The mean amount paid by the insurer is:

$$\boldsymbol{E(Y) = \int_0^{M/k} kx\, f(x)\, dx + M\, P(X > M/k)} \qquad \textbf{(18.5)}$$

For example, if $X \sim Exp(\lambda)$, then:

$$E(Y) = \int_0^{M/k} kx\, \lambda e^{-\lambda x}\, dx + M\, P(X > M/k)$$

Integrating by parts:

$$E(Y) = \left[-kx\, e^{-\lambda x}\right]_0^{M/k} + \int_0^{M/k} ke^{-\lambda x}\, dx + M\, P(X > M/k)$$

$$= -Me^{-\lambda M/k} + \int_0^{M/k} ke^{-\lambda x}\, dx + Me^{-\lambda M/k}$$

The first and last terms in the line above cancel to give:

$$E(Y) = \int_0^{M/k} ke^{-\lambda x}\, dx = \left[-\frac{k}{\lambda} e^{-\lambda x}\right]_0^{M/k} = \frac{k}{\lambda}\left(1 - e^{-\lambda M/k}\right)$$

One important general point that can be made is that the new mean claim amount paid by the insurer is not k times the mean claim amount paid by the insurer without inflation.

The insurer's mean claim amount will inflate by less than k. We can see this by considering different sizes of claim. From the insurer's point of view, the amount it has to pay out on small claims (those that are nowhere near the retention limit) will increase by k. However, the amount paid on claims that were already above the limit will not increase at all (and the amount paid on claims that didn't reach the limit before but now do will increase, but by less than k).

The reinsurer's mean claim amount will increase by a factor of more than k to compensate.

A similar approach can also be taken in situations where the retention limit is linked to some index of inflation.

Of course if the retention limit increases by a factor of k as well, both mean claim amounts (for the insurer and reinsurer) will increase by the same factor.

When examining the details of a reinsurance arrangement in real life it is very important to check whether the retention limits are fixed or are linked to an agreed inflation index. There are special published indices specifically for use in connection with general insurance claims.

The payments of the insurer, *y* , and the reinsurer, *z* , with retention *M* and inflation factor *k* would be:

```
y <- pmin(k*x,M)
z <- pmax(0,k*x-M)
```

We can then calculate moments, probabilities and quantiles as before.

4 Estimation

Consider the problem of estimation in the presence of excess of loss reinsurance. Suppose that the claims record shows only the net claims paid by the insurer. A typical claims record might be:

$$x_1, x_2, M, x_3, M, x_4, x_5, \ldots \qquad (18.6)$$

and an estimate of the underlying gross claims distribution is required.

As before, we wish to estimate the parameters for the distribution we have assumed for the claims.

The method of moments is not available since even the mean claim amount cannot be computed. On the other hand, it may be possible to use the method of percentiles without alteration; this would happen if the retention level M is high and only the higher sample percentiles were affected by the (few) reinsurance claims.

The statistical terminology for a sample of the form (18.6) is censored. In general, a censored sample occurs when some values are recorded exactly and the remaining values are known only to exceed a particular value, here the retention level M.

Maximum likelihood can be applied to censored samples. The likelihood function is made up of two parts. If the values of $x_1, x_2, \ldots, x_n$ are recorded exactly these contribute a factor of:

$$L_1(\underline{\theta}) = \prod_{i=1}^{n} f(x_i; \underline{\theta})$$

If a further m claims are referred to the reinsurer, then the insurer records a payment of M for each of these claims. These censored values then contribute a factor:

$$L_2(\underline{\theta}) = \prod_{j=1}^{m} P(X > M) \quad ie \left[P(X > M)\right]^m$$

The complete likelihood function is:

$$L(\underline{\theta}) = \prod_{i=1}^{n} f(x_i; \underline{\theta}) \times \left[1 - F(M; \underline{\theta})\right]^m$$

where $F(.;\underline{\theta})$ is the CDF of the claims distribution.

The reason for multiplying is that the likelihood reflects the probability of getting the n claims with known values *and* m claims exceeding M. Also, we are assuming that the claims are independent.

In R, we can define the censored log-likelihood function and use the function `nlm` on the negative value of this as before.

Question

Claims from a portfolio are believed to follow an $Exp(\lambda)$ distribution. The insurer has effected individual excess of loss reinsurance with a retention limit of 1,000.

The insurer observes a random sample of 100 claims, and finds that the average amount of the 90 claims that do not exceed 1,000 is 82.9. There are 10 claims that do exceed the retention limit.

Calculate the maximum likelihood estimate of the parameter λ.

Solution

Here $X \sim Exp(\lambda)$ and $P(X > 1{,}000) = e^{-1{,}000\lambda}$. So the likelihood function is:

$$L(\lambda) = \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2} \ldots \lambda e^{-\lambda x_{90}} \times (e^{-1{,}000\lambda})^{10} = \lambda^{90} e^{-(10{,}000+\sum x_i)\lambda}$$

Taking logs:

$$\ln L = 90 \ln \lambda - (10{,}000 + \sum x_i)\lambda$$

Differentiating with respect to λ:

$$\frac{\partial}{\partial \lambda} \ln L = \frac{90}{\lambda} - (10{,}000 + \sum x_i)$$

This is equal to 0 when:

$$\lambda = \frac{90}{10{,}000 + \sum x_i} = \frac{90}{10{,}000 + (90 \times 82.9)} = 0.005154$$

Differentiating again:

$$\frac{\partial^2}{\partial \lambda^2} \ln L = -\frac{90}{\lambda^2}$$

This is negative when $\lambda = 0.005154$. (In fact it is always negative.) So we have a maximum turning point and hence $\hat{\lambda} = 0.005154$.

5 Policy excess

Insurance policies with an excess are common in motor insurance and many other kinds of property and accident insurance. Under this kind of policy, the insured agrees to carry the full burden of the loss up to a limit, L, called the excess. If the loss is an amount X, greater than L, then the policyholder will claim only $X-L$. If Y is the amount actually paid by the insurer, then:

$$Y = 0 \qquad \text{if } X \le L$$

$$Y = X - L \qquad \text{if } X > L$$

Clearly, the premium due on any policy with an excess will be less than that on a policy without an excess.

This assumes that some of the saving is actually passed on to the policyholder. A policy excess may also be referred to as a *deductible*.

The position of the insurer for a policy with an excess is exactly the same as that of the reinsurer under excess of loss reinsurance. The position of the policyholder as far as losses are concerned is exactly the same as that of an insurer with an excess of loss reinsurance contract.

In practice, expenses form a significant part of the insurance cost. So the presence of an excess might not affect the premium as much as might be expected. A premium calculated ignoring expenses is called a 'risk premium'.

Question

An insurer believes that claims from a particular type of policy follow a Pareto distribution with parameters $\alpha = 2$ and $\lambda = 900$. The insurer wishes to introduce a policy excess so that 20% of losses result in no claim to the insurer.

Calculate the size of the excess.

Solution

Let L be the size of the excess. The insurer wants to set L so that $P(X<L)=0.2$. Using the given loss distribution, we have:

$$P(X<L) = 1 - \left(\frac{900}{900+L}\right)^2$$

So we require:

$$1 - \left(\frac{900}{900+L}\right)^2 = 0.2$$

Rearranging:

$$\left(\frac{900}{900+L}\right)^2 = 0.8$$

$$\Rightarrow \frac{900}{900+L} = \sqrt{0.8}$$

$$\Rightarrow 900+L = \frac{900}{\sqrt{0.8}}$$

$$\Rightarrow L = \frac{900}{\sqrt{0.8}} - 900 = 106.23$$

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 18 Summary

Reinsurance

Reinsurance is insurance for insurance companies. By using reinsurance, the insurer seeks to protect itself from large claims. The mean amount paid by the insurer is reduced, and the variance of the amount paid by the insurer is reduced.

Reinsurance may be proportional or non-proportional (*ie* excess of loss).

We use the following notation:

X is the gross claim amount random variable

Y is the net claim amount, *ie* the amount of the claim paid by the insurer

Z is the amount paid by the reinsurer

Proportional reinsurance

Under proportional reinsurance, the insurer and the reinsurer split the claim in pre-defined proportions. For a claim amount X, the amount paid by the insurer is $Y = \alpha X$ and the amount paid by the reinsurer is $Z = (1-\alpha)X$ where α is known as the retained proportion or retention level, $0 < \alpha < 1$.

$$E(Y) = \alpha E(X) \qquad E(Z) = (1-\alpha)E(X)$$

$$\text{var}(Y) = \alpha^2 \text{var}(X) \qquad \text{var}(Z) = (1-\alpha)^2 \text{var}(X)$$

Non-proportional reinsurance (individual excess of loss)

Under individual excess of loss, the insurer will pay any claim in full up to an amount M, the retention level. Any amount above M will be met by the reinsurer.

$$Y = \begin{cases} X & \text{if } X \le M \\ M & \text{if } X > M \end{cases} \qquad Z = \begin{cases} 0 & \text{if } X \le M \\ X - M & \text{if } X > M \end{cases}$$

$$E(Y) = \int_0^M x\, f_X(x)\,dx + \int_M^\infty M\, f_X(x)\,dx \qquad E(Z) = \int_M^\infty (x-M)\, f_X(x)\,dx$$

$$E(Y^2) = \int_0^M x^2\, f_X(x)\,dx + \int_M^\infty M^2\, f_X(x)\,dx \qquad E(Z^2) = \int_M^\infty (x-M)^2\, f_X(x)\,dx$$

Reinsurer's conditional claims distribution

It may be the case that the reinsurer is only informed of claims greater than the retention level M. In this case, the reinsurer observes claims from a truncated (or conditional) distribution. Let W be the random variable associated with this distribution, then:

$$W = Z \mid Z > 0 = X - M \mid X > M$$

$$f_W(w) = \frac{f_X(w+M)}{P(X>M)} = \frac{f_X(w+M)}{P(Z>0)}$$

$$E(W) = \frac{E(Z)}{P(Z>0)}$$

Excesses

When a policy excess applies, the policyholder pays for the first part of each loss up to an excess level L. Any amount greater than L will be met by the insurer. The positions of the policyholder and the insurer as far as losses are concerned are the same as those of the insurer and the reinsurer respectively under individual excess of loss reinsurance. When a policy excess applies, the insurer's conditional distribution takes the same form as that of the reinsurer's conditional distribution above.

Inflation and individual excess of loss reinsurance

If claims are inflated by a factor of k but the retention level remains fixed at M then the amount paid by the insurer is:

$$Y = \begin{cases} kX & \text{if } X \le \frac{M}{k} \\ M & \text{if } X > \frac{M}{k} \end{cases}$$

The amount paid by the reinsurer is:

$$Z = \begin{cases} 0 & \text{if } X \le \frac{M}{k} \\ kX - M & \text{if } X > \frac{M}{k} \end{cases}$$

Estimation of parameters from a censored sample

The likelihood function of a vector of parameters $\underline{\theta}$, based on a sample of n exact observations and m censored observations known to exceed M is:

$$L(\underline{\theta}) = \left[\prod_{i=1}^{n} f_X(x_i)\right]\left[P(X>M)\right]^m$$

assuming that the observations are realisations of $n+m$ IID random variables.

Chapter 18 Practice Questions

18.1 An insurer insures a risk for which individual claim sizes (in £000s) have mean 500 and standard deviation 250. The insurer arranges excess of loss reinsurance for this risk with a retention limit of £1,000,000.

Calculate the proportion of claims from this risk for which the insurer expects to receive a payment from the reinsurer if the loss distribution is:

(a) gamma

(b) lognormal.

18.2 Claims arising from a particular portfolio have a Pareto distribution with parameters $\alpha = 6$ and $\lambda = 200$. The insurer effects individual excess of loss reinsurance with a retention limit of 80.

(i) Calculate the insurer's expected claim amount before and after reinsurance.

(ii) Calculate the mean amount paid by the reinsurer on claims in which it is involved.

18.3 A sample of a reinsurer's payments made under a proportional reinsurance arrangement consists of the following values, in units of thousands of pounds:

4.6, 6.8, 22.9, 1.4, 3.8, 10.2, 19.4, 32.1

If the original claim amounts have a $Gamma(\alpha,\lambda)$ distribution, and the retained proportion is 80%, determine the distribution of the reinsurer's claim payments. Hence estimate the parameters α and λ using the method of moments.

18.4 If $X \sim \log N(7.5, 0.85^2)$, calculate:

(a) $$\int_{1,000}^{5,000} f(x)\,dx$$

(b) $$\int_{0}^{1,000} x\, f(x)\,dx$$

(c) $$\int_{5,000}^{\infty} x^2 f(x)\,dx$$

18.5 Claims from a portfolio are believed to have a Pareto distribution with parameters α and λ. In Year 0, $\alpha = 6$ and $\lambda = 1{,}000$. An excess of loss reinsurance arrangement is in force, with a retention limit of 500. Inflation is a constant 10% *pa*.

(i) Determine the distribution of the gross claim amounts in Years 1 and 2.

(ii) Calculate the reinsurer's mean claim payment on all claims in Years 0, 1 and 2.

18.6 Claim amounts from a portfolio follow a Weibull distribution with PDF:

$$f(x) = 2cxe^{-cx^2}, x \geq 0$$

An individual excess of loss reinsurance arrangement with retention limit $M = 3$ is in force. A sample of the *reinsurer's* non-zero payment amounts gives the following values:

$$n = 10 \qquad \sum w_i = 8.7 \qquad \sum w_i^2 = 92.3$$

where the units are millions of pounds. Calculate the maximum likelihood estimate of c.

18.7 (i) A random variable X has the lognormal distribution with density function $f(x)$ and parameters μ and σ. Show that for $a > 0$:

Exam style

$$\int_a^\infty x\, f(x)dx = \exp\left(\mu + \frac{\sigma^2}{2}\right)\left(1 - \Phi\left(\frac{\log a - \mu - \sigma^2}{\sigma}\right)\right)$$

where Φ is the cumulative distribution function of the standard normal distribution. [4]

(ii) Claims under a particular class of insurance follow a lognormal distribution with mean 9.070 and standard deviation of 10.132 (figures in £000s). In any one year 20% of policies are expected to give rise to a claim.

An insurance company has 200 policies on its books and wishes to take out individual excess of loss reinsurance to cover all the policies in the portfolio. The reinsurer has quoted premiums for two levels of reinsurance as follows (figures in £000s):

Retention limit	Premium
25	50
30	40

(a) Calculate the probability, under each reinsurance arrangement, that a claim arising will involve the reinsurer.

(b) By investigating the average amount of each claim ceded to the reinsurer, calculate which of the retention levels gives the best value for money for the insurer (ignoring the insurer's attitude to risk).

(c) The following year, assuming all other things equal, the insurer believes that inflation will increase the mean and standard deviation of the claims in its portfolio by 8%. If the reinsurer charges the same premiums as before, determine which of the retention levels will give best value for money next year. [18]

[Total 22]

18.8 Exam style

(i) Loss amounts from a particular type of insurance have a Pareto distribution with parameters α and λ. If the company applies a policy excess, *E*, derive the distribution function of claim amounts paid by the insurer. [3]

(ii) Assuming that $\alpha = 4$ and $\lambda = 15$, calculate the mean claim amount paid by the insurer:

(a) with no policy excess ($E = 0$),

(b) with an excess of 10 ($E = 10$). [2]

(iii) Using your answers to (ii), comment on the effect of introducing a policy excess. [2]

[Total 7]

18.9 Exam style

Losses from a group of travel insurance policies are assumed to follow a Pareto distribution with parameters $\alpha = 4.5$ and $\lambda = 3{,}000$.

Next year losses are expected to increase by 3%, and the insurer has decided to introduce a policy excess of 100 per claim.

Calculate the probability that a loss next year is borne entirely by the policyholder. [2]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 18 Solutions

18.1 (a) ***Gamma distribution***

Let X denote the loss random variable and suppose that $X \sim Gamma(\alpha, \lambda)$. Then:

$$E(X) = \frac{\alpha}{\lambda} = 500 \qquad \text{and} \qquad \text{var}(X) = \frac{\alpha}{\lambda^2} = 250^2$$

Solving these equations simultaneously gives:

$$\alpha = 4 \qquad \text{and} \qquad \lambda = 0.008$$

The reinsurer will make a payment if the claim size exceeds £1m. Since we are working in £000s, we have to calculate:

$$P(X > 1{,}000)$$

To do this, we can use the relationship between the gamma and chi-squared distributions:

$$X \sim Gamma(\alpha, \lambda) \Rightarrow 2\lambda X \sim \chi^2_{2\alpha}$$

So:

$$P(X > 1{,}000) = P(2\lambda X > 2{,}000\lambda) = P(\chi^2_8 > 16) = 1 - 0.9576 = 0.0424$$

(b) ***Lognormal distribution***

If $X \sim \log N(\mu, \sigma^2)$, then:

$$E(X) = e^{\mu + ½\sigma^2} = 500 \qquad \text{and} \qquad \text{var}(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) = 250^2$$

Squaring the first equation and substituting this into the second gives:

$$e^{\sigma^2} - 1 = \frac{250^2}{500^2} = 0.25 \quad \Rightarrow \quad \sigma^2 = \ln 1.25 = 0.22314$$

Then, from the equation for $E(X)$ we have:

$$\mu = \ln 500 - ½\sigma^2 = 6.1030$$

and:

$$P(X > 1{,}000) = P\left(Z > \frac{\ln 1{,}000 - 6.103036}{\sqrt{0.2231436}}\right) = P(Z > 1.70354)$$

$$= 1 - 0.95576 = 0.04424$$

18.2 (i) ***Insurer's expected claim payments***

Let X denote the gross claim amount random variable. Then $X \sim Pa(6,200)$ and:

$$E(X) = \frac{200}{6-1} = 40$$

ie the mean claim amount before reinsurance is 40.

The insurer's net claim amount is:

$$Y = \begin{cases} X & \text{if } X \le 80 \\ 80 & \text{if } X > 80 \end{cases}$$

So:

$$E(Y) = \int_0^{80} x \frac{6(200)^6}{(200+x)^7}\,dx + 80P(X > 80)$$

The integral can be evaluated by substitution. Let $u = 200 + x$. Then:

$$\int_0^{80} x \frac{6(200)^6}{(200+x)^7}\,dx = \int_{200}^{280} (u-200)\frac{6(200)^6}{u^7}\,du$$

$$= 6(200)^6 \int_{200}^{280} (u^{-6} - 200u^{-7})\,du$$

$$= 6(200)^6 \left[\frac{u^{-5}}{-5} - \frac{200u^{-6}}{-6} \right]_{200}^{280}$$

$$= 6(200)^6 \left[\left(-\frac{280^{-5}}{5} + \frac{200 \times 280^{-6}}{6} \right) - \left(-\frac{200^{-5}}{5} + \frac{200 \times 200^{-6}}{6} \right) \right]$$

$$= 21.9378$$

Alternatively, we could integrate by parts.

Also:

$$P(X > 80) = 1 - F(80) = \left(\frac{200}{200+80} \right)^6 = 0.1328$$

So the mean claim amount after reinsurance is:

$$E(Y) = 21.9378 + 80 \times 0.1328 = 32.5626$$

(ii) ***Mean amount paid by the reinsurer on claims in which it is involved***

The mean amount paid by the reinsurer on all claims (including those where the reinsurer makes no payment) is:

$$E(Z) = E(X) - E(Y) = 40 - 32.5626 = 7.4374$$

and the mean amount paid by the reinsurer on claims in which it is involved is:

$$\frac{E(Z)}{P(X > 80)} = \frac{7.4374}{0.1328} = 56$$

Alternatively, we could say that W, the reinsurer's conditional claim payment random variable has a $Pa(6,280)$ distribution. The mean of this distribution is $\frac{280}{6-1} = 56$.

18.3 Let X be the gross claim amount random variable and Z be the reinsurer's claim payment. Then $X \sim Gamma(\alpha,\lambda)$ and $Z = 0.2X$. The moment generating function of Z is:

$$M_Z(t) = E(e^{tZ}) = E(e^{0.2tX}) = M_X(0.2t) = \left(1 - \frac{0.2t}{\lambda}\right)^{-\alpha} = \left(1 - \frac{t}{5\lambda}\right)^{-\alpha}, \quad t < 5\lambda$$

This is the MGF of the $Gamma(\alpha,5\lambda)$ distribution. By the uniqueness property of MGFs, it follows that $Z \sim Gamma(\alpha,5\lambda)$. So:

$$E(Z) = \frac{\alpha}{5\lambda} \qquad \text{and} \qquad \text{var}(Z) = \frac{\alpha}{(5\lambda)^2} = \frac{\alpha}{25\lambda^2}$$

The sample mean and n-denominator variance are:

$$\bar{z} = \frac{\sum z_i}{8} = \frac{101.2}{8} = 12.65$$

and:

$$s^2 = \frac{\sum z_i^2}{8} - \bar{z}^2 = \frac{2,119.02}{8} - 12.65^2 = 104.855$$

The method of moments estimates of α and λ are the solutions of the equations:

$$\frac{\alpha}{5\lambda} = 12.65 \qquad \text{and} \qquad \frac{\alpha}{25\lambda^2} = 104.855$$

Solving these gives $\hat{\alpha} = 1.526$ and $\hat{\lambda} = 0.02413$.

18.4 (a) Using the truncated moments formula with $k=0$:

$$\int_{1,000}^{5,000} f(x)\,dx = \Phi\left(\frac{\ln 5,000-7.5}{0.85}\right)-\Phi\left(\frac{\ln 1,000-7.5}{0.85}\right)$$

$$=\Phi(1.19670)-\Phi(-0.69676)$$

$$=0.88429-0.24298$$

$$=0.6413$$

This is $P(1,000<X<5,000)$.

(b) Using the formula with $k=1$:

$$\int_{0}^{1,000} x\, f(x)\,dx = e^{7.5+½\times 0.85^2}\left[\Phi\left(\frac{\ln 1,000-7.5}{0.85}-0.85\right)-\Phi(-\infty)\right]$$

$$=e^{7.86125}[\Phi(-1.54676)-0]$$

$$=2,594.76\times 0.06096$$

$$=158.2$$

(c) Using the formula with $k=2$:

$$\int_{5,000}^{\infty} x^2 f(x)\,dx = e^{2(7.5)+2\times 0.85^2}\left[\Phi(\infty)-\Phi\left(\frac{\ln 5,000-7.5}{0.85}-2(0.85)\right)\right]$$

$$=e^{16.445}[1-\Phi(-0.50330)]$$

$$=13,866,688(1-0.30738)$$

$$=9.604\text{m}$$

18.5 (i) ***Distribution of insurer's claim payments before reinsurance***

Let X_j be the gross claim amount random variable in Year j. Then $X_0 \sim Pa(6,1000)$ and $X_1 = 1.1X_0$. The Pareto distribution does not have a moment generating function, but we can determine the distribution of X_1 by considering its CDF. For $x>0$:

$$F_{X_1}(x)=P(X_1\le x)=P(1.1X_0\le x)=P\left(X_0\le \tfrac{x}{1.1}\right)=F_{X_0}\left(\tfrac{x}{1.1}\right)=1-\left(\frac{1,000}{1,000+\frac{x}{1.1}}\right)^6$$

Multiplying the numerator and the denominator of the bracketed fraction by 1.1, we see that:

$$F_{X_1}(x)=1-\left(\frac{1,100}{1,100+x}\right)^6 \quad , \quad x>0$$

This is the CDF of the $Pa(6,1100)$ distribution. So $X_1 \sim Pa(6,1100)$.

Inflation has no effect on the first parameter, but the second parameter has increased by 10%.

Similarly, $X_2 \sim Pa(6,1210)$.

(ii) ***Reinsurer's expected claim payments***

Let Z_j be the reinsurer's claim payment random variable in Year j. Then:

$$E(Z_0) = \int_{500}^{\infty} (x-500)\, f_{X_0}(x)\, dx = \int_{500}^{\infty} (x-500) \frac{6(1{,}000)^6}{(1{,}000+x)^7}\, dx$$

Substituting $t = x - 500$, we see that:

$$E(Z_0) = \int_0^{\infty} t \frac{6(1{,}000)^6}{(1{,}500+t)^7}\, dt$$

We can rewrite this as follows:

$$E(Z_0) = \left(\frac{1{,}000}{1{,}500}\right)^6 \int_0^{\infty} t \frac{6(1{,}500)^6}{(1{,}500+t)^7}\, dt$$

The integrand is of the form $t\, f_T(t)$, where $T \sim Pa(6,1500)$. So:

$$E(Z_0) = \left(\frac{1{,}000}{1{,}500}\right)^6 E(T) = \left(\frac{1{,}000}{1{,}500}\right)^6 \times \frac{1{,}500}{5} = 26.337$$

The only change from Year 0 to Year 1 is in the λ parameter. So, using the same approach:

$$E(Z_1) = \int_{500}^{\infty} (x-500) \frac{6(1{,}100)^6}{(1{,}100+x)^7}\, dx = \int_0^{\infty} t \frac{6(1{,}100)^6}{(1{,}600+t)^7}\, dt = \left(\frac{1{,}100}{1{,}600}\right)^6 \int_0^{\infty} t \frac{6(1{,}600)^6}{(1{,}600+t)^7}\, dt$$

The final integral is the mean of the $Pa(6,1600)$ distribution. So:

$$E(Z_1) = \left(\frac{1{,}100}{1{,}600}\right)^6 \times \frac{1{,}600}{5} = 33.790$$

Similarly:

$$E(Z_2) = \int_{500}^{\infty} (x-500) \frac{6(1{,}210)^6}{(1{,}210+x)^7}\, dx = \int_0^{\infty} t \frac{6(1{,}210)^6}{(1{,}710+t)^7}\, dt = \left(\frac{1{,}210}{1{,}710}\right)^6 \int_0^{\infty} t \frac{6(1{,}710)^6}{(1{,}710+t)^7}\, dt$$

$$= \left(\frac{1{,}210}{1{,}710}\right)^6 \times \frac{1{,}710}{5} = 42.930$$

The percentage increase from Year 0 to Year 1 is 28.3%, and the percentage increase from Year 1 to Year 2 is 27.1%. These figures are more than 10% as expected.

18.6 Since we are given information about claim payments made by the reinsurer, we need to consider the reinsurer's conditional claim amount random variable. This has PDF:

$$g(w)=\frac{f(w+M)}{1-F(M)}$$

The gross claim amount random variable has a Weibull distribution with parameters c and 2. So:

$$g(w)=\frac{2c(w+M)e^{-c(w+M)^2}}{e^{-cM^2}}=2c(w+3)e^{-c(w^2+6w)}$$

So the likelihood function based on a random sample of n payments made by the reinsurer is:

$$L(c)=2^n c^n \prod_{i=1}^{n}(w_i+3)\exp\left(-c\sum_{i=1}^{n}(w_i^2+6w_i)\right)$$

Taking logs:

$$\ln L = n\ln 2 + n\ln c + \sum_{i=1}^{n}\ln(w_i+3) - c\sum_{i=1}^{n}(w_i^2+6w_i)$$

Differentiating with respect to c :

$$\frac{\partial}{\partial c}\ln L=\frac{n}{c}-\sum_{i=1}^{n}(w_i^2+6w_i)$$

This is 0 when:

$$c=\frac{n}{\sum_{i=1}^{n}(w_i^2+6w_i)}$$

Differentiating again:

$$\frac{\partial^2}{\partial c^2}\ln L=-\frac{n}{c^2}$$

This is negative, so we have a maximum.

Substituting in the given numerical values, we find that:

$$\hat{c}=\frac{10}{92.3+6\times 8.7}=0.0692$$

18.7 (i) ***Proof***

We want to simplify the integral:

$$\int_a^\infty x \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2} dx$$

Making the substitution $u = \dfrac{\log x - \mu}{\sigma} - \sigma$, the integral becomes:

$$\int_{\frac{\log a-\mu}{\sigma}-\sigma}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(u+\sigma)^2} \sigma e^{\mu+u\sigma+\sigma^2} du \qquad [1]$$

Multiplying out the brackets in the exponent and simplifying, we have:

$$e^{\mu+\frac{1}{2}\sigma^2} \int_{\frac{\log a-\mu}{\sigma}-\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \qquad [1]$$

This integrand is the PDF of the standard normal distribution and the integral is:

$$P\left(N(0,1) > \frac{\log a - \mu}{\sigma} - \sigma\right) \qquad [1]$$

So we have:

$$\int_a^\infty x\, f(x)\, dx = e^{\mu+\frac{1}{2}\sigma^2}\left[1 - \Phi\left(\frac{\log a - \mu}{\sigma} - \sigma\right)\right] \qquad [1]$$

This is the required result.

(ii)(a) ***Probability***

We first need the parameter values for the lognormal distribution. Using the formulae for the mean and variance of the lognormal distribution from page 14 of the *Tables* we have the following equations:

$$e^{\mu+\frac{1}{2}\sigma^2} = 9.070 \quad \text{and} \quad e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) = 10.132^2$$

Solving these simultaneous equations (by squaring the first equation and then substituting into the second equation), we obtain the values:

$$\sigma^2 = 0.80999 \quad \text{and} \quad \mu = 1.79998 \qquad [2]$$

The probability that a claim involves the reinsurer is the probability that it exceeds the retention limit. So if X represents the amount of a claim, we have, for the first reinsurance arrangement:

$$P(X>25)=P\left(\log N(\mu,\sigma^2)>25\right)=P\left(N(\mu,\sigma^2)>\log 25\right)$$

$$=P\left(N(0,1)>\frac{\log 25-\mu}{\sigma}\right)=1-\Phi\left(\frac{\log 25-\mu}{\sigma}\right) \quad [1]$$

Substituting in the values for μ and σ^2, we get:

$$P(X>25)=1-\Phi(1.57656)=0.05745 \quad [1]$$

So the probability that a claim will involve the reinsurer if the first arrangement is in force is 5.745%.

Using exactly the same argument for the second arrangement, we get:

$$P(X>30)=1-\Phi\left(\frac{\log 30-\mu}{\sigma}\right)=1-\Phi(1.77914)=0.03761 \quad [1]$$

So the probability that a claim will involve the reinsurer if the second arrangement is in force is 3.761%.

(ii)(b) ***Better arrangement***

Consider the first arrangement. The amount ceded to the reinsurer (*ie* the amount paid by the reinsurer on a claim) is:

$$Z=\begin{cases}0 & \text{if } X\le 25\\ X-25 & \text{if } X>25\end{cases}$$

So:

$$E(Z)=\int_{25}^{\infty}(x-25)\ f(x)\ dx=\int_{25}^{\infty}x\ f(x)\ dx-25\int_{25}^{\infty}f(x)\ dx \quad [1]$$

where $f(x)$ is the PDF of the original lognormal distribution.

We can calculate the first of these integrals by using the result from the first part of the question:

$$\int_{25}^{\infty}x\ f(x)\ dx=e^{\mu+½\sigma^2}\left[1-\Phi\left(\frac{\log 25-\mu-\sigma^2}{\sigma}\right)\right]$$

$$=9.070\left[1-\Phi(0.67657)\right]$$

$$=9.070\times 0.24934$$

$$=2.2615 \quad [1]$$

The second integral is just the probability that we worked out in part (ii)(a). So:

$$E(Z) = 2.2615 - 25 \times 0.05745 = 0.8253$$ [1]

Working in exactly the same way for the second arrangement (where Z is now the amount paid by the reinsurer in excess of 30), we have:

$$E(Z) = 9.070\left[1 - \Phi(0.87915)\right] - 30 \times 0.03761$$

$$= 9.070 \times 0.18966 - 30 \times 0.03761$$

$$= 0.5919$$ [1]

So the expected amount paid out by the reinsurer per £1 of premium is (under the first arrangement):

$$\frac{200 \times 0.2 \times 0.8253}{50} = £0.660$$ [1]

Under the second arrangement, it is:

$$\frac{200 \times 0.2 \times 0.5919}{40} = £0.592$$ [1]

So, other things being equal, the first arrangement looks better value.

(ii)(c) ***Better arrangement under new circumstances***

The new mean and standard deviation are now 9.7956 and 10.94256 respectively. So we can calculate the new parameter values:

$$e^{\mu + ½\sigma^2} = 9.7956 \qquad \text{and} \qquad e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) = 10.94256^2$$

Solving these exactly as we did before, we find that $\mu = 1.87694$ and σ^2 is unchanged at 0.80999. [2]

So the value for $E(Z)$ is now (under the first arrangement):

$$E(Z) = 9.7956\left[1 - \Phi\left(\frac{\log 25 - 1.87694 - 0.80999}{\sqrt{0.80999}}\right)\right]$$

$$- 25 \times \left[1 - \Phi\left(\frac{\log 25 - 1.87694}{\sqrt{0.80999}}\right)\right]$$

$$= 9.7956\left[1 - \Phi(0.59106)\right] - 25\left[1 - \Phi(1.49105)\right]$$

$$= 9.7956 \times 0.27724 - 25 \times 0.06797$$

$$= 1.0164$$ [2]

Under the second arrangement:

$$E(Z) = 9.7956\left[1-\Phi\left(\frac{\log 30 - 1.87694 - 0.80999}{\sqrt{0.80999}}\right)\right]$$

$$-30\times\left[1-\Phi\left(\frac{\log 30 - 1.87694}{\sqrt{0.80999}}\right)\right]$$

$$= 9.7956\left[1-\Phi(0.79364)\right] - 30\left[1-\Phi(1.69363)\right]$$

$$= 9.7956\times 0.21370 - 30\times 0.04517$$

$$= 0.73830$$ [2]

So working exactly as before, the payment per £1 of premium under the first arrangement is now:

$$\frac{200\times 0.2\times 1.0164}{50} = £0.813$$

In the second arrangement the corresponding figure is:

$$\frac{200\times 0.2\times 0.73830}{40} = £0.738$$

So the first arrangement is still better value for the insurer. [1]

18.8 (i) ***Distribution function***

Let X denote the amount of the loss and Y denote the amount paid by the insurer in respect of the loss. With a policy excess of E in force, we have:

$$Y = X - E \mid X > E$$

The CDF of Y is given by:

$$F_Y(y) = P(Y\le y) = P(X - E \le y \mid X > E)$$

$$= \frac{P(X-E\le y \text{ and } X > E)}{P(X>E)}$$

$$= \frac{P(X\le y+E \text{ and } X > E)}{P(X>E)}$$

$$= \frac{P(E < X \le y+E)}{P(X>E)}$$

$$= \frac{F_X(y+E) - F_X(E)}{1-F_X(E)}$$ [2]

Since $X \sim Pa(\alpha,\lambda)$:

$$F_Y(y) = \frac{\left[1-\left(\frac{\lambda}{\lambda+y+E}\right)^{\alpha}\right]-\left[1-\left(\frac{\lambda}{\lambda+E}\right)^{\alpha}\right]}{1-\left[1-\left(\frac{\lambda}{\lambda+E}\right)^{\alpha}\right]}$$

$$= \frac{\left(\frac{\lambda}{\lambda+E}\right)^{\alpha}-\left(\frac{\lambda}{\lambda+y+E}\right)^{\alpha}}{\left(\frac{\lambda}{\lambda+E}\right)^{\alpha}}$$

$$= 1-\left(\frac{\lambda+E}{\lambda+y+E}\right)^{\alpha}, \quad y>0$$ [1]

So $Y \sim Pa(\alpha,\lambda+E)$.

(ii) ***Mean values***

The mean of the $Pa(\alpha,\lambda+E)$ distribution is $\dfrac{\lambda+E}{\alpha-1}$.

If $E=0$ then $E(Y)=\dfrac{15}{3}=5$. [1]

If $E=10$ then $E(Y)=\dfrac{25}{3}=8\dfrac{1}{3}$. [1]

(iii) ***Effect of introducing a policy excess***

Introducing a policy excess of E increases the mean claim amount paid by the insurer by $\dfrac{E}{\alpha-1}$.

This is because small losses are met in full by the policyholder. [2]

It may still be advantageous to the insurer to introduce a policy excess, since although the average claim payment will increase, fewer claim payments will be made.

18.9 Let X be the loss amount random variable for this year, and let X' be the loss amount random variable for next year. Then:

$$X \sim Pa(4.5,3000) \quad \text{and} \quad X' = 1.03X$$

The probability that a loss next year is borne entirely by the policyholder is:

$$P(X' \le 100) = P(X \le \tfrac{100}{1.03}) = 1-\left(\frac{3,000}{3,000+\frac{100}{1.03}}\right)^{4.5} = 0.13353$$ [2]

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Risk models 1

Syllabus objectives

1.2 Compound distributions and their application in risk modelling

1.2.1 Construct models appropriate for short-term insurance contracts in terms of the numbers of claims and the amounts of individual claims.

1.2.2 Describe the major simplifying assumptions underlying the models in 1.2.1.

1.2.3 Define a compound Poisson distribution and show that the sum of independent random variables each having a compound Poisson distribution also has a compound Poisson distribution.

1.2.4 Derive the mean, variance and coefficient of skewness for compound binomial, compound Poisson and compound negative binomial random variables.

0 Introduction

In the first section of this chapter, we describe the main features of general insurance policies. There is no mathematics in this section, and you should be able to read it through fairly quickly in order to obtain a good overview of the different types of product available.

In the remaining sections of the chapter we introduce the idea of a *compound distribution*. We will define and use the compound Poisson, compound geometric, compound negative binomial and compound binomial distributions.

We will also start to look at two models, the *individual risk model* and the *collective risk model*, which are used to describe *aggregate claims*, *ie* the total claims that arise during a period from a group of policies.

In the simplest case of a life assurance benefit (often referred to as 'long-term business'), each policy can result in at most one claim and claims will be for amounts specified in advance (*ie* the sum assured). The benefit level may be the same for all policies or it may vary between policies.

In general insurance (often referred to as 'short-term business'), policies can give rise to more than one claim and the amounts will not usually be known in advance.

In this chapter we look at the theory of risk models. In the next chapter we explain how to adapt these models when reinsurance is in place.

1 General features of a product

1.1 Insurable interest

Generally, for a risk to be insurable:

- **the policyholder must have an interest in the risk being insured, to distinguish between insurance and a wager**
- **a risk must be of a financial and reasonably quantifiable nature.**

1.2 Insurable risk

Ideally risk events also need to meet the following criteria if they are to be insurable:

- **Individual risk events should be independent of each other.**

  In practice we won't often get strict independence but a low correlation is desirable.

- **The probability of the event should be relatively small. In other words, an event that is nearly certain to occur is not conducive to insurance.**

  For example, a house would not be insured if it stood on the edge of a crumbling cliff.

- **Large numbers of potentially similar risks should be pooled in order to reduce the variance and hence achieve more certainty.**

  The similar risks should still be independent.

- **There should be an ultimate limit on the liability undertaken by the insurer.**

  This would help the risk event meet the above criteria that it must be of a reasonably quantifiable nature.

- **Moral hazards should be eliminated as far as possible because these are difficult to quantify, result in selection against the insurer and lead to unfairness in treatment between one policyholder and another.**

  Moral hazards occur when a person takes more risks because another party bears the cost of those risks.

However, the desire for income means that an insurer or reinsurer will usually be found to provide cover when these ideal criteria are not met.

Other characteristics that most general insurance products share are:

- **Cover is normally for a fixed period, most commonly one year, after which it has to be renegotiated. There is normally no obligation on the insurer or insured to continue the arrangement thereafter although in most cases a need for continuing cover may be assumed to exist.**
- **Claims are not of fixed amounts, and the amount of loss as well as the fact needs to be proved before a claim can be settled.**
- **A claim occurring does not bring the policy to an end.**

- **Claims may occur at any time during the policy period. Although there is normally a contractual obligation on the policyholder to report a claim to the insurer as quickly as possible, notification may take some time if the loss is not evident immediately. Settlement of the claim may take a long time if protracted legal proceedings are needed or if it is not straightforward to determine the extent of the loss. However, from the moment of the event giving rise to the claim the ultimate settlement amount is a liability of the insurer. Estimating the amounts of money that need to be reserved to settle these liabilities is one of the most important areas of actuarial involvement in general insurance.**

Classes of insurance in which claims tend to take a long time to settle are known as long-tail. Those which tend to take a short time to settle are known as short-tail, although the dividing line between the two categories is not always distinct.

2 Models for short-term insurance contracts

2.1 The basic model

Many forms of non-life insurance can be regarded as short-term contracts, for example motor insurance. Some forms of life insurance also fall into this category, for example group life and one-year term assurance policies.

A short-term insurance contract can be defined as having the following attributes:

- **The policy lasts for a fixed, and relatively short, period of time, typically one year.**
- **The insurance company receives from the policyholder(s) a premium.**
- **In return, the insurer pays claims that arise during the term of the policy.**

At the end of the policy's term the policyholder may or may not renew the policy. If it is renewed, the premium payable by the policyholder may or may not be the same as in the previous period.

The insurer may choose to pass part of the premium to a reinsurer. In return, the reinsurer will reimburse the insurer for part of the cost of the claims during the policy's term according to some agreed formula.

An important feature of a short-term insurance contract is that the premium is set at a level to cover claims arising during the (short) term of the policy only. This contrasts with life assurance policies where mortality rates increasing with age mean that the (level) annual premium in the early years would be more than sufficient to cover the expected claims in the early years. The excess amount would then be accumulated as a reserve to be used in the later years when the premium on its own would be insufficient to meet the expected cost of claims.

Now to be more specific, a short-term insurance contract covering a risk will be considered. A risk includes either a single policy or a specified group of policies. For ease of terminology the term of the contract is assumed to be one year, but it could equally well be any other short period, for example six months. The random variable S denotes the aggregate claims paid by the insurer in the year in respect of this risk. Models will be constructed for this random variable S. In Section 3 collective risk models will be studied. Later, in the next chapter, the idea of a collective risk model is extended to an individual risk model.

We will see shortly what these terms mean.

A first step in the construction of a collective risk model is to write S in terms of the number of claims arising in the year, denoted by the random variable N, and the amount of each individual claim. Let the random variable X_i denote the amount of the ith claim. Then:

$$S = \sum_{i=1}^{N} X_i \qquad \textbf{(19.1)}$$

where the summation is taken to be zero if N is zero.

This decomposition of S allows consideration of claim numbers and claim amounts separately. A practical advantage of this is that the factors affecting claim numbers and claim amounts may well be different. Take motor insurance as an example. A prolonged spell of bad weather may have a significant effect on claim numbers but little or no effect on the distribution of individual claim amounts. On the other hand, inflation may have a significant effect on the cost of repairing cars, and hence on the distribution of individual claim amounts, but little or no effect on claim numbers.

This approach is referred to as a *collective risk model* because it is considering the *claims* arising from a group of policies taken as a whole, rather than by considering the claims arising from each individual policy.

The random variable S is the sum of a random number of random quantities, and is said to have a *compound distribution*.

Because compound distributions arise commonly in general insurance examples, the random variable N is often referred to as the 'number of claims' and the distribution of the random variables $X_1, X_2, \ldots$ is referred to as the 'individual claim size distribution', even where the compound distribution arises in another context.

To define a compound distribution, we need to know:

- the distribution of N (which is a *discrete* distribution) and
- the distribution of the X_i's (which may be *any* distribution).

If the distribution of the X_i's is continuous, then S will have a *mixed distribution*, *ie* partly discrete and partly continuous. This is because of the possibility that $N = 0$.

The problems that will be studied are the derivation of the moments and distribution of S in terms of the moments and distributions of N and the X_i's. Both will be studied with and without simple forms of reinsurance. The corresponding problems for the reinsurer will also be studied, *ie* the derivation of the moments and distribution of the aggregate claims paid in the year in respect of this risk by the reinsurer.

2.2 Discussion of the simplifications in the basic model

The model for short-term insurance described in the previous subsection contains a number of simplifications as compared to a real insurance operation. The first of these is that it is usually assumed that the moments, and sometimes the distributions, of N and the X_i's are known with certainty. In practice these would probably be estimated from some relevant data.

For example, we might assume that claim amounts have a $Gamma(500, 4)$ distribution.

In practice it may not be possible to make such simple assumptions. For example:

- There may not be an appropriate theoretical distribution that models the distribution of claim amounts actually paid sufficiently well.
- Even if the shape of the distribution is satisfactory, appropriate parameter values may change over time, even in the short term.

- There may not be sufficient homogeneity in the portfolio. For example, different policies may produce claim amounts that have different sizes. This leads to the idea of a mixture distribution.

Another simplification is to assume, at least implicitly, that claims are settled more or less as soon as the incident causing the claim occurs, so that, for example, the insurer's profit is known at the end of the year. In practice, there will be at least a short delay in the settlement of claims and in some cases the delay can amount to many years. This will be especially true when the extent of the loss is difficult to determine, for example if it is to be decided in a court of law.

Delays will often lead to higher payments being made, owing to inflationary factors. (The relevant inflation rate may well be very different from that normally used to measure inflation.)

The model does not in general include any mention of expenses. The premium is assumed to pay the claims and include a loading for profit. In practice, the premium paid by the policyholder(s) will also include a loading for expenses. It is possible to include expenses in the model in a very simple way.

The simplest way to allow for expenses would be to use a claim size distribution that was artificially inflated to allow for some sort of claim expense amount (*eg* adding an extra 20%), although this might not give the right 'shape'. Alternatively we might express the random variable X as the sum of two other random variables, one to represent the actual claim amount and the other to represent the corresponding claim expense.

An important element in models for long-term insurance is a rate of interest since, as explained above, excess premium income would be invested to build up reserves. Interest is a relatively less important, but still important, feature of short-term insurance. It is possible to include interest in models for short-term insurance but it is more usual to ignore it, at least in elementary models.

We will ignore interest in the models used in this chapter.

There are a number of additional elements included when setting the premium to be charged to policyholders, including the policyholders' previous claims record and these are covered in Subject CP1 – Actuarial Practice. The allowance for policyholders' claim experience could be based on claim frequency or total claim amounts. This is beyond the scope of this subject.

In fact, we have already looked at models that make allowance for policyholders' claims experience in Chapter 2.

2.3 Notation and assumptions

Throughout this chapter the following two important assumptions will be made:

- **the random variables $\{X_i\}_{i=1}^N$ are independent and identically distributed**
- **the random variable N is independent of $\{X_i\}_{i=1}^N$.**

In words these assumptions mean that:

1. **the number of claims is not affected by the amount of individual claims**
2. **the amount of a given individual claim is not affected by the amount of any other individual claim**
3. **the distribution of the amounts of individual claims does not change over the (short) term of the policy.**

Point 1 follows from the second of the two assumptions above. Points 2 and 3 follow from the first.

Throughout this chapter it will be assumed that all claims are for non-negative amounts, so that $P(X_i \leq x) = 0$ for $x < 0$. Many of the formulae in this chapter will be derived using the moment generating functions (from now on abbreviated to MGFs) of S, N and X_i. These MGFs will be denoted $M_S(t)$, $M_N(t)$ and $M_X(t)$, respectively, and will be assumed to exist for some positive values of the dummy variable t. The existence of the MGF of a non-negative random variable for positive values of t cannot generally be taken for granted; for example the MGFs of the Pareto and of the lognormal distributions do not exist for any positive value of t. However, all the formulae derived in this chapter with the help of MGFs can be derived, although less easily, without assuming the MGFs exist for positive values of t.

One method would be to use characteristic functions, $E(e^{itX})$, which don't have the same convergence problems as MGFs. However the Core Reading does not cover these.

$G(x)$ and $F(x)$ shall denote the distribution functions of S and X_i, respectively, so that:

$$G(x) = P(S \leq x) \text{ and } F(x) = P(X_i \leq x)$$

For convenience it will often be assumed that the density of $F(x)$ exists and it will be denoted $f(x)$. In cases where this density does not exist, so that X_i has a discrete or a mixed continuous/discrete distribution, expressions such as:

$$\int_0^\infty x\,f(x)\,dx$$

should be interpreted appropriately. The meaning should always be clear from the context.

The kth moment, ($k = 1,2,3...$) of X_i about zero will be denoted m_k.

Using this notation:

$$E(X_i) = m_1 \quad \text{and} \quad \text{var}(X_i) = m_2 - m_1^2$$

3 The collective risk model

3.1 The collective risk model

Recall from Section 2.1 that S is represented as the sum of N random variables X_i, where X_i denotes the amount of the ith claim. Thus:

$$S = X_1 + X_2 + \cdots + X_N$$

and $S = 0$ if $N = 0$.

S is said to have a *compound distribution*.

Note that it is the number of claims, N, from the risk as a collective (as opposed to counting the number of claims from individual policies) that is being considered and this gives the name 'collective risk model'. Within this framework, expressions in general terms for the distribution function, mean, variance and MGF of S can be developed.

3.2 Distribution functions and convolutions

An expression for $G(x)$, the distribution function of S, can be derived by considering the event $\{S \le x\}$. Note that if this event occurs, then one, and only one, of the following events must occur:

	$\{S \le x \text{ and } N = 0\}$	(*ie* no claims)
or	$\{S \le x \text{ and } N = 1\}$	(*ie* one claim of amount $\le x$)
or	$\{S \le x \text{ and } N = 2\}$	(*ie* two claims which total $\le x$)
	$\vdots$	
or	$\{S \le x \text{ and } N = r\}$	(ie r claims which total $\le x$)
	$\vdots$	

and so on. These events are mutually exclusive and exhaustive.

Thus:

$$\{S \le x\} = \bigcup_{n=0}^{\infty} \{S \le x \text{ and } N = n\}$$

and hence:

$$P(S \le x) = \sum_{n=0}^{\infty} P(S \le x \text{ and } N = n)$$

$$= \sum_{n=0}^{\infty} P(N = n) P(S \le x \mid N = n)$$

Question

A group of policies can give rise to at most two claims in a year. The probability function for the number of claims is as follows:

Number of claims, n	0	1	2
$P(N=n)$	0.6	0.3	0.1

Each claim is either for an amount of 1 or an amount of 2, with equal probability. Claim amounts are independent of one another and are independent of the number of claims.

Determine the distribution function of the aggregate annual claim amount, S.

Solution

S can take the values 0, 1, 2, 3 or 4.

S will only equal 0 if $N=0$ and this has probability 0.6. So:

$$P(S \leq 0) = P(S=0) = P(N=0) = 0.6$$

S will equal 1 if there is one claim for amount 1. So:

$$\begin{aligned} P(S=1) &= P(N=1, X=1) \\ &= P(N=1)P(X=1) \qquad \text{by independence} \\ &= 0.3 \times 0.5 \\ &= 0.15 \end{aligned}$$

and:

$$P(S \leq 1) = P(S=0) + P(S=1) = 0.6 + 0.15 = 0.75$$

The other values of the CDF can be calculated in a similar way and are given below:

$$P(S \leq 2) = P(S \leq 1) + P(S=2) = 0.75 + 0.3 \times 0.5 + 0.1 \times 0.5^2 = 0.925$$

$$P(S \leq 3) = P(S \leq 2) + P(S=3) = 0.925 + 0.1 \times 2 \times 0.5^2 = 0.975$$

$$P(S \leq 4) = P(S \leq 3) + P(S=4) = 1$$

The distribution of a sum of independent random variables can be found using *convolutions*.

If $Z = X + Y$, where X and Y are independent random variables with PDFs (or PFs) $f_X(x)$ and $f_Y(y)$, then $f_Z(z)$, the PDF (or PF) of Z, is called the *convolution* of X and Y.

This is written mathematically as $f_Z = f_X * f_Y$.

A formula for a convolution can be found by summing over all possible values of x and y that give a particular value z.

Finding a convolution

$$f_Z(z) = \sum_x f_X(x) f_Y(z-x) \qquad \text{for } \textit{discrete} \text{ random variables}$$

$$f_Z(z) = \int f_X(x) f_Y(z-x)\,dx \qquad \text{for } \textit{continuous} \text{ random variables}$$

'Sum or integrate over all values of x that could lead to a total of z.'

Similar formulae can be used to find the distribution function of a sum.

$$F_Z(z) = \sum_x f_X(x) F_Y(z-x) \qquad \text{or} \qquad \sum_x F_X(x) f_Y(z-x)$$

$$F_Z(z) = \int f_X(x) F_Y(z-x)\,dx \qquad \text{or} \qquad F_Z(z) = \int F_X(x) f_Y(z-x)\,dx$$

Question

Suppose that $N \sim Poisson(\lambda)$, $M \sim Poisson(\mu)$, and N and M are independent.

Use a convolution approach to derive the probability function of $N+M$.

Solution

Let $V = N+M$. Then, for $v = 0,1,2,\ldots$:

$$\begin{aligned}
P(V=v) &= \sum_{n=0}^{v} P(N=n)P(M=v-n) \\
&= \sum_{n=0}^{v} \frac{\lambda^n e^{-\lambda}}{n!} \frac{\mu^{v-n} e^{-\mu}}{(v-n)!} \\
&= \frac{e^{-(\lambda+\mu)}}{v!} \sum_{n=0}^{v} \frac{v!}{n!(v-n)!} \lambda^n \mu^{v-n} \\
&= \frac{e^{-(\lambda+\mu)}}{v!} \sum_{n=0}^{v} \binom{v}{n} \lambda^n \mu^{v-n} \\
&= \frac{e^{-(\lambda+\mu)}}{v!} (\lambda+\mu)^v
\end{aligned}$$

This is the probability function of the $Poisson(\lambda+\mu)$ distribution. So $N+M \sim Poisson(\lambda+\mu)$.

This result can also be proved using MGFs.

We can now consider the distribution function for a compound distribution.

For convolutions of distribution functions, suppose that $\{X_i\}_{i=1}^n$ are independent and identically distributed (IID) random variables with common distribution function $F(x)$.

Then the distribution function of $\sum_{i=1}^{n} X_i$ is denoted by $F^{n^*}(x)$, so that:

$$F^{n^*}(x) = P(X_1 + X_2 + \cdots + X_n \le x)$$

Now note that if $N = n$, then S is the sum of a fixed number n, of random variables, $\{X_i\}_{i=1}^n$, and hence:

$$P(S \le x \mid N = n) = F^{n^*}(x)$$

where $F^{n^*}(x)$ is the *n*-fold convolution of the distribution $F(x)$.

In other words, $F^{3^*}(x)$ would be the convolution $F(x) * F(x) * F(x)$ *etc*.

(Note that $F^{1^*}(x)$ is just $F(x)$ and, for convenience, $F^{0^*}(x)$ is defined to equal 1 for all non-negative values of x. Otherwise $F^{0^*}(x) = 0$.) Thus:

$$G(x) = P(S \le x) = \sum_{n=0}^{\infty} P(N = n) F^{n^*}(x) \qquad \textbf{(19.2)}$$

Formula (19.2) is a general expression for the distribution function of S. Neither the distribution of N nor of X_i has been specified.

Note that when X_i is distributed on the positive integers it is easy to calculate $P(S = x)$ for $x = 1, 2, 3, \ldots$ since:

$$\begin{aligned} P(S = x) &= G(x) - G(x-1) \\ &= \sum_{n=1}^{\infty} P(N = n)\left[F^{n^*}(x) - F^{n^*}(x-1)\right] \end{aligned}$$

ie $$P(S = x) = \sum_{n=1}^{\infty} P(N = n) f_x^{n^*} \qquad \textbf{(19.3)}$$

where $f_x^{n^*} = F^{n^*}(x) - F^{n^*}(x-1)$ is the probability function of $\sum_{i=1}^{n} X_i$.

This is just saying that $f_x^{n^*} = P(\sum_{i=1}^{n} X_i = x)$.

As in the case when X_i is a continuous random variable, $P(S = 0) = P(N = 0)$.

When the number of claims is large, and the claim amount distribution is not too skewed, we can approximate the distribution of S by a normal distribution with mean $E(S)$ and variance $var(S)$. We explain how to calculate moments of S in the next section.

3.3 Moments of compound distributions

To calculate the moments of S, conditional expectation results are used, conditioning on the number of claims, N. To find $E[S]$, apply the identity:

$$\boldsymbol{E[S] = E[E[S|N]]}$$

Here we are using the conditional expectation formula, which is given on page 16 of the *Tables*.

Now $E[S|N=n] = \sum_{i=1}^{n} E[X_i] = nm_1$. Hence:

$$\boldsymbol{E[S|N] = Nm_1}$$

and:

$$\boldsymbol{E[S] = E[Nm_1] = E[N]m_1} \qquad \textbf{(19.4)}$$

This can also be written as follows:

$$E(S) = E(N)E(X)$$

where $E(X) = E(X_i)$, $i = 1,2,...,N$.

Formula (19.4) has a very natural interpretation. It says that the expected aggregate claim amount is the product of the expected number of claims and the expected individual claim amount.

This formula is also given on page 16 of the *Tables.*

Question

If X has a Pareto distribution with parameters $\lambda = 400$ and $\alpha = 3$, and N has a $Poisson(50)$ distribution, calculate the expected value of S.

Solution

The expected value is:

$$E(S) = E(N)E(X) = 50 \times \frac{400}{3-1} = 10,000$$

Variance

To find an expression for $\text{var}[S]$, apply the identity:

$$\text{var}[S] = E[\text{var}[S \mid N]] + \text{var}[E[S \mid N]]$$

Here we are using the conditional variance formula, which is given on page 16 of the *Tables*.

Since $E(S \mid N) = Nm_1$, we have:

$$\text{var}[S] = E[\text{var}[S \mid N]] + \text{var}[Nm_1]$$

$\text{var}[S \mid N]$ can be found by using the fact that individual claim amounts are independent.

Now:

$$\text{var}[S \mid N = n] = \text{var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \text{var}[X_i] = n(m_2 - m_1^2)$$

and so $\text{var}[S \mid N] = N(m_2 - m_1^2)$. Hence:

$$\text{var}[S] = E[N(m_2 - m_1^2)] + \text{var}[Nm_1]$$

ie: $$\text{var}[S] = E[N](m_2 - m_1^2) + \text{var}[N]m_1^2 \qquad \textbf{(19.5)}$$

Alternatively, writing this solely in terms of means and variances:

$$\text{var}(S) = E(N)\text{var}(X) + \text{var}(N)[E(X)]^2$$

where $\text{var}(X) = \text{var}(X_i)$, $i = 1, 2, \ldots, N$.

This formula can also be found on page 16 of the *Tables*. We will use it to determine the variances of the various compound distributions in the next few sections.

Unlike the expression for $E[S]$, formula (19.5) does not have a natural interpretation. The variance of S is expressed in terms of the mean and variance of both N and X_i.

However, this formula shows that the variability of the overall aggregate claim distribution is a function of both the variability in the number of claims and the variability in the claim amounts.

Moment generating function

The MGF of S is also found using conditional expectation. By definition, $M_S(t) = E[\exp(tS)]$, so:

$$M_S(t) = E\big[E[\exp(tS) \mid N]\big] \qquad \textbf{(19.6)}$$

Again, we are conditioning on the number of claims, exactly as we did before.

Now $E[\exp(tS)|N=n]=E[\exp(tX_1+tX_2+\ldots+tX_n)]$, and as $\{X_i\}_{i=1}^n$ are independent random variables:

$$E[\exp(tX_1+tX_2+\ldots+tX_n)]=\prod_{i=1}^{n}E[\exp(tX_i)]$$

Also, since $\{X_i\}_{i=1}^n$ are identically distributed, they have common MGF, $M_X(t)$, so that:

$$\prod_{i=1}^{n}E[\exp(tX_i)]=\prod_{i=1}^{n}M_X(t)=[M_X(t)]^n$$

Hence:

$$E[\exp(tS)\,|\,N]=[M_X(t)]^N \tag{19.7}$$

These conditional expectations are random variables because they are functions of N.

Hence, inserting (19.7) in (19.6):

$$M_S(t)=E[M_X(t)^N]=E[\exp(N\log M_X(t))]=M_N(\log M_X(t)) \tag{19.8}$$

We can see this last step by observing that $E[\exp(N\log M_X(t))]$ is of the same form as $E(e^{Nt})$ but with t replaced by $\log M_X(t)$. So it is the MGF of N evaluated at $\log M_X(t)$.

Again, this is given on page 16 of the *Tables*.

Thus, the MGF of S is expressed in terms of the MGFs of N and of X_i. As with the previous results, the distributions of neither N nor of X_i have been specified.

A summary of the results for the mean, variance and MGF of S is given below.

Mean, variance and MGF of *S*

$$E(S)=E(N)E(X)$$

$$\text{var}(S)=E(N)\text{var}(X)+\text{var}(N)[E(X)]^2$$

$$M_S(t)=M_N\left(\log M_X(t)\right)$$

There is one special case that is of some interest. This is when all claims are for the same fixed amount.

Example

Consider a portfolio of one-year term assurances each with the same sum assured. Suppose that the amount of a claim is B with probability one (assuming that a claim occurs at all), ***ie* $P(X_i=B)=1$ so that $m_1=B$ and $m_2=B^2$.**

B is a constant here, not a random variable. So the expected value of an individual claim amount is B and its variance is 0.

Then S is distributed on $0, B, 2B, \dots$. In fact, $S = BN$ so:

$$P(S \le Bx) = P(N \le x)$$

Formulae (19.4) and (19.5) give the mean and variance of S, but as $S = BN$ it is easier to note that $E[S] = E[N]B$ and $\text{var}[S] = \text{var}[N]B^2$.

The next three sections consider compound distributions using various models for the number of claims, N.

3.4 The compound Poisson distribution

First consider aggregate claims when N has a Poisson distribution with mean λ denoted $N \sim Poi(\lambda)$. S then has a compound Poisson distribution with parameter λ, and $F(x)$ is the CDF of the individual claim amount random variable.

S is sometimes referred to as a compound Poisson random variable.

The results required for this distribution for N are:

$$E[N] = \text{var}[N] = \lambda$$

$$M_N(t) = \exp\left[\lambda(e^t - 1)\right]$$

Note that these results are given in the *Tables*.

These results can be combined with those of Section 3.1 as follows.

From (19.4):

$$E(S) = E(N)E(X) = \lambda E(X)$$

ie:

$$E[S] = \lambda m_1 \qquad \textbf{(19.9)}$$

From (19.5):

$$\text{var}(S) = E(N)\,\text{var}(X) + \text{var}(N)[E(X)]^2 = \lambda\,\text{var}(X) + \lambda[E(X)]^2 = \lambda E(X^2)$$

ie:

$$\text{var}[S] = \lambda m_2 \qquad \textbf{(19.10)}$$

and from (19.8):

$$M_S(t) = \exp\left[\lambda(M_X(t) - 1)\right] \qquad \textbf{(19.11)}$$

The results for the mean and variance have a very simple form. Note that the variance of S is expressed in terms of the second moment of X_i about zero (and not in terms of the variance of X_i).

Note also that the formula for the skewness of S has a simple form when S is a compound Poisson random variable:

$$skew[S] = \lambda m_3 \tag{19.12}$$

***ie*:**

$$skew(S) = \lambda E(X^3)$$

The easiest way to show that the third central moment of S is λm_3 is to use the cumulant generating function:

$$C_S(t) = \log M_S(t)$$

To determine the skewness, we differentiate it three times with respect to t and set $t = 0$, *ie*:

$$E[(S - \lambda m_1)^3] = \left.\frac{d^3}{dt^3}\log M_S(t)\right|_{t=0}$$

In other words:

$$skew(S) = C_S'''(0)$$

Recall also that:

$$E(S) = C_S'(0)$$

$$\text{var}(S) = C_S''(0)$$

Since $M_S(t) = \exp\left[\lambda(M_X(t)-1)\right]$, it follows that:

$$\log M_S(t) = \lambda\left(M_X(t) - 1\right)$$

So:

$$\frac{d^3}{dt^3}\log M_S(t) = \lambda\left[\frac{d^3}{dt^3}M_X(t) - 1\right]_{t=0} = \lambda m_3$$

ie $\quad E[(S - \lambda m_1)^3] = \lambda m_3$

This is because $M_X'''(0) = E(X^3)$ for any random variable.

The coefficient of skewness of S is given by:

$$\frac{skew(S)}{[\text{var}(S)]^{3/2}}$$

Hence the coefficient of skewness $= \lambda m_3 / (\lambda m_2)^{3/2}$.

This result shows that the distribution of S is positively skewed, since m_3 is the third moment about zero of X_i and hence is greater than zero because X_i is a non-negative valued random variable. Note that the distribution of S is positively skewed even if the distribution of X_i is negatively skewed. The coefficient of skewness of S is $\lambda m_3 / (\lambda m_2)^{3/2}$, and hence goes to 0 as $\lambda \to \infty$. Thus for large values of λ, the distribution of S is almost symmetric.

Mean, variance and skewness of a compound Poisson random variable

If $N \sim Poisson(\lambda)$, then S is a compound Poisson random variable and:

$$E(S) = \lambda E(X) = \lambda m_1$$

$$\text{var}(S) = \lambda E(X^2) = \lambda m_2$$

$$skew(S) = \lambda E(X^3) = \lambda m_3$$

These results are all given on page 16 of the *Tables*.

Sums of independent compound Poisson random variables

A very useful property of the compound Poisson distribution is that the sum of independent compound Poisson random variables is itself a compound Poisson random variable. A formal statement of this property is as follows.

Let $S_1, S_2, ..., S_n$ be independent random variables. Suppose that each S_i has a compound Poisson distribution with parameter λ_i, and that the CDF of the individual claim amount random variable for each S_i is $F_i(x)$.

Define $A = S_1 + S_2 + \cdots + S_n$. Then A has a compound Poisson distribution with parameter Λ, and $F(x)$ is the CDF of the individual claim amount random variable for A, where:

$$\Lambda = \sum_{i=1}^{n} \lambda_i \quad \text{and} \quad F(x) = \frac{1}{\Lambda} \sum_{i=1}^{n} \lambda_i F_i(x)$$

Recall that Λ is the capital form of the Greek letter λ.

This is a very important result.

To prove the result, first note that $F(x)$ is a weighted average of distribution functions and that these weights are all positive and sum to one. This means that $F(x)$ is a distribution function and this distribution has MGF:

$$M(t) = \int_0^{\infty} e^{tx} f(x)\,dx$$

where $f(x) = F'(x)$ is the PDF of the individual claim amount random variable for A.

So:

$$\boldsymbol{M(t) = \int_0^{\infty} \exp(tx) \frac{1}{\Lambda} \sum_{i=1}^{n} \lambda_i f_i(x)\,dx}$$

where $f_i(x)$ is the density of $F_i(x)$. Hence:

$$\boldsymbol{M(t) = \frac{1}{\Lambda} \sum_{i=1}^{n} \lambda_i \int_0^{\infty} \exp\{tx\} f_i(x)\,dx = \frac{1}{\Lambda} \sum_{i=1}^{n} \lambda_i M_i(t)} \qquad \textbf{(19.13)}$$

where $M_i(t)$ is the MGF for the distribution with CDF $F_i(x)$.

Let $M_A(t)$ denote the MGF of A. Then:

$$\boldsymbol{M_A(t) = E[\exp(tA)] = E[\exp(tS_1 + tS_2 + \cdots + tS_n)]}$$

By independence of $\{S_i\}_{i=1}^{n}$:

$$\boldsymbol{M_A(t) = \prod_{i=1}^{n} E(\exp(tS_i))}$$

As S_i is a compound Poisson random variable, its MGF is of the form given by formula (19.11), so:

$$\boldsymbol{E[\exp(tS_i)] = \exp\left[\lambda_i (M_i(t) - 1)\right]}$$

Thus:

$$\boldsymbol{M_A(t) = \prod_{i=1}^{n} \exp\{\lambda_i (M_i(t) - 1)\} = \exp\left\{\sum_{i=1}^{n} \lambda_i (M_i(t) - 1)\right\}}$$

***ie*:**

$$\boldsymbol{M_A(t) = \exp\{\Lambda(M(t) - 1)\}} \qquad \textbf{(19.14)}$$

where:

$$\boldsymbol{\Lambda = \sum_{i=1}^{n} \lambda_i \quad \text{and} \quad M(t) = \frac{1}{\Lambda} \sum_{i=1}^{n} \lambda_i M_i(t)}$$

By the one-to-one relationship between distributions and MGFs, formula (19.14) shows that A has a compound Poisson distribution with Poisson parameter Λ. By (19.13), the individual claim amount distribution has CDF $F(x)$.

Question

The distributions of aggregate claims from two risks, denoted by S_1 and S_2, are as follows:

- S_1 has a compound Poisson distribution with parameter 100 and distribution function $F_1(x) = 1 - \exp(-x/\alpha)$, $x > 0$.
- S_2 has a compound Poisson distribution with parameter 200 and distribution function $F_2(x) = 1 - \exp(-x/\beta)$, $x > 0$.

Assuming that S_1 and S_2 are independent, determine the distribution of $S_1 + S_2$.

Solution

Let $S = S_1 + S_2$. Then S has a compound Poisson distribution with parameters $\Lambda = 300$ and $F(x)$, where:

$$F(x) = \tfrac{1}{3}F_1(x) + \tfrac{2}{3}F_2(x) = 1 - \tfrac{1}{3}\exp(-x/\alpha) - \tfrac{2}{3}\exp(-x/\beta)$$

We can use R to simulate values from a compound Poisson distribution.

The R code to simulate 10,000 values from a compound Poisson distribution with parameter 1,000 and a gamma claims distribution with $\alpha = 750$ and $\lambda = 0.25$ is:

```
set.seed(123)
n <- rpois(10000,1000)
s <- numeric(10000)
for(i in 1:10000)
{x <- rgamma(n[i],shape=750,rate=0.25)
s[i] <- sum(x)}
```

We can obtain a mean of 2,997,651, a standard deviation of 93,719.71 and a coefficient of skewness of 0.02655921 as follows:

```
mean(s)
sd(s)
skewness<-sum((s-mean(s))^3)/length(s)
coeff.of.skew<-skewness/var(s)^(3/2)
```

We can estimate P(S>3,000,000) to be 0.4881 as follows:

```
length(s[s>3000000])/length(s)
```

Finally we could estimate the 90th percentile to be 3,115,719 as follows::

```
quantile(s,0.9)
```

We can plot a histogram of the compound distribution using the `hist` **function and an empirical density function using** `density` **in the** `plot` **function. We can then superimpose a normal or other distribution to see if they provide a good approximation.**

However, a better way to check the fit with a normal distribution is to use the `qqnorm` **function:**

```
qqnorm(<simulated values>)
```

or the `qqplot` **function to compare the sample data to simulated values from a fitted model distribution:**

```
qqplot(<simulated theoretical values>,
<simulated compound distribution values>)
```

Note we have used `set.seed(123)` **so you can obtain the same values as this example.**

3.5 The compound binomial distribution

Under certain circumstances, the binomial distribution is a natural choice for N. For example, under a group life insurance policy covering n lives, the distribution of the number of deaths in a year is binomial if it is assumed that each insured life is subject to the same mortality rate, and that lives are independent with respect to mortality.

The notation $N \sim Bin(n,p)$ is used to denote the binomial distribution for N. The key results for this distribution are:

$$E[N] = np$$

$$\text{var}[N] = np(1-p)$$

$$M_N(t) = (pe^t + 1 - p)^n$$

Note that these results are given in the *Tables*.

However, the notation for the MGF is slightly different.

When N has a binomial distribution, S has a compound binomial distribution. One important point about choosing the binomial distribution for N is that there is an upper limit, n, to the number of claims.

Expressions for the mean, variance and MGF of S are now found in terms of n, p, m_1, m_2 and $M_X(t)$ when $N \sim Bin(n,p)$.

There is no need to memorise the formulae in this section. However, it is important to be able to derive them.

Formula (19.4) gives the mean:

$$E(S) = E(N)E(X)$$

$$\Rightarrow E[S] = npm_1 \qquad \textbf{(19.15)}$$

Formula (19.5) gives the variance:

$$\text{var}(S) = E(N)\,\text{var}(X) + \text{var}(N)[E(X)]^2$$

$$\Rightarrow \textbf{var}[S] = np(m_2 - m_1^2) + np(1-p)\, m_1^2$$

$$= npm_2 - np^2 m_1^2 \qquad \textbf{(19.16)}$$

Lastly, formula (19.8) gives the MGF:

$$M_S(t) = M_N\left(\log M_X(t)\right)$$

$$\Rightarrow M_S(t) = (pM_X(t) + 1 - p)^n$$

We can also find expressions for the skewness and the coefficient of skewness.

The third central moment is found from the cumulant generating function:

$$C_S(t) = \ln M_S(t)$$

In the next few steps, liberal use is made of the chain rule $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$ and the product rule for differentiation $\left(\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}\right)$. The third derivative of the cumulant generating function is:

$$\frac{d^3}{dt^3}\log M_S(t) = \frac{d^3}{dt^3} n\log\left(pM_X(t) + q\right) \quad \textbf{where } q = 1-p$$

$$= \frac{d^2}{dt^2}\left\{np\left(\frac{d}{dt}M_X(t)\right)(pM_X(t)+q)^{-1}\right\}$$

$$= \frac{d}{dt}\left\{np\left(\frac{d^2}{dt^2}M_X(t)\right)(pM_X(t)+q)^{-1} - n\left(p\frac{d}{dt}M_X(t)\right)^2(pM_X(t)+q)^{-2}\right\}$$

$$= np\left(\frac{d^3}{dt^3}M_X(t)\right)(pM_X(t)+q)^{-1}$$

$$- 3np^2\left(\frac{d^2}{dt^2}M_X(t)\right)(pM_X(t)+q)^{-2}\left(\frac{d}{dt}M_X(t)\right)$$

$$+ 2n\left(p\frac{d}{dt}M_X(t)\right)^3(pM_X(t)+q)^{-3}$$

Setting $t = 0$ gives:

$$skew(S) = E[(S - npm_1)^3] = npm_3 - 3np^2 m_2 m_1 + 2np^3 m_1^3 \qquad \textbf{(19.17)}$$

The coefficient of skewness is then given by:

$$\frac{skew(S)}{[\text{var}(S)]^{3/2}} = \frac{npm_3 - 3np^2 m_2 m_1 + 2np^3 m_1^3}{(npm_2 - np^2 m_1^2)^{3/2}}$$

It can be deduced from formula (5.17) that it is possible for the compound binomial distribution to be negatively skewed. The simplest illustration of this fact is when all claims are of (a fixed) **amount B. Then $S = BN$ and:**

$$E\left[(S-E[S])^3\right] = B^3 E\left[(N-E[N])^3\right]$$

ie:

$$skew(S) = B^3 skew(N)$$

So the coefficient of skewness of S is a multiple of that for N.

In fact:

$$coeff\ of\ skew(S) = \frac{skew(S)}{[\text{var}(S)]^{3/2}} = \frac{B^3 skew(N)}{[B^2\ \text{var}(N)]^{3/2}} = coeff\ of\ skew(N)$$

If $p > 0.5$, then the binomial distribution for N is negatively skewed.

So the coefficient of skewness of S will also be negative in this case.

Question

Determine an expression for the MGF of the aggregate claim amount random variable if the number of claims has a $Bin(100, 0.01)$ distribution and individual claim sizes have a $Gamma(10, 0.2)$ distribution.

Solution

Since $N \sim Bin(100, 0.01)$ and $X \sim Gamma(10, 0.2)$, we have:

$$M_N(t) = (0.99 + 0.01e^t)^{100} \quad \text{and} \quad M_X(t) = \left(1 - \frac{t}{0.2}\right)^{-10} = (1-5t)^{-10}$$

So:

$$\begin{aligned} M_S(t) &= M_N[\log M_X(t)] \\ &= \left[0.99 + 0.01e^{\log M_X(t)}\right]^{100} \\ &= \left[0.99 + 0.01M_X(t)\right]^{100} \\ &= \left[0.99 + 0.01(1-5t)^{-10}\right]^{100} \end{aligned}$$

3.6 The compound negative binomial distribution

The final choice of distribution for N is the negative binomial distribution, which has probability function:

$$P(N=n)=\binom{k+n-1}{n}p^k q^n \quad \text{for } n=0,1,2,\ldots$$

This is the Type 2 negative binomial distribution. See page 9 of the *Tables*.

The Type 1 negative binomial distribution has probability function:

$$P(N=n)=\binom{n-1}{k-1}p^k q^{n-k} \quad \text{for } n=k,\ k+1,\ k+2,\ \ldots$$

It is not likely to be appropriate here, unless there is a specific reason why the number of claims must be at least k.

The parameters of the distribution are k (>0) and p, where $p+q=1$ and $0<p<1$. This distribution is denoted by $NB(k,p)$. When $N \sim NB(k,p)$:

$$E[N]=\frac{kq}{p}$$

$$\text{var}[N]=\frac{kq}{p^2}$$

$$M_N(t)=p^k(1-qe^t)^{-k}$$

The special case $k=1$ leads to the geometric distribution. Once again, note that these results are given in the *Tables*.

The negative binomial distribution is an alternative to the Poisson distribution for N.

This is because the negative binomial distribution can take any non-negative integer value, unlike the binomial distribution which has an upper limit.

One advantage that the negative binomial distribution has over the Poisson distribution is that its variance exceeds its mean. These two quantities are equal for the Poisson distribution. Thus, the negative binomial distribution may give a better fit to a data set which has a sample variance in excess of the sample mean. This is often the case in practice. In the next chapter a situation leading to the negative binomial distribution for N is discussed. When N has a negative binomial distribution, S has a compound negative binomial distribution.

Expressions for the mean, variance and MGF of S when $N \sim NB(k,p)$ come immediately from formulae (19.4), (19.5) and (19.8):

$$E(S)=E(N)E(X) \Rightarrow \mathbf{E[S]=\frac{kq}{p}m_1}$$

$$\text{var}(S)=E(N)\text{var}(X)+\text{var}(N)[E(X)]^2 \Rightarrow \mathbf{var[S]=\frac{kq}{p}(m_2-m_1^2)+\frac{kq}{p^2}m_1^2}$$

Multiplying out the brackets and regrouping the terms, we see that:

$$\begin{aligned}\frac{kq}{p}(m_2-m_1^2)+\frac{kq}{p^2}m_1^2 &= \frac{kq}{p}m_2-\frac{kq}{p}m_1^2+\frac{kq}{p^2}m_1^2\\ &=\frac{kq}{p}m_2-\frac{kpq}{p^2}m_1^2+\frac{kq}{p^2}m_1^2\\ &=\frac{kq}{p}m_2+\frac{kq-kpq}{p^2}m_1^2\\ &=\frac{kq}{p}m_2+\frac{kq(1-p)}{p^2}m_1^2\\ &=\frac{kq}{p}m_2+\frac{kq^2}{p^2}m_1^2\end{aligned}$$

So:

$$\mathbf{var[S]=\frac{kq}{p}m_2+\frac{kq^2}{p^2}m_1^2}$$

and:

$$M_S(t)=M_N\left(\log M_X(t)\right) \Rightarrow \mathbf{M_S(t)=\frac{p^k}{(1-qM_X(t))^k}}$$

As before, the third central moment of S can be found from the cumulant generating function of S, as follows:

$$\begin{aligned}\frac{d}{dt}\log M_S(t) &= \frac{d}{dt}\left(k\log p-k\log\left[1-qM_X(t)\right]\right)\\ &=\frac{kq}{1-qM_X(t)}\left(\frac{d}{dt}M_X(t)\right)\end{aligned}$$

Then:

$$\frac{d^2}{dt^2}\log M_S(t) = kq^2\left(\frac{d}{dt}M_X(t)\right)^2 \frac{1}{(1-qM_X(t))^2} + \frac{kq}{1-qM_X(t)}\left(\frac{d^2}{dt^2}M_X(t)\right)$$

and:

$$\frac{d^3}{dt^3}\log M_S(t) = 3kq^2\left(\frac{d}{dt}M_X(t)\right)\left(\frac{d^2}{dt^2}M_X(t)\right)\frac{1}{(1-qM_X(t))^2}$$

$$+\frac{2kq^3}{(1-qM_X(t))^3}\left(\frac{d}{dt}M_X(t)\right)^3 + \frac{kq}{1-qM_X(t)}\left(\frac{d^3}{dt^3}M_X(t)\right)$$

Setting $t=0$ in the third derivative gives:

$$skew(S) = E[(S-E[S])^3] = \frac{3kq^2 m_1\, m_2}{p^2} + \frac{2kq^3\, m_1^3}{p^3} + \frac{kqm_3}{p} \qquad \textbf{(19.18)}$$

The parameters k and p are positive, as are the moments of X. It therefore follows from formula (19.18) that the compound negative binomial distribution is positively skewed. The coefficient of skewness can be found from $E((S-E(S))^3)/(\text{var}(S))^{3/2}$.

Question

The distribution of the number of claims from a motor portfolio is negative binomial with parameters $k=4{,}000$ and $p=0.9$. The claim size distribution is Pareto with parameters $\alpha=5$ and $\lambda=1{,}200$. Calculate the mean and standard deviation of the aggregate claim distribution.

Solution

The first two moments of the Pareto distribution are:

$$m_1 = E(X) = \frac{\lambda}{\alpha-1} = \frac{1{,}200}{4} = 300$$

$$m_2 = E(X^2) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} + m_1^2 = \frac{5\times 1{,}200^2}{4^2\times 3} + 300^2 = 240{,}000$$

So, using the formulae for the mean and variance of a compound negative binomial distribution:

$$E(S) = \frac{kq}{p}\times m_1 = \frac{4{,}000\times 0.1}{0.9}\times 300 = 133{,}333$$

$$\text{var}(S) = \frac{kq}{p}m_2 + \frac{kq^2}{p^2}m_1^2 = \frac{4{,}000\times 0.1}{0.9}\times 240{,}000 + \frac{4{,}000\times 0.1^2}{0.9^2}\times 300^2 = 111{,}111{,}111$$

So the standard deviation is 10,541.

4 Appendix

There is some repetition in the Core Reading in Section 3.3. To improve the flow of the chapter, we have removed the repeated section from the main part of the text and placed it below.

Let $S = X_1 + X_2 + \dots + X_N$ (and $S = 0$ if $N = 0$) where the X_i's are independent, identically distributed (as a variable X) and are also independent of N. S is said to have a *compound distribution*.

***Illustration*: N is the number of claims which arise in a portfolio of business and X_i is the amount of the ith claim. S is the total claim amount.**

The mean and variance of S are easily found:

$$E(S \mid N = n) = E(X_1 + \dots + X_N \mid N = n) = E(X_1 + \dots + X_n) = nE(X)$$

Similarly:

$$\text{var}(S \mid N = n) = n\,\text{var}(X)$$

Therefore:

$$E(S) = E[E(S \mid N)] = E[NE(X)] = E(N)E(X)$$

***ie*:**

$$\mu_S = \mu_N \mu_X$$

and:

$$\begin{aligned}\text{var}(S) &= E[\text{var}(S \mid N)] + \text{var}[E(S \mid N)] \\ &= E[N\,\text{var}(X)] + \text{var}[NE(X)] \\ &= E(N)\,\text{var}(X) + \text{var}(N)[E(X)]^2\end{aligned}$$

***ie*:**

$$\sigma_S^2 = \mu_N \sigma_X^2 + \sigma_N^2 \mu_X^2$$

The MGF of S is given by:

$$M_S(t) = E(e^{tS}) = E\left[E\left(e^{tS} \mid N\right)\right]$$

and:

$$\begin{aligned}E(e^{tS} \mid N = n) &= E\left[\exp\left(t(X_1 + X_2 + \dots + X_N)\right) \mid N = n\right] \\ &= E\left[\exp\left(t(X_1 + X_2 + \dots + X_n)\right)\right] \\ &= \prod E[\exp(tX_i)] = [M_X(t)]^n\end{aligned}$$

Therefore:

$$M_S(t) = E\left[\left(M_X(t)\right)^N\right] = E\left[\exp\left(N \log M_X(t)\right)\right] = M_N\left(\log M_X(t)\right)$$

Compound Poisson distribution

An important illustration is provided by the compound Poisson distribution, which is the case in which $N \sim Poisson(\lambda)$. In this case $\mu_N = \sigma_N^2 = \lambda$.

***Properties*:**

$$E(S) = \lambda E(X)$$

$$\text{var}(S) = \lambda \,\text{var}(X) + \lambda\left[E(X)\right]^2 = \lambda E(X^2)$$

$$M_N(t) = \exp\left[\lambda(e^t - 1)\right]$$

so:

$$M_S(t) = \exp\left[\lambda\left(M_X(t) - 1\right)\right]$$

from which the mean and variance can be obtained and the results above verified.

Chapter 19 Summary

Insurable risks

For a risk to be insurable the policyholder should have an interest in the risk being insured to distinguish between insurance and a wager, and it should be of a financial and reasonably quantifiable nature. Ideally, risk events should:

- be independent
- have low probability of occurring
- be pooled with similar risks
- have an ultimate liability
- avoid moral hazards.

Characteristics of general insurance products

Most general insurance contracts share the following characteristics:

- Cover is normally for a fixed period, typically a year, after which it needs to be renegotiated.
- There is usually no obligation to continue cover although in most cases a need for continuing cover may be assumed to exist.
- Claims are not of fixed amounts.
- The existence of a claim and its amount have to be proved before a claim can be settled.
- A claim occurring does not bring the policy to an end.
- Claims that take a long time to settle are known as long-tailed and those that take a short time to settle are known as short-tailed.

Features of short-term insurance contracts

A short-term insurance contract can be defined as having the following attributes:

- The policy lasts for a fixed, and relatively short time period, typically one year.
- The insurance company receives a premium from the policyholder.
- In return the insurer pays claims that arise during the term of the policy.
- At the end of the policy term, the policyholder may or may not renew the policy. If it is renewed, the premium may or may not be the same as in the previous period.
- The insurer may pass part of the premium to a reinsurer, who, in return, will reimburse the insurer for part of the claims cost.

Collective risk model

Aggregate claim amounts may be modelled using a compound distribution. The aggregate claim amount S is the sum of a random number of IID random variables:

$$S = X_1 + X_2 + \cdots + X_N$$

where S is taken to be zero if $N = 0$. We assume that the random variable N is independent of the random variables X_i so that the distributions of the claim numbers and the individual claim amounts can be analysed separately. The distribution of S is said to be a compound distribution.

Other simplifying assumptions include:

- The moments (and sometimes the distributions) of N and X_i are known.
- Claims are settled more or less as soon as the claims occur.
- Expenses and investment returns are ignored.

Specific types of compound distributions include the compound Poisson, compound binomial, compound negative binomial, and compound geometric. Formulae for the MGF and the moments of a compound random variable are given on page 16 of the *Tables*.

Convolutions

If $Z = X + Y$, and X and Y are independent, then:

$$P(Z = z) = \sum_x P(X = x)P(Y = z - x) \text{ if } X \text{ and } Y \text{ are discrete}$$

$$f_Z(z) = f_X * f_Y(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\,dx \text{ if } X \text{ and } Y \text{ are continuous}$$

Sums of independent compound Poisson random variables

Let $S_1, S_2, \ldots, S_n$ be a set of independent random variables where S_i has a compound Poisson distribution with parameter λ_i and $F_i(x)$ is the CDF of the individual claim amount random variable for S_i. Then $A = S_1 + \cdots + S_n$ is compound Poisson with parameter $\Lambda = \sum \lambda_i$. The CDF of the individual claim amount random variable for A is:

$$F(x) = \frac{1}{\Lambda} \sum \lambda_i \, F_i(x)$$

The MGF of the individual claim amounts for A is:

$$M(t) = \frac{1}{\Lambda} \sum \lambda_i M_i(t)$$

where $M_i(t)$ is the MGF of the individual claim amounts for S_i.

Chapter 19 Practice Questions

19.1 (i) State the two conditions that must hold for a risk to be insurable.

(ii) List five other risk criteria that would be considered desirable by a general insurer.

19.2 A group of policies can give rise to at most 2 claims. The probabilities of 0, 1 or 2 claims are ½, ¼ and ¼ respectively. Claim amounts are IID $U(0,10)$ random variables. Let S denote the aggregate claim amount random variable.

Sketch the frequency distribution of S.

19.3 The random variable S has a compound Poisson distribution with Poisson parameter 4. The individual claim amounts are either 1, with probability 0.3, or 3, with probability 0.7.

Calculate the probability that $S=4$.

19.4 A compound random variable $S=X_1+X_2+\cdots+X_N$ has claim number distribution:

$$P(N=n)=9(n+1)4^{-n-2}\ , \quad n=0,1,2,\ldots$$

The individual claim size random variable, X, is exponentially distributed with mean 2.

Calculate $E(S)$ and $\text{var}(S)$.

19.5 Write down a formula for the MGF of a compound Poisson distribution with individual claim size distribution $Gamma(\alpha,\beta)$ and Poisson parameter λ.

19.6 S_1 and S_2 are independent random variables each with a compound Poisson distribution. The distribution of S_i, $i=1,2$, has Poisson parameter λ_i and individual claim amount distribution $F_i(x)$.

Which one of the following statements about the distribution of S_1+S_2 is correct?

A S_1+S_2 has a compound Poisson distribution with Poisson parameter $\lambda_1\lambda_2$ and individual claim amount distribution $F_1(x)+F_2(x)$.

B S_1+S_2 has a compound Poisson distribution with Poisson parameter $(\lambda_1+\lambda_2)$ and individual claim amount distribution $\left(F_1(x)+F_2(x)\right)/2$.

C S_1+S_2 has a compound Poisson distribution with Poisson parameter $(\lambda_1+\lambda_2)$ and individual claim amount distribution $\left(\lambda_1 F_1(x)+\lambda_2 F_2(x)\right)/\left(\lambda_1+\lambda_2\right)$.

D S_1+S_2 does not have a compound Poisson distribution.

19.7 Claims on a group of policies of a certain type arise as a Poisson process with parameter λ_1. Claims on a second, independent, group of policies arise as a Poisson process with parameter λ_2. The aggregate claim amounts on the respective groups are denoted S_1 and S_2.

Exam style

Using moment generating functions (or otherwise), show that S (the sum of S_1 and S_2) also has a compound Poisson distribution and hence derive the Poisson parameter for S. [4]

19.8 The aggregate claim amount from a portfolio has a compound negative binomial distribution.

Exam style

(i) Show that if $S = X_1 + \cdots + X_N$, then:

$$M_S(t) = M_N\left[\log M_X(t)\right]$$ [3]

(ii) If N has Type 2 negative binomial distribution with $k = 2$ and $p = 0.9$, and X has a gamma distribution with $\alpha = 10$ and $\lambda = 0.1$, determine an expression for $M_S(t)$. [2]

(iii) (a) Calculate the mean and variance of S.

(b) Using a suitable approximation, estimate the aggregate amount which will be exceeded with probability 0.1%. [4]

(iv) The insurer in fact has 100 identical independent portfolios of this type. Let:

$$T = S_1 + \cdots + S_{100}$$

(a) Determine the moment generating function for T.

(b) Using a normal approximation, estimate the total aggregate claim amount from the whole business which will be exceeded with probability 0.1%.

(c) Comment on your answers to parts (iii)(b) and (iv)(b). [4]

[Total 13]

Chapter 19 Solutions

19.1 (i) ***Criteria for an insurable risk***

The two conditions are:

- The policyholder must have an interest in the risk being insured.
- The risk must be of a financial and reasonably quantifiable nature.

(ii) ***Other desirable features of a risk***

Other desirable features are:

- Individual risks should be independent of one another.
- The probability that the insured event will occur should be small.
- Large numbers of similar risks should be pooled in order to reduce the variance and achieve greater certainty.
- The insurer's liability should be limited.
- Moral hazards should be eliminated as far as possible since these are difficult to quantify, result in selection against the insurer and lead to unfairness in the treatment of some policyholders.

19.2 If $N=0$, *ie* there are no claims, then $S=0$. So there is a point mass (or a 'blob' of probability) at $S=0$.

If $N=1$, *ie* there is exactly one claim, then S has a $U(0,10)$ distribution. This will happen with probability ¼.

If $N=2$, *ie* there are exactly 2 claims, then $S=X_1+X_2$ where X_1 and X_2 are independent random variables with PDFs:

$$f_{X_1}(x)=f_{X_2}(x)=\begin{cases}0.1 & \text{if } 0\le x\le 10\\ 0 & \text{otherwise}\end{cases}$$

The PDF of S is:

$$f_S(s)=\int_{-\infty}^{\infty} f_{X_1}(s-x)f_{X_2}(x)\,dx=0.1\int_0^{10} f_{X_1}(s-x)\,dx$$

The integrand is 0.1 if $0\le s-x\le 10$ and 0 otherwise. So for $0\le s\le 10$:

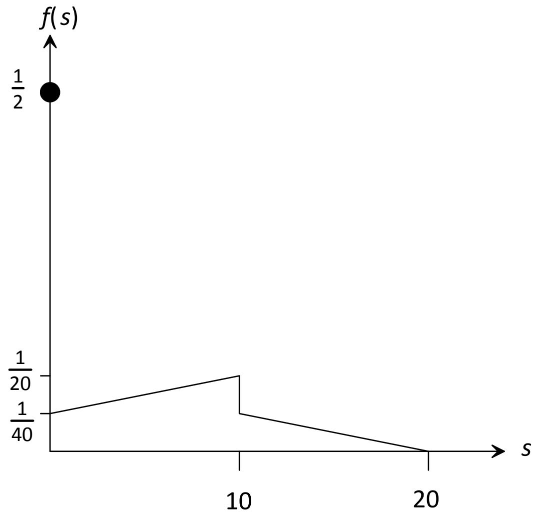
$$f_S(s)=0.1\int_0^{s} 0.1\,dx=0.01s$$

and for $10\le s\le 20$:

$$f_S(s)=0.1\int_{s-10}^{10} 0.1\,dx=0.2-0.01s$$

So for $N=2$, S has a symmetrical triangular shaped distribution on the interval (0, 20).

A graph of the distribution is shown below:



This graph is the combination of a blob at zero, a uniform distribution on (0,10) and a triangular distribution on (0,20).

19.3 We need to consider how we could get an aggregate claim amount of 4. This could happen in two ways:

(a) 2 claims, one for 1 and one for 3

(b) 4 claims, all for an amount of 1.

The probability of this happening is:

$$P(S=4)=P(N=2)P(X_1=1)P(X_2=3)+P(N=2)P(X_1=3)P(X_2=1)$$
$$+P(N=4)P(X_1=1)P(X_2=1)P(X_3=1)P(X_4=1)$$

Since the X_i's are identical, this simplifies to:

$$P(S=4)=2P(N=2)P(X=1)P(X=3)+P(N=4)[P(X=1)]^4$$
$$=2\times\frac{e^{-4}4^2}{2!}\times 0.3\times 0.7+\frac{e^{-4}4^4}{4!}\times 0.3^4$$
$$=0.06312$$

19.4 The probability function of N can be written as:

$$P(N=n)=9(n+1)4^{-n-2}=\binom{n+1}{n}(3/4)^2(1/4)^n$$

We can see from this formula that N has a Type 2 negative binomial distribution with parameters $k=2$ and $p=3/4$.

Hence:

$$E(N) = \frac{kq}{p} = \frac{2 \times 1/4}{3/4} = 2/3$$

and: $$\text{var}(N) = \frac{kq}{p^2} = \frac{2 \times 1/4}{(3/4)^2} = 8/9$$

The individual claim amounts have an exponential distribution with $\lambda = ½$. So the mean and variance of the individual claims are:

$$E(X) = \frac{1}{\lambda} = 2 \quad \text{and} \quad \text{var}(X) = \frac{1}{\lambda^2} = 4$$

Hence:

$$E(S) = E(N)\ E(X) = \frac{2}{3} \times 2 = \frac{4}{3}$$

and: $$\text{var}(S) = E(N)\ \text{var}(X) + \text{var}(N)[E(X)]^2 = \frac{2}{3} \times 4 + \frac{8}{9} \times 2^2 = \frac{56}{9}$$

19.5 Since $N \sim Poisson(\lambda)$:

$$M_N(u) = \exp[\lambda(e^u - 1)]$$

So:

$$M_S(t) = M_N[\log M_X(t)] = \exp\left[\lambda\left(e^{\log M_X(t)} - 1\right)\right] = \exp\left[\lambda\left(M_X(t) - 1\right)\right]$$

Since $X \sim Gamma(\alpha, \beta)$:

$$M_X(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha}$$

So:

$$M_S(t) = \exp\left[\lambda\left(\left(1 - \frac{t}{\beta}\right)^{-\alpha} - 1\right)\right]$$

19.6 Option C is correct.

$S_1 + S_2$ has a compound Poisson distribution with Poisson parameter $(\lambda_1 + \lambda_2)$ and individual claim amount distribution $\left(\lambda_1 F_1(x) + \lambda_2 F_2(x)\right) / \left(\lambda_1 + \lambda_2\right)$.

19.7 Let N_i denote the number of claims on policies of type i and let X_i denote the claim amount random variable for policies of type i, for $i=1,2$. Then:

$$M_{S_i}(t)=M_{N_i}\left(\ln M_{X_i}(t)\right)=\exp\left(\lambda_i\left(M_{X_i}(t)-1\right)\right)$$

By independence:

$$M_S(t)=E\left(e^{tS}\right)=E\left(e^{t(S_1+S_2)}\right)=E\left(e^{tS_1}e^{tS_2}\right)=E\left(e^{tS_1}\right)E\left(e^{tS_2}\right)=M_{S_1}(t)M_{S_2}(t) \qquad [1]$$

Hence:

$$\begin{aligned} M_S(t)&=\exp\left(\lambda_1\left(M_{X_1}(t)-1\right)\right)\exp\left(\lambda_2\left(M_{X_2}(t)-1\right)\right) \\ &=\exp\left(\lambda_1 M_{X_1}(t)+\lambda_2 M_{X_2}(t)-(\lambda_1+\lambda_2)\right) \\ &=\exp\left(\lambda(M_W(t)-1)\right) \qquad [1] \end{aligned}$$

where:

$$\lambda=\lambda_1+\lambda_2 \qquad [1]$$

and:

$$M_W(t)=\frac{\lambda_1 M_{X_1}(t)+\lambda_2 M_{X_2}(t)}{\lambda_1+\lambda_2} \qquad [1]$$

Hence S is a compound Poisson random variable with Poisson parameter $\lambda=\lambda_1+\lambda_2$.

19.8 (i) ***MGF of S***

The MGF of S is:

$$M_S(t)=E(e^{tS})=E\left[E(e^{tS}\mid N)\right]$$

using the standard result for conditional means from page 16 of the *Tables*.

Looking at the inner expression, we have:

$$E(e^{tS}\mid N=n)=E\left(e^{t(X_1+\cdots+X_n)}\right)=E(e^{tX_1})\ldots E(e^{tX_n}) \qquad [1]$$

Now each of these terms is just the MGF of the random variable X.

So:

$$E(e^{tS} \mid N=n) = \left[M_X(t)\right]^n$$

and hence:

$$E(e^{tS} \mid N) = [M_X(t)]^N \qquad [1]$$

So:

$$M_S(t) = E\left[\{M_X(t)\}^N\right] = E\left(e^{N\log M_X(t)}\right) = M_N\left[\log M_X(t)\right] \qquad [1]$$

(ii) ***Compound negative binomial distribution***

We first need the individual MGFs. Using results from the *Tables*, we have:

$$M_X(t) = \left(1-\frac{t}{0.1}\right)^{-10} = (1-10t)^{-10} \qquad [½]$$

and: $$M_N(t) = \left(\frac{0.9}{1-0.1e^t}\right)^2 \qquad [½]$$

Combining these, using the result from part (i):

$$M_S(t) = M_N\left[\log M_X(t)\right] = \left(\frac{0.9}{1-0.1(1-10t)^{-10}}\right)^2 \qquad [1]$$

(iii)(a) ***Mean and variance of S***

We could differentiate this expression to find the mean and variance of S. However, it is much easier to use the standard compound distribution formulae:

$$E(S) = E(X)E(N)$$

and: $$\text{var}(S) = \left[E(X)\right]^2 \text{var}(N) + \text{var}(X)E(N)$$

Using the results from the *Tables* for the individual distributions:

$$E(X) = \frac{\alpha}{\lambda} = \frac{10}{0.1} = 100$$

$$\text{var}(X) = \frac{\alpha}{\lambda^2} = \frac{10}{0.1^2} = 1{,}000 \qquad [1]$$

$$E(N) = \frac{kq}{p} = \frac{2\times 0.1}{0.9} = 0.22222$$

$$\text{var}(N) = \frac{kq}{p^2} = \frac{2\times 0.1}{0.9^2} = 0.24691 \qquad [1]$$

Using the formulae above:

$$E(S) = E(X)E(N) = 22.222$$

and: $\text{var}(S) = [E(X)]^2 \text{var}(N) + \text{var}(X)E(N) = 100^2 \times 0.24691 + 1{,}000 \times 0.22222 = 2{,}691.358$ [1]

(iii)(b) ***Aggregate amount***

We now assume that S has an approximate normal distribution with this mean and variance. So, standardising in the usual way, we have:

$$P(S > k) = 0.001 \quad \Rightarrow \quad P\left(N(0,1) > \frac{k - E(S)}{\sqrt{\text{var}(S)}}\right) = 0.001$$

Using the percentage points of the standard normal distribution, we find that:

$$\frac{k - E(S)}{\sqrt{\text{var}(S)}} = 3.0902 \quad \Rightarrow \quad k = 22.222 + 3.0902\sqrt{2{,}691.358} = 182.54$$ [1]

(iv)(a) ***MGF of T***

The MGF of T is:

$$M_T(t) = E(e^{tT}) = E[e^{t(S_1 + \dots + S_{100})}] = E(e^{tS_1}) \dots E(e^{tS_{100}}) = [M_S(t)]^{100}$$

$$= \left(\frac{0.9}{1 - 0.1(1 - 10t)^{-10}}\right)^{200}$$ [1]

The mean and variance of T are 100 times the corresponding results for S, *ie*:

$$E(T) = 100E(S) = 2{,}222$$

and: $\text{var}(T) = 100\,\text{var}(S) = 269{,}135.8$ [1]

(iv)(b) ***Total aggregate amount***

So the corresponding figure for the aggregate amount exceeded with probability 0.001 is:

$$2{,}222 + 3.0902\sqrt{269{,}135.8} = 3{,}825.37$$ [1]

(iv)(c) ***Comment***

This is substantially less than one hundred times the corresponding answer to part (iii)(b). The Central Limit Theorem tells us that as the number of portfolios increases, bad experience in some of the portfolios will be offset by better experience in others, leading to a situation where the overall variation is relatively smaller. Pooling of similar risks reduces the overall variance. We can see this happening here. [1]

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Risk models 2

Syllabus objectives

1.2 Compound distributions and their application in risk modelling

1.2.1 Construct models appropriate for short-term insurance contracts in terms of the numbers of claims and the amounts of individual claims.

1.2.2 Describe the major simplifying assumptions underlying the models in 1.2.1.

1.2.4 Derive the mean, variance and coefficient of skewness for compound binomial, compound Poisson and compound negative binomial random variables.

1.2.5 Repeat 1.2.4 for both the insurer and the reinsurer after the operation of simple forms of proportional and excess of loss reinsurance.

0 Introduction

In this chapter we will look at some of the practical applications of risk models. We start by looking at how the models can be adapted for situations involving reinsurance. A section on the individual risk model is followed by some more complex examples of the use of risk models in practice.

1 Aggregate claim distributions under proportional and individual excess of loss reinsurance

In Chapter 19, we introduced the notation S to denote the aggregate claim amount random variable, *ie*:

$$S = X_1 + X_2 + \cdots + X_N$$

where N denotes the number of claims and X_i denotes the amount of the i th claim.

Here we extend this concept to consider the situation when reinsurance is in force. We will use the following notation:

- Y_i is the amount paid by the insurer in respect of the i th claim
- Z_i is the amount paid by the reinsurer in respect of the i th claim
- $S_I = Y_1 + Y_2 + \cdots + Y_N$ is the aggregate claim amount paid by the insurer
- $S_R = Z_1 + Z_2 + \cdots + Z_N$ is the aggregate claim amount paid by the reinsurer.

The formulae that we derived for the mean, variance and MGF of S can be adapted to cover the reinsurance situation by replacing X by Y or Z, as appropriate. For example:

$$E(S_I) = E(N)E(Y)$$

$$\text{var}(S_I) = E(N)\text{var}(Y) + \text{var}(N)[E(Y)]^2$$

$$M_{S_I}(t) = M_N(\log M_Y(t))$$

1.1 Proportional reinsurance

The distribution of the number of claims involving the reinsurer is the same as the distribution of the number of claims involving the insurer, as each pays a defined proportion of every claim.

For a retention level α ($0 \le \alpha \le 1$), the i th individual claim amount for the insurer is αX_i and for the reinsurer is $(1-\alpha)X_i$.

In other words:

$$Y_i = \alpha X_i$$

$$Z_i = (1-\alpha)X_i$$

So:

$$\begin{aligned} S_I &= Y_1 + Y_2 + \cdots + Y_N \\ &= \alpha X_1 + \alpha X_2 + \cdots + \alpha X_N \\ &= \alpha(X_1 + X_2 + \cdots + X_N) \\ &= \alpha S \end{aligned}$$

and:

$$\begin{aligned} S_R &= Z_1 + Z_2 + \cdots + Z_N \\ &= (1-\alpha)X_1 + (1-\alpha)X_2 + \cdots + (1-\alpha)X_N \\ &= (1-\alpha)(X_1 + X_2 + \cdots + X_N) \\ &= (1-\alpha)S \end{aligned}$$

***ie* the aggregate claims amounts are αS and $(1-\alpha)S$ respectively.**

Question

(i) Show that under a proportional reinsurance arrangement where the direct writer retains a proportion α, the MGF of the net individual claim amount Y paid by the direct insurer is $M_X(\alpha t)$.

(ii) Hence give a formula for $M_{S_I}(t)$ when the number of claims follows a Poisson distribution with mean 25 and individual claim amounts are exponentially distributed with mean 1,000.

Solution

(i) ***MGF of Y***

Under this arrangement, $Y = \alpha X$. So:

$$M_Y(t) = E(e^{tY}) = E(e^{t\alpha X}) = E[e^{(t\alpha)X}] = M_X(\alpha t)$$

(ii) ***MGF of insurer's aggregate claim amount***

Since $N \sim Poisson(25)$:

$$M_N(t) = e^{25(e^t - 1)}$$

and the MGF of S_I is:

$$M_{S_I}(t) = M_N(\log M_Y(t)) = e^{25(\exp(\log M_Y(t))-1)} = e^{25(M_Y(t)-1)}$$

In addition, since X is exponentially distributed with mean 1,000 (*ie* with parameter $\frac{1}{1,000}$):

$$M_X(t) = (1-1,000t)^{-1}$$

and hence:

$$M_Y(t) = M_X(\alpha t) = (1-1,000\alpha t)^{-1}$$

So the MGF of S_I is:

$$M_{S_I}(t) = e^{25\left[(1-1,000\alpha t)^{-1}-1\right]}$$

1.2 Individual excess of loss reinsurance

The amount that an insurer pays on the ith claim under individual excess of loss reinsurance with retention level M is $Y_i = \min\{X_i, M\}$.

Equivalently:

$$Y = \begin{cases} X & \text{if } X \le M \\ M & \text{if } X > M \end{cases}$$

The amount that the reinsurer pays is $Z_i = \max\{0, X_i - M\}$.

We can also write this as:

$$Z = \begin{cases} 0 & \text{if } X \le M \\ X-M & \text{if } X > M \end{cases}$$

As previously stated, **the insurer's aggregate claims net of reinsurance can be represented as:**

$$S_I = Y_1 + Y_2 + \cdots + Y_N$$

and the reinsurer's aggregate claims as:

$$S_R = Z_1 + Z_2 + \cdots + Z_N \qquad \textbf{(20.1)}$$

If, for example, $N \sim Poi(\lambda)$, S_I has a compound Poisson distribution with Poisson parameter λ, and the ith individual claim amount is Y_i. Similarly, S_R has a compound Poisson distribution with Poisson parameter λ, and the ith individual claim amount is Z_i.

Hence, for this compound Poisson distribution with reinsurance, we have:

$$E(S_I) = \lambda E(Y)$$

$$\text{var}(S_I) = \lambda E(Y^2)$$

$$skew(S_I) = \lambda E(Y^3)$$

Similar formulae can be obtained for S_R by replacing Y with Z.

Note, however that if $F(M) > 0$, as will usually be the case, then Z_i may take the value 0.

Here, $F(x)$ denotes the distribution function of the original claim amount random variable, X. So:

$$F(M) = P(X \le M)$$

If this is greater than 0, then there is a non-zero probability that the reinsurer will not be involved in a claim.

In other words, 0 is counted as a possible claim amount for the reinsurer. From a practical point of view, this definition of S_R is rather artificial. The insurer will know the observed value of N, but the reinsurer will probably know only the number of claims above the retention level M, since the insurer may notify the reinsurer only of claims above the retention level.

Example

The annual aggregate claim amount from a risk has a compound Poisson distribution with Poisson parameter 10. Individual claim amounts are uniformly distributed on $(0, 2000)$. The insurer of this risk has effected excess of loss reinsurance with retention level 1,600.

Calculate the mean, variance and coefficient of skewness of both the insurer's and reinsurer's aggregate claims under this reinsurance arrangement.

Solution

Let S_I and S_R be as above. To find $E[S_i]$ calculate $E[Y_i]$. Now:

$$E(Y_i) = \int_0^M x f(x) dx + M P(X_i > M)$$

where $f(x) = 0.0005$ is the U(0,2000) density function and M = 1,600.

This gives:

$$E[Y_i] = \left[\frac{0.0005x^2}{2}\right]_0^M + 0.2M = 960$$

So:

$$E[S_I] = 10E[Y_i] = 9,600$$

To find $\text{var}[S_I]$, we must calculate $E[Y_i^2]$:

$$E[Y_i^2] = \int_0^M x^2 f(x)\,dx + M^2 P(X_i > M)$$

$$= \left[\frac{0.0005x^3}{3}\right]_0^M + 0.2M^2$$

$$= 1,194,666.7$$

So:

$$\text{var}[S_I] = 10E[Y_i^2] = 11,946,667$$

To find the coefficient of skewness of the insurer's claims, we must calculate $E[Y_i^3]$:

$$E[Y_i^3] = \int_0^M x^3 f(x)\,dx + M^3 P(X_i > M)$$

$$= \left[\frac{0.0005x^4}{4}\right]_0^M + 0.2M^3$$

$$= 1,638,400,000$$

So:

$$E\left[\left(S_I - E(S_I)\right)^3\right] = 10E[Y_i^3] = 16,384,000,000$$

and the coefficient of skewness of S_I is:

$$\frac{16,384,000,000}{11,946,667^{3/2}} = 0.397$$

To find $E[S_R]$, note that the expected annual aggregate claim amount from the risk is $E[S] = \lambda E[X] = 10 \times 1,000 = 10,000$. Then:

$$E[S_R] = 10,000 - E[S_I] = 400$$

To find $\text{var}[S_R]$, calculate $E[Z_i^2]$ from:

$$E[Z_i^2] = \int_M^{2,000} (x-M)^2 f(x)\,dx$$

$$= \int_0^{2,000-M} 0.0005y^2\,dy \quad \text{where } y = x - M$$

$$= \left[\frac{0.0005\,y^3}{3}\right]_0^{2,000-M}$$

$$= 10,666.7$$

So:

$$\text{var}[S_R] = 10E[Z_i^2] = 106,667$$

To find the coefficient of skewness of the reinsurer's claims, we need to calculate $E[Z_i^3]$:

$$E[Z_i^3] = \int_M^{2,000} (x-M)^3 f(x)\,dx$$

$$= \int_0^{2,000-M} 0.0005y^3\,dy \quad \text{where } y = x - M$$

$$= 3,200,000$$

So:

$$E\left[\left(S_R - E(S_R)\right)^3\right] = 10E[Z_i^3] = 32,000,000$$

and the coefficient of skewness of S_R is:

$$\frac{32,000,000}{106,667^{3/2}} = 0.92$$

Question

Calculate the variance of S, the aggregate claim amount before reinsurance for the example above and explain why:

$$\text{var}(S_I) + \text{var}(S_R) \neq \text{var}(S)$$

Solution

We have $\text{var}(S) = 10E(X^2)$, where:

$$E(X^2) = \int_0^{2000} \frac{x^2}{2000} dx = \frac{4{,}000{,}000}{3}$$

So $\text{var}(S) = 13{,}333{,}333$.

This is not equal to $11{,}946{,}667 + 106{,}667$ because S_I and S_R are not independent.

To simulate the collective risk model with *individual* reinsurance we can combine the R code from Chapters 15, 18 and 19.

For example, to simulate 10,000 values for a reinsurer where claims have a compound Poisson distribution with parameter 1,000 and a gamma claims distribution with $\alpha = 750$ and $\lambda = 0.25$ under *individual* excess of loss with retention 2,500 we would use:

```
set.seed(123)
M <-2500
n <- rpois(10000,1000)
sR <- numeric(10000)
for(i in 1:10000)
{x <- rgamma(n[i],shape=750,rate=0.25)
z <- pmax(0,x-M)
sR[i] <- sum(z)}
```

We can now find moments, the coefficient of skewness, probabilities and quantiles as before.

Earlier we mentioned that using $S_R = Z_1 + \cdots + Z_N$ is a bit artificial. We now look at an alternative way of modelling the reinsurer's compound claim amount distribution.

The reinsurer's aggregate claims can also be represented by:

$$S_R = W_1 + W_2 + \cdots + W_{NR} \qquad \textbf{(20.2)}$$

where the random variable NR denotes the actual number of (non-zero) payments made by the reinsurer.

Here:

$$W_i = X_i - M \mid X_i > M$$

For example, suppose that the risk above gave rise to the following eight claim amounts in a particular year:

403 1,490 1,948 443 1,866 1,704 1,221 823

The retention limit is 1,600.

Then in formula (20.1) the observed value of N is 8, and the third, fifth and sixth claims require payments from the reinsurer of 348, 266 and 104 respectively. The reinsurer makes a 'payment' of 0 on the other five claims.

In formula (20.2), the observed value of NR is 3 and the observed values of W_1, W_2 and W_3 are 348, 266 and 104 respectively. Note that the observed value of S_R is the same (*ie* 718) under each definition.

W_i has density function:

$$g(w) = \frac{f(x+M)}{1-F(M)} \ , \quad w > 0$$

We saw this result in Section 1.2 of Chapter 18. It can also be written as:

$$f_W(w) = \frac{f_X(w+M)}{1-F_X(M)}$$

To specify the distribution for S_R as given in formula (20.2) the distribution of NR is needed.

In some contexts it may be obvious what this distribution is, but here is a general method for establishing the distribution.

This is found as follows. Define:

$$NR = I_1 + I_2 + \cdots + I_N$$

where N denotes the number of claims from the risk (as usual). I_j is an indicator random variable which takes the value 1 if the reinsurer makes a (non-zero) payment on the j th claim, and takes the value 0 otherwise. Thus NR gives the number of payments made by the reinsurer.

From its definition, we see that NR is a compound random variable. However, instead of being the sum of individual claims amounts, NR is a sum of indicator random variables.

Since I_j takes the value 1 only if $X_j > M$:

$$P(I_j = 1) = P(X_j > M) = \pi \text{ , say}$$

and:

$$P(I_j = 0) = 1 - \pi$$

In other words, I_j has a $Binomial(1,\pi)$ distribution.

Further, I_j has MGF:

$$M_I(t) = \pi e^t + 1 - \pi$$

The formula for the MGF of a binomial distribution is given on page 6 of the *Tables*.

By formula (19.8) in Chapter 19 (the formula for the MGF of a compound random variable), ***NR*** **has MGF:**

$$M_{NR}(t) = M_N(\log M_I(t))$$

Question

If N has a $Poisson(\lambda)$ distribution and $P(X > M) = ½$, show that NR has a $Poisson(½\lambda)$ distribution.

Solution

Here $\pi = ½$, so:

$$M_I(t) = ½e^t + ½$$

and:

$$M_{NR}(t) = M_N[\log M_I(t)] = \exp\{\lambda[M_I(t) - 1]\} = \exp\left\{\lambda(½e^t - ½)\right\} = \exp\left\{½\lambda(e^t - 1)\right\}$$

This is the MGF of the $Poisson(½\lambda)$ distribution. By the uniqueness property of MGFs, it follows that $NR \sim Poisson(½\lambda)$.

We now continue the above Core Reading example where the annual aggregate claim amount from a risk has a compound Poisson distribution with Poisson parameter 10, individual claim amounts are uniformly distributed on $(0, 2000)$, and the insurer of this risk has effected excess of loss reinsurance with retention level 1,600.

Example

Continuing the above example and using formula (20.2) as the model for S_R, it can be seen that S_R has a compound Poisson distribution with Poisson parameter $0.2 \times 10 = 2$. Individual claims, W_i, have density function:

$$g(w) = \frac{f(w+M)}{1-F(M)} = \frac{0.0005}{0.2} = 0.0025 \text{ , for } 0 < w < 400$$

ie **W_i is uniformly distributed on (0,400).**

Using the formulae for the moments of a continuous uniform distribution from page 13 of the *Tables*, we have:

$$E[W_i] = 200 \text{ , } E[W_i^2] = 53{,}333.33 \text{ and } E[W_i^3] = 16{,}000{,}000$$

giving the same results as before.

If we multiply these figures by 2 (the Poisson parameter of S_R), we get $E(S_R) = 400$, $\text{var}(S_R) = 106,667$ and $\text{skew}(S_R) = 32,000,000$, which agree with the answers obtained previously.

Thus, there are two ways to specify and evaluate the distribution of S_R.

1.3 Aggregate excess of loss reinsurance

Under an aggregate excess of loss arrangement with retention limit M, the insurer pays all the claims if the total claim amount is less than or equal to the retention limit. The maximum payment made by the insurer is M. So the insurer's aggregate claim payment is:

$$S_I = \begin{cases} S & \text{if } S \le M \\ M & \text{if } S > M \end{cases}$$

The reinsurer's aggregate claim payment is:

$$S_R = \begin{cases} 0 & \text{if } S \le M \\ S - M & \text{if } S > M \end{cases}$$

Calculations involving aggregate excess of loss reinsurance are done using a first principles approach.

Question

The annual number of claims from a small group of policies has a Poisson distribution with a mean of 2. Individual claim amounts have the following distribution:

Amount	200	400
Probability	0.7	0.3

Individual claim amounts are independent of each other and are also independent of the number of claims. The insurer has purchased aggregate excess of loss reinsurance with a retention limit of 600.

Calculate the probability that the reinsurer is involved in paying the claims that arise in the next policy year.

Solution

The reinsurer will be involved if the total claim amount is more than 600. Since the total claim amount must be a multiple of 200, the probability is:

$$P(S > 600) = 1 - P(S = 0) - P(S = 200) - P(S = 400) - P(S = 600)$$

The total claim amount will be 0 only if there are no claims. So:

$$P(S=0)=P(N=0)=\frac{e^{-2}2^0}{0!}=e^{-2}$$

Using the assumption that individual claim amounts are independent of the number of claims, we have:

$$P(S=200)=P(N=1,X_1=200)=P(N=1)P(X_1=200)=\frac{e^{-2}2^1}{1!}\times 0.7=1.4e^{-2}$$

$$\begin{aligned}P(S=400)&=P(N=2,X_1=200,X_2=200)+P(N=1,X_1=400)\\&=\frac{e^{-2}2^2}{2!}\times 0.7^2+\frac{e^{-2}2^1}{1!}\times 0.3\\&=1.58e^{-2}\end{aligned}$$

and:

$$\begin{aligned}P(S=600)&=P(N=3,X_1=200,X_2=200,X_3=200)\\&\quad+P(N=2,X_1=200,X_2=400)+P(N=2,X_1=400,X_2=200)\\&=\frac{e^{-2}2^3}{3!}\times 0.7^3+2\times\frac{e^{-2}2^2}{2!}\times 0.7\times 0.3\\&=1.29733e^{-2}\end{aligned}$$

So:

$$P(S>600)=1-e^{-2}(1+1.4+1.58+1.29733)=0.28579$$

We can also simulate the collective risk model with *aggregate* reinsurance. For example to simulate 10,000 values for a reinsurer where claims have a compound Poisson distribution with parameter 1,000 and a gamma claims distribution with $\alpha=750$ and $\lambda=0.25$ under *aggregate* excess of loss with retention 3,000,000 we would take our S from Section 3.4 of Chapter 19 and then use:

```
sR <- pmax(0,s-3000000)
```


2 The individual risk model

Under this model a portfolio consisting of a fixed number of risks is considered. It will be assumed that:

- **these risks are independent**
- **claim amounts from these risks are not** (necessarily) **identically distributed random variables**
- **the number of risks does not change over the period of insurance cover.**

As before, aggregate claims from this portfolio are denoted by S. So:

$$S = Y_1 + Y_2 + \cdots + Y_n$$

where Y_j denotes the claim amount under the j th risk and n denotes the number of risks. It is possible that some risks will not give rise to claims. Thus, some of the observed values of $\{Y_j\}_{j=1}^{n}$ may be 0.

In fact in most forms of insurance most policies would not give rise to any claims during a given year.

This approach is referred to as an *individual risk model* because it is considering the claims arising from each individual policy.

For each risk, the following assumptions are made:

- **the number of claims from the j th risk, N_j, is either 0 or 1** **(20.3)**
- **the probability of a claim from the j th risk is q_j.** **(20.4)**

If a claim occurs under the j th risk, the claim amount is denoted by the random variable X_j. Let $F_j(x)$, μ_j and σ_j^2 denote the distribution function, mean and variance of X_j respectively.

Assumption (20.3) is very restrictive. It means that a maximum of one claim from each risk is allowed for in the model. This includes risks such as one-year term assurance (since a policyholder can only die once)**, but excludes many types of general insurance policy. For example, there is no restriction on the number of claims that could be made in a policy year under household contents insurance.**

There are three important differences between this model and the collective risk model:

(1) The number of risks in the portfolio has been specified. In the collective risk model, there was no need to specify this number, nor to assume that it remained fixed over the period of insurance cover (not even when it was assumed that $N \sim Bin(n,q)$).

On the other hand we could argue that there was an implicit assumption of a constant number of risks in the very fact that we were using a binomial distribution to model the number of claims.

(2) The number of claims from each individual risk has been restricted. There was no such restriction in the collective risk model.

(3) It is assumed that individual risks are independent. In the collective risk model it was individual claim amounts that were independent.

The contrast here is between the occurrence of claims and the size of claims.

Assumptions (20.3) and (20.4) say that $N_j \sim Bin(1, q_j)$. Thus, the distribution of Y_j is compound binomial, with individual claim amount random variable X_j. From formulae (19.15) and (19.16) in Chapter 19 (for the mean and variance of a compound random variable) **it follows immediately that:**

$$E[Y_j] = q_j\mu_j \qquad (20.5)$$

$$\text{var}[Y_j] = q_j\sigma_j^2 + q_j(1-q_j)\mu_j^2 \qquad (20.6)$$

S is the sum of n independent compound binomial random variables. The distribution of S can be stated only when the compound binomial variables are identically distributed, as well as independent. It is possible, but complicated, to compute the distribution function of S under certain conditions.

However, it is easy to find the mean and variance of S :

$$E[S] = E\left[\sum_{j=1}^{n} Y_j\right] = \sum_{j=1}^{n} E[Y_j] = \sum_{j=1}^{n} q_j\mu_j \qquad (20.7)$$

The assumption that individual risks are independent is needed to write:

$$\text{var}[S] = \text{var}\left[\sum_{j=1}^{n} Y_j\right] = \sum_{j=1}^{n} \text{var}[Y_j] = \sum_{j=1}^{n} (q_j\sigma_j^2 + q_j(1-q_j)\mu_j^2) \qquad (20.8)$$

In the special case when $\{Y_j\}_{j=1}^{n}$ is a sequence of identically distributed, as well as independent, random variables, then for each policy the values of q_j, μ_j and σ_j^2 are identical, say q, μ and σ^2. Since $F_j(x)$ is independent of j, we can refer to it simply as $F(x)$. Hence, S is compound binomial, with binomial parameters n and q, and individual claims have distribution function $F(x)$. In this special case, it reduces to the collective risk model, and it can be seen from (20.7) and (20.8) that:

$$E[S] = nq\mu$$

$$\text{var}[S] = nq\sigma^2 + nq(1-q)\mu^2$$

which correspond to (19.15) and (19.16) respectively in Chapter 19.

Question

The probability of a claim arising on any given policy in a portfolio of 1,000 one-year term assurance policies is 0.004. Claim amounts have a $Gamma(5, 0.002)$ distribution. Calculate the mean and variance of the aggregate claim amount.

Solution

We have:

$$\mu_j = \frac{5}{0.002} = 2,500 \qquad \text{and:} \qquad \sigma_j^2 = \frac{5}{0.002^2} = 1,250,000$$

So the mean and variance of the aggregate claim amount are:

$$E(S) = nq\,\mu = 1,000 \times 0.004 \times 2,500 = 10,000$$

and:

$$\begin{aligned} \text{var}(S) &= nq\sigma^2 + nq(1-q)\mu^2 \\ &= 1,000 \times \left[0.004 \times 1,250,000 + 0.004 \times 0.996 \times 2,500^2\right] \\ &= (£5,468)^2 \end{aligned}$$

We can use R to simulate the total claim amount payable under the individual risk model.

Suppose we have n life policies, with the probabilities of death for each policy contained in the vector `q` and simulated claim amounts for each policy contained in the vector `claim`. Then:

```
S <- q*claim
```

We can now find moments, the coefficient of skewness, probabilities and quantiles as before, and also apply reinsurance if appropriate.

3 Parameter variability / uncertainty

This section forms part of the Core Reading, but does not address any specific syllabus objectives. However, the material here provides useful practice in applying the models we have studied.

3.1 Introduction

So far risk models have been studied assuming that the parameters, that is the moments and in some cases even the distributions, of the number of claims and of the amount of individual claims are known with certainty. In general, these parameters would not be known but would have to be estimated from appropriate sets of data. In this section it will be seen how the models introduced earlier can be extended to allow for parameter uncertainty / variability. This will be done by looking at a series of examples. Most, but not all, of these examples will consider uncertainty in the claim number distribution since this, rather than the individual claim amount distribution, has received more attention in the actuarial literature. All the examples will be based on claim numbers having a Poisson distribution.

3.2 Variability in a heterogeneous portfolio

Consider a portfolio consisting of n independent policies. The aggregate claims from the ith policy are denoted by the random variable S_i, where S_i has a compound Poisson distribution with parameters λ_i, and the CDF of the individual claim amounts distribution is $F(x)$. Notice that, for simplicity, the CDF of the distribution of individual claim amounts, $F(x)$, is assumed to be identical for all the policies.

In this example the CDF of individual claim amounts, *ie* $F(x)$, is assumed to be known but the values of the Poisson parameters, *ie* the λ_i s, are not known. In this subsection the λ_i s are assumed to be (sample values of) independent random variables with the same (known) distribution. In other words $\{\lambda_i\}_{i=1}^{n}$ is treated as a set of independent and identically distributed random variables with a known distribution. This means that if a policy is chosen at random from the portfolio it is assumed that the Poisson parameter for the policy is not known but that probability statements can be made about it. For example, 'there is a 50% chance that its Poisson parameter lies between 3 and 5'. It is important to understand that the Poisson parameter for a policy chosen from the portfolio is a fixed number; the problem is that this number is not known.

Example

Suppose that the Poisson parameters of policies in a portfolio are not known but are equally likely to be 0.1 or 0.3.

(i) Find the mean and variance (in terms of m_1 and m_2) of the aggregate claims from a policy chosen at random from the portfolio.

(ii) Find the mean and variance (in terms of m_1, m_2 and n) of the aggregate claims from the whole portfolio.

It may be helpful to think of this as a model of part of a motor insurance portfolio. The policies in the whole portfolio have been subdivided according to their values for rating factors such as 'age of driver', 'type of car' and even 'past claims experience'. The policies in the part of the portfolio being considered have identical values for these rating factors. However, there are some factors, such as 'driving ability', that cannot easily be measured and so they cannot be taken explicitly into account. It is supposed that some of the policyholders in this part of the portfolio are 'good' drivers and the remainder are 'bad' drivers. The individual claim amount distribution is the same for all drivers but 'good' drivers make fewer claims (0.1 *pa* on average) than 'bad' drivers (0.3 *pa* on average). It is assumed that it is known, possibly from national data, that a policyholder in this part of the portfolio is equally likely to be a 'good' driver or a 'bad' driver but that it cannot be known whether a particular policyholder is a 'good' driver or a 'bad' driver.

Solution

Let λ_i, $i = 1, 2, ..., n$ be the Poisson parameter of the ith policy in the portfolio. $\{\lambda_i\}_{i=1}^{\infty}$ is regarded as a set of independent and identically distributed random variables, each with the following distribution:

$$P(\lambda_i = 0.1) = 0.5$$

$$P(\lambda_i = 0.3) = 0.5$$

From this:

$$E[\lambda_i] = 0.2$$

$$\text{var}[\lambda_i] = 0.01$$

(i) **The moments of S_i can be calculated by conditioning on the value of λ_i. Since $S_i \mid \lambda_i$ has a straightforward compound Poisson distribution, formulae (19.8) and (19.9) in Chapter 19 can be used to write:**

$$E[S_i] = E[E[S_i \mid \lambda_i]] = E[\lambda_i m_1] = 0.2 m_1$$

$$\begin{aligned}\text{var}[S_i] &= E[\text{var}[S_i \mid \lambda_i]] + \text{var}[E[S_i \mid \lambda_i]] \\ &= E[\lambda_i m_2] + \text{var}[\lambda_i m_1] \\ &= 0.2 m_2 + 0.01 m_1^2\end{aligned}$$

(ii) **The random variables $\{S_i\}_{i=1}^{n}$ are independent and identically distributed, each with the distribution of S_i given in part (i). Hence, the result in (i) above can be used to write:**

$$E\left[\sum_{i=1}^{n} S_i\right] = nE[S_i] = 0.2 n m_1$$

$$\text{var}\left[\sum_{i=1}^{n} S_i\right] = n\,\text{var}[S_i] = 0.2 n m_2 + 0.01 n m_1^2$$

Example

Suppose the Poisson parameters for individual policies are drawn from a gamma distribution with parameters α and δ. Find the distribution of the number of claims from a policy chosen at random from the portfolio.

Solution

Let N_i denote the number of claims from the i th policy in the portfolio and let λ_i be its Poisson parameter. Then N_i has a Poisson distribution with parameter λ_i but the problem is that (by assumption) the value of λ_i is not known. What *is* known is the distribution from which λ_i has been chosen.

The problem can be summarised as follows:

Given that:

$$N_i \mid \lambda_i \sim Poisson(\lambda_i) \text{ and } \lambda_i \sim Gamma(\alpha,\delta)$$

find the marginal distribution of N_i.

The marginal distribution of N_i is its unconditional distribution. In this example, N_i can only take whole number values, so it is a discrete random variable. To determine its marginal distribution, we need to derive a formula for the unconditional probability $P(N_i = x)$.

This problem can be solved by removing the conditioning in the usual way.

Recall that if X and Y are discrete random variables, then the unconditional probability $P(X = x)$ is given by:

$$P(X=x)=\sum_y P(X=x,Y=y)=\sum_y P(X=x \mid Y=y)P(Y=y)$$

In this example X is being replaced by N_i and Y is being replaced by λ_i. Since λ_i is a continuous random variable, we turn the sum into an integral and formula becomes:

$$P(N_i=x)=\int_{\lambda_i} P(N_i=x \mid \lambda_i) f(\lambda_i)\, d\lambda_i$$

For $x = 0,1, 2, \ldots$:

$$P(N_i=x)=\int_0^\infty \exp\{-\lambda\}\frac{\lambda^x}{x!}\frac{\delta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}\exp\{-\delta\lambda\}\, d\lambda$$

$$=\frac{\delta^\alpha}{\Gamma(\alpha)x!}\int_0^\infty \exp\{-\lambda(\delta+1)\}\lambda^{x+\alpha-1}\, d\lambda$$

Evaluate the integral by comparing the integrand with a gamma density.

We can make the integrand look like the PDF of the $Gamma(x+\alpha,\ \delta+1)$ distribution by inserting a factor of $\dfrac{(\delta+1)^{x+\alpha}}{\Gamma(x+\alpha)}$ inside the integral. We need to compensate for doing this by inserting a factor of $\dfrac{\Gamma(x+\alpha)}{(\delta+1)^{x+\alpha}}$ outside the integral. This gives:

$$P(N_i = x) = \frac{\delta^\alpha}{\Gamma(\alpha)x!}\frac{\Gamma(x+\alpha)}{(\delta+1)^{x+\alpha}}\int_0^\infty \frac{(\delta+1)^{x+\alpha}}{\Gamma(x+\alpha)}\lambda^{x+\alpha-1}e^{-\lambda(\delta+1)}\,d\lambda$$

$$= \frac{\delta^\alpha}{\Gamma(\alpha)x!}\frac{\Gamma(x+\alpha)}{(\delta+1)^{x+\alpha}}\int_0^\infty f(\lambda)d\lambda \quad \text{where } \lambda \sim Gamma(x+\alpha,\delta+1)$$

The integral in the line above is 1 (as we are integrating a PDF over all possible values of the random variable).

So:

$$\boldsymbol{P(N_i = x) = \frac{\delta^\alpha}{\Gamma(\alpha)x!}\frac{\Gamma(x+\alpha)}{(\delta+1)^{x+\alpha}}}$$

which shows that the marginal distribution of N_i is negative binomial with parameters α and $\dfrac{\delta}{\delta+1}$.

3.3 Variability in a homogeneous portfolio

Now a different example is considered. Suppose, as before, there is a portfolio of n policies. The aggregate claims from a single policy have a compound Poisson distribution with parameters λ, and the CDF of the individual claim amounts random variable is $F(x)$. The Poisson parameters are the same for all policies in the portfolio. If the value of λ were known, the aggregate claims from different policies would be independent of each other. It is assumed that the value of λ is not known, possibly because it changes from year to year, but that there *is* some indication of the probability that λ will be in any given range of values. As in the previous example, it is assumed for simplicity that there is no uncertainty about the moments or distribution of the individual claim amounts, *ie* about $F(x)$. The uncertainty about the value of λ can be modelled by regarding λ as a random variable (with a known distribution).

Example

Suppose that the Poisson parameter, λ, will be equal to 0.1 or to 0.3 with equal probability.

(i) Calculate the mean and variance (in terms of m_1 and m_2) of the aggregate claims from a policy chosen at random from the portfolio.

(ii) Calculate the mean and variance (in terms of m_1, m_2 and n) of the aggregate claims from the whole portfolio.

Solution

Using the same notation as before let S_i denote the aggregate claims from the i th policy in the portfolio. The situation can be summarised as follows:

The random variables $\{S_i \mid \lambda\}_{i=1}^{n}$ are independent and identically distributed, each with a compound Poisson distribution with parameters λ and $F(x)$. The random variable λ has the following distribution:

$$P(\lambda = 0.1) = 0.5$$

$$P(\lambda = 0.3) = 0.5$$

(i) Conditioning on the value of λ:

$$E[S_i] = E[E(S_i \mid \lambda)] = E[\lambda m_1] = 0.2\, m_1$$

$$\operatorname{var}[S_i] = E[\operatorname{var}(S_i \mid \lambda)] + \operatorname{var}[E(S_i \mid \lambda)] = E[\lambda m_2] + \operatorname{var}[\lambda m_1]$$

$$= 0.2\, m_2 + 0.01 m_1^2$$

(ii) $$E\left[\sum_{i=1}^{n} S_i\right] = n\,E[S_1] = 0.2\, n\, m_1$$

(since $\{S_i\}_{i=1}^{n}$ are identically distributed)

$$\operatorname{var}\left[\sum_{i=1}^{n} S_i\right] = E\left[\operatorname{var}\left(\sum_{i=1}^{n} S_i \mid \lambda\right)\right] + \operatorname{var}\left[E\left(\sum_{i=1}^{n} S_i \mid \lambda\right)\right]$$

$$= E[n \lambda m_2] + \operatorname{var}[n \lambda m_1]$$

$$= 0.2\, n\, m_2 + 0.01 n^2 m_1^2$$

Note that $S_1 \mid \lambda, \ldots, S_n \mid \lambda$ are independent but $S_1, \ldots, S_n$ are not unconditionally independent (since they all depend on the value of λ), so $\operatorname{var}(\sum_{i=1}^{n} S_i) \neq \sum_{i=1}^{n} \operatorname{var}(S_i)$.

It is useful to compare the answers to the above example with those to the first example in the Section 3.2. The values of the mean are in all cases the same, as are the variances when a single policy is considered (part (i)). The difference occurs when variances for more than one policy are considered (part (ii)), in which case the second example gives the greater variance. It is important to understand the differences (and the similarities) between the two examples. A practical situation where the second example could be appropriate would be a portfolio of policies insuring buildings in a certain area. The number of claims could depend on, among other factors, the weather during the year; an unusually high number of storms resulting in a high expected number of claims (*ie* a high value of λ) and vice versa for all the policies together.

3.4 Variability in claim numbers and claim amounts and parameter uncertainty

This section contains two more examples. The first is a rather complicated example involving uncertainty over claim amounts as well as claim numbers.

Example

An insurance company models windstorm claims under household insurance policies using the following assumptions.

The number of storms arising each year, K, is assumed to have a Poisson distribution with parameter λ.

The number of claims arising from the ith storm, N_i, $i = 1, 2, \ldots, K$, is assumed to have a Poisson distribution with parameter Θ_i.

The parameters Θ_i, $i = 1, 2, \ldots, K$, are assumed to be independent and identically distributed random variables, with $E(\Theta_i) = n$ and $\text{var}(\Theta_i) = s_1^2$.

The amount of the jth claim arising from the ith storm, X_{ij}, $j = 1, 2, \ldots, N_i$, has a lognormal distribution with parameters μ_i and σ^2, where σ^2 is assumed to be known. The mean claim amounts, $\Lambda_i = \exp(\mu_i + \sigma^2/2)$ are assumed to be independent and identically distributed random variables with mean p and variance s_2^2.

It is also assumed that Θ_i and Λ_i are independent.

(i) Show that $E[X_{ij}] = p$ and $\text{var}[X_{ij}] = \exp\{\sigma^2\}(p^2 + s_2^2) - p^2$.

(ii) Let S_i denote aggregate claims outgo from the i-th storm, so that $S_i \mid \{\Theta_i, \Lambda_i\}$ is a compound Poisson random variable. Show that:

$$E[S_i] = np$$

and:

$$\text{var}[S_i] = (p^2 + s_2^2)(n^2 + s_1^2 + n\exp\{\sigma^2\}) - n^2p^2$$

(iii) Find expressions for the mean and variance of the annual aggregate claims outgo from all storms.

Solution

(i) $E[X_{ij}] = E[E(X_{ij} \mid \Lambda_i)] = E[\Lambda_i] = p$

$$\text{var}[X_{ij}] = E[\text{var}(X_{ij} \mid \Lambda_i)] + \text{var}[E(X_{ij} \mid \Lambda_i)]$$

$$= E[\Lambda_i^2(\exp\{\sigma^2\} - 1)] + \text{var}(\Lambda_i)$$

$$= (p^2 + s_2^2)(\exp\{\sigma^2\} - 1) + s_2^2$$

$$= (p^2 + s_2^2)\exp\{\sigma^2\} - p^2$$

(ii) $E[S_i] = E[E(S_i \mid \Theta_i, \Lambda_i)] = E[\Theta_i \Lambda_i] = np$ **since** Θ_i **and** Λ_i **are independent.**

Now, since $S_i \mid \{\Theta_i, \Lambda_i\}$ **has a compound Poisson distribution:**

$$\text{var}[S_i \mid \Theta_i, \Lambda_i] = \Theta_i E[X_{ij}^2 \mid \Lambda_i] = \Theta_i (\Lambda_i^2 \exp\{\sigma^2\})$$

and so:

$$E[\text{var}(S_i \mid \Theta_i, \Lambda_i)] = n(p^2 + s_2^2)\exp\{\sigma^2\}$$

Also:

$$\text{var}[E(S_i \mid \Theta_i, \Lambda_i)] = \text{var}[\Theta_i \Lambda_i] = E[\Theta_i^2 \, \Lambda_i^2] - n^2 p^2$$
$$= (n^2 + s_1^2)(p^2 + s_2^2) - n^2 p^2$$

Putting these last two results together:

$$\text{var}[S_i] = (n^2 + s_1^2)(p^2 + s_2^2) - n^2 p^2 + n(p^2 + s_2^2)\exp\{\sigma^2\}$$

(iii) **Let** R **be a random variable denoting the annual aggregate claims outgo from all storms. Then** R **can be written:**

$$R = \sum_{i=1}^{K} S_i$$

where K **has a Poisson distribution and the random variables** S_i **are IID (independent and identically distributed).**

Hence, R **has a compound Poisson distribution and so:**

$$E[R] = \lambda E(S_i) = \lambda n p$$

$$\text{var}[R] = \lambda E(S_i^2) = \lambda(\text{var}[S_i] + E[S_i]^2)$$
$$= \lambda(p^2 + s_2^2)(n^2 + s_1^2 + n\exp\{\sigma^2\})$$

Example

Each year an insurance company issues a number of household contents insurance policies, for each of which the annual premium is £80. The aggregate annual claims from a single policy have a compound Poisson distribution; the Poisson parameter is 0.4 and individual claim amounts have a gamma distribution with parameters α **and** λ**. The expense involved in settling a claim is a random variable uniformly distributed between £50 and £**b **(>£50). The amount of the expense is independent of the amount of the associated claim. The random variable** S **represents the total aggregate claims and expenses in one year from this portfolio. It may be assumed that** S **has approximately a normal distribution.**

(i) Suppose that:

$$\alpha = 1;\ \lambda = 0.01;\ b = 100$$

Show that the company must sell at least 884 policies in a year to be at least 99% sure that the premium income will exceed the claims and expenses outgo.

(ii) Now suppose that the values of α, λ and b are not known with certainty but could be anywhere in the following ranges:

$$0.95 \le \alpha \le 1.05;\ 0.009 \le \lambda \le 0.011;\ 90 \le b \le 110$$

By considering what, for the insurance company, would be the worst possible combination of values for α, λ and b, calculate the number of policies the company must sell to be at least 99% sure that the premium income will exceed the claims and expenses outgo.

Solution

Let X_i be the amount of the i th claim and Y_i be the amount of the associated expense. Let N be the total number of claims from the portfolio and let n be the number of policies in the portfolio. Then N has a Poisson distribution with parameter $0.4n$ and S can be written:

$$S = \sum_{i=1}^{N}(X_i + Y_i)$$

where $\{X_i + Y_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables, independent of N. From this it can be seen that S has a compound Poisson distribution with $X_i + Y_i$ representing the 'amount of the i th individual claim'. Standard results can now be used to write down the following formulae for the moments of S:

$$E[S] = 0.4n\, E[X_i + Y_i]$$

$$\text{var}[S] = 0.4n\, E[(X_i + Y_i)^2] = 0.4n(E[X_i^2] + 2E[X_iY_i] + E[Y_i^2])$$

In terms of α, λ and b, the moments of X_i and Y_i are as follows:

$$E[X_i] = \alpha / \lambda \qquad E[Y_i] = (b+50)/2$$

$$E[X_i^2] = \alpha(\alpha+1)/\lambda^2 \qquad E[Y_i^2] = (b^2 + 50b + 2{,}500)/3$$

$$E[X_iY_i] = E[X_i]\,E[Y_i]$$

where the final relationship follows from the independence of X_i and Y_i.

(i) **Now put:**

$$\alpha = 1\,;\ \lambda = 0.01\,;\ b = 100$$

into these formulae to show that:

$$E[S] = 70n \text{ and } \operatorname{var}[S] = 127.80^2 n$$

Hence, S has approximately a normal distribution with mean $70n$ and standard deviation $127.80\sqrt{n}$. The premium income is $80n$ and the smallest value of n is required such that:

$$P(S < 80n) \ge 0.99$$

Standardising S in the usual way for a normal distribution, this becomes:

$$P\left[\frac{S - 70n}{127.80\sqrt{n}} < \frac{80n - 70n}{127.80\sqrt{n}}\right] \ge 0.99$$

The upper 99% point of a standard normal distribution is 2.326 and so the condition for n is:

$$\frac{80n - 70n}{127.80\sqrt{n}} \ge 2.326$$

which gives:

$$n \ge 883.7$$

(or $n \ge 884$ to the next higher integer).

(ii) **For the insurance company, the worst possible combination of values for α, λ and b is the combination which gives the highest possible values for $E[S]$ and var[S]. To see this, let μ and σ denote the mean and the standard deviation of aggregate claims and expenses from a single policy. Both μ and σ will be functions of α, λ and b and:**

$$E[S] = n\mu \text{ and } \operatorname{var}[S] = n\sigma^2$$

Following the same steps as in part (i), the condition for n is:

$$\frac{(80 - \mu)\sqrt{n}}{\sigma} \ge 2.326$$

which becomes:

$$n \ge [2.326\sigma / (80 - \mu)]^2$$

Hence, the highest value of n results from the highest values for μ and σ (provided the highest value for μ is less than 80). Now note that:

$$\mu = 0.4E[X_i + Y_i] \quad \text{and} \quad \sigma^2 = 0.4E[(X_i + Y_i)^2]$$

From the formulae for the moments of X_i and Y_i given above, μ and σ are maximised when α and b are as large as possible and λ is as small as possible, *ie* when:

$$\alpha = 1.05 \; ; \quad \lambda = 0.009 \; ; \quad b = 110$$

This combination of values gives:

$$\mu = 78.67 \quad \text{and} \quad \sigma = 144.14$$

so that n must be at least 63,546 for the insurance company to be at least 99% sure that premium income will exceed claims and expenses outgo.

Chapter 20 Summary

Collective risk model with reinsurance

In the collective risk model, individual claims can be subject to a reinsurance agreement, either proportional or excess of loss.

Under the collective risk model, the aggregate claim amount S is given by:

$$S = X_1 + X_2 + \cdots + X_N$$

where X_i is the amount of the i th claim and N is the total number of claims.

If reinsurance is in place, the insurer's aggregate claims net of reinsurance can be represented as:

$$S_I = Y_1 + Y_2 + \cdots + Y_N$$

where Y_i is the amount of the i th claim paid by the insurer and N is defined as above. S_I is a compound random variable and:

$$E(S_I) = E(N)E(Y)$$

$$\text{var}(S_I) = E(N)\text{var}(Y) + \text{var}(N)[E(Y)]^2$$

$$M_{S_I}(t) = M_N[\ln M_Y(t)]$$

The reinsurer's aggregate claims can be represented as:

$$S_R = Z_1 + Z_2 + \cdots + Z_N$$

where Z_i is the amount of the i th claim paid by the reinsurer and N is defined as above. S_R is a compound random variable and:

$$E(S_R) = E(N)E(Z)$$

$$\text{var}(S_R) = E(N)\text{var}(Z) + \text{var}(N)[E(Z)]^2$$

$$M_{S_R}(t) = M_N[\ln M_Z(t)]$$

Under individual excess of loss reinsurance, some of the claims may fall below the retention level M. If this is the case, then some of the Z_i will be zero. An alternative way of expressing the reinsurer's aggregate claims is as:

$$S_R = W_1 + W_2 + \cdots + W_{NR}$$

where $W_i = Z_i \mid Z_i > 0$ and NR is the number of non-zero claims, *ie* the number of claims in which the reinsurer is involved.

Under an aggregate excess of loss arrangement with retention limit M, the maximum payment made by the insurer is M. The insurer's aggregate claim payment is:

$$S_I = \begin{cases} S & \text{if } S \le M \\ M & \text{if } S > M \end{cases}$$

The reinsurer's aggregate claim payment is:

$$S_R = \begin{cases} 0 & \text{if } S \le M \\ S-M & \text{if } S > M \end{cases}$$

Individual risk model

The individual risk model considers the payments made under each risk (*eg* policy) separately. The model assumes that:

- the number of risks is fixed
- the risks are independent
- claim amounts from these risks are not necessarily IID
- N_j, the number of claims from the j th risk is either 0 or 1.

For a portfolio containing n risks, the aggregate claim amount is given by:

$$S = Y_1 + Y_2 + \cdots + Y_n$$

where Y_j denotes the aggregate claims from risk j. Since each Y_j is the sum of a random number (0 or 1) of random claim amounts, each Y_j has a compound binomial distribution. Suppose that q_j is the probability of a claim from the j th risk. If a claim arises from the j th risk, suppose that the claim amount random variable is X_j. Then:

$$E(S) = \sum_{j=1}^{n} q_j \mu_j$$

$$\text{var}(S) = \sum_{j=1}^{n} \left[q_j \sigma_j^2 + q_j (1-q_j) \mu_j^2 \right]$$

$$M_S(t) = \prod_{j=1}^{n} \left[q_j M_{X_j}(t) + (1-q_j) \right]$$

where $\mu_j = E(X_j)$ and $\sigma_j^2 = \text{var}(X_j)$.

If, for a group of n risks, the probability of a claim is fixed and the claim amounts are IID random variables, then the individual risk model is equivalent to a collective risk model where S has a compound binomial distribution with $N \sim Bin(n,q)$.

Chapter 20 Practice Questions

20.1 The annual aggregate claims from a risk have a compound Poisson distribution with parameter 250. Individual claim amounts have a Pareto distribution with parameters $\alpha = 4$ and $\lambda = 900$. The insurer effects proportional reinsurance with a retained proportion of 75%.

Calculate the variances of the total amounts paid by the insurer and by the reinsurer.

20.2 The aggregate claims from a risk have a compound Poisson distribution with parameter μ. Individual claim amounts (in £) have a Pareto distribution with parameters $\alpha = 3$ and $\lambda = 1{,}000$.

The insurer of this risk calculates the premium using a premium loading factor of 0.2 (*ie* it charges 20% in excess of the risk premium).

The insurer is considering effecting individual excess of loss reinsurance with retention limit £1,000. The reinsurance premium would be calculated using a premium loading factor of 0.3.

The insurer's profit is defined to be the premium charged by the insurer less the reinsurance premium and less the claims paid by the insurer, net of reinsurance.

(i) Show that the insurer's expected profit before reinsurance is 100μ.

(ii) Calculate the insurer's expected profit after effecting the reinsurance, and hence find the percentage reduction in the insurer's expected profit.

(iii) Calculate the percentage reduction in the standard deviation of the insurer's profit as a result of effecting the reinsurance.

20.3 Aggregate annual claims from a portfolio of general insurance policies have a compound Poisson distribution with Poisson parameter 20. Individual claim amounts have a uniform distribution over the interval $(0, 200)$. Excess of loss reinsurance is arranged so that the expected amount paid by the insurer on any claim is 50.

Calculate the variance of the aggregate annual claims paid by the insurer.

20.4 A portfolio of policies consists of one-year term assurances on 100 lives aged exactly 30 and 200 lives aged exactly 40. The probability of a claim during the year on any one of the lives is 0.0004 for the 30 year olds and 0.001 for the 40 year olds.

If the sum assured on a life aged x is uniformly distributed between $1{,}000(x-10)$ and $1{,}000(x+10)$, calculate the variance of the aggregate claims from this portfolio during the year (assuming that policies are independent with regard to claims).

20.5 The number of claims from a given portfolio has a Poisson distribution with a mean of 1.5 per month. Individual claim amounts have the following distribution:

Amount	200	300
Probability	0.65	0.35

An aggregate reinsurance contract has been arranged so that the insurer pays no more than 400 per month in total.

Assuming that the individual claim amounts are independent of each other and are also independent of the number of claims, calculate the expected aggregate monthly claim amounts for the insurer and the reinsurer.

20.6 *Exam style*

A portfolio consists of 500 independent risks. For the i th risk, with probability $1-q_i$ there are no claims in one year, and with probability q_i there is exactly one claim ($0<q_i<1$). For all risks, if there is a claim, it has mean μ, variance σ^2 and moment generating function $M(t)$. Let T be the total amount claimed on the whole portfolio in one year.

(i) Determine the mean and variance of T. [4]

The amount claimed in one year on risk i is approximated by a compound Poisson random variable with Poisson parameter q_i and claims with the same mean μ, the same variance σ^2, and the same moment generating function $M(t)$ as above. Let $\tilde{T}$ denote the total amount claimed on the whole portfolio in one year in this approximate model.

(ii) Determine the mean and variance of $\tilde{T}$, and compare your answers to those in part (i). [4]

Assume that $q_i = 0.02$ for all i, and if a claim occurs, it is of size μ with probability one.

(iii) Derive the moment generating function of T, and show that T has a compound binomial distribution. [2]

(iv) Determine the moment generating function of the approximating $\tilde{T}$, and show that $\tilde{T}$ has a compound Poisson distribution. [2]

[Total 12]

20.7 Exam style

A company is analysing the number of accidents that occur each year on the factory floor. It believes that the number of accidents per year N has a geometric distribution with parameter 0.8, so that:

$$P(N=n)=0.8\times 0.2^{n}, \quad n=0,1,2,...$$

For each accident, the number of employees injured is Y, where $Y=X+1$, and X is believed to have a Poisson distribution with parameter 2.2.

The company has taken out an insurance policy, which provides a benefit of £1,000 to each injured employee, up to a maximum of three employees per accident, irrespective of the level of injury. There is no limit on the number of accidents that may be claimed for in a year.

(i) Show that $E(S)=0.634$ and $\text{var}(S)=2.125$, where S is the total number of employees claiming benefit in a year under this policy. [7]

(ii) Hence find the mean and variance of the aggregate amount paid out under this policy in a year. [1]

[Total 8]

20.8 Exam style

An insurance company offers accident insurance for employees. A total of 650 policies have been issued split between two categories of employees. The first category contains 400 policies, and claims occur on each policy according to a Poisson process at a rate of one claim per 20 years, on average. In this category all claim amounts are £3,000. In the second category, claims occur on each policy according to a Poisson process at a rate of one claim per 10 years, on average. In this category, the claim amount is either £2,000 or £3,000 with probabilities 0.4 and 0.6, respectively. All policies are assumed to be independent. Let S denote the aggregate annual claims from the portfolio.

(i) Calculate the mean, variance and coefficient of skewness of S. [4]

(ii) Using the normal distribution as an approximation to the distribution of S, calculate Y such that the probability of S exceeding Y is 10%. [3]

(iii) The insurance company decides to effect reinsurance cover with aggregate retention £100,000, so that the insurance company then pays out no more than this amount in claims each year. In the year following the inception of this reinsurance, the numbers of policies in each of the two groups remains the same but, because of changes in the employment conditions of which the company was unaware, the probability of a claim in group 2 falls to zero. Using the normal distribution as an approximation to the distribution of S, calculate the probability of a claim being made on the reinsurance treaty. [3]

[Total 10]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 20 Solutions

20.1 The mean and variance of the gross claim amounts are:

$$E(X)=\frac{\lambda}{\alpha-1}=\frac{900}{3}=300$$

$$\text{var}(X)=\frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)}=\frac{4\times 900^2}{3^2\times 2}=180,000$$

So the mean and variance of the net claims for the direct insurer and the reinsurer are:

$$E(Y)=0.75\times 300=225$$

$$\text{var}(Y)=0.75^2\times 180,000=101,250$$

$$E(Z)=0.25\times 300=75$$

$$\text{var}(Z)=0.25^2\times 180,000=11,250$$

Using the formula for the variance of a compound Poisson random variable, the variances of the aggregate claim payments made by the insurer and the reinsurer are:

$$\text{var}(S_I)=\lambda E[Y^2]=\lambda\left[\text{var}(Y)+[E(Y)]^2\right]=250[101,250+225^2]=37,968,750$$

$$\text{var}(S_R)=\lambda E[Z^2]=\lambda\left[\text{var}(Z)+[E(Z)]^2\right]=250[11,250+75^2]=4,218,750$$

20.2 (i) ***Expected profit before reinsurance***

We have:

$$E(X)=\frac{\lambda}{\alpha-1}=500$$

So the expected aggregate claim amount is:

$$E(S)=500\mu$$

The insurer's premium income is:

$$1.2E(S)=1.2\times 500\mu=600\mu$$

So the expected profit before reinsurance is:

$$600\mu-500\mu=100\mu$$

(ii) ***Expected profit after reinsurance***

The reinsurance premium is given by $1.3E(S_R)$, where:

$$E(S_R) = E(Z)E(N) = \mu E(Z)$$

Now:

$$E(Z) = \int_{1,000}^{\infty} (x - 1,000)\frac{3 \times 1,000^3}{(1,000 + x)^4}dx$$

Setting $u = x - 1,000$:

$$E(Z) = \int_0^{\infty} u\frac{3 \times 1,000^3}{(2,000 + u)^4}du = \left(\frac{1,000}{2,000}\right)^3 \int_0^{\infty} u\frac{3 \times 2,000^3}{(2,000 + u)^4}du$$

Recognising this integral as the mean of the $Pareto(3, 2\,000)$ distribution, we see that:

$$E(Z) = \left(\frac{1}{2}\right)^3 \times \frac{2,000}{3 - 1} = 125$$

and:

$$E(S_R) = 125\mu$$

So the reinsurance premium is $1.3 \times 125\mu = 162.5\mu$.

Alternatively we could evaluate this integral using the substitution $t = 1000 + x$ or using integration by parts.

The insurer's expected aggregate claim payment is:

$$E(S_I) = E(S) - E(S_R) = 500\mu - 125\mu = 375\mu$$

So the insurer's expected profit after reinsurance is:

$$600\mu - 162.5\mu - 375\mu = 62.5\mu$$

This is the insurer's premium income, minus the premium paid by the insurer to the reinsurer, minus the insurer's expected aggregate claim payment.

The percentage reduction in the expected profit (which was 100μ without reinsurance) is 37.5%.

(iii) ***Percentage reduction in standard deviation***

In the absence of reinsurance, the insurer's profit is equal to its premium income minus the aggregate claim amount. Since the premium income is a fixed amount and only the cost of claims is random, the variance of the profit is:

$$\text{var}(S) = \mu E(X^2)$$

We have:

$$E(X)=\frac{\lambda}{\alpha-1}=500 \qquad \text{and} \qquad \text{var}(X)=\frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)}=750{,}000$$

So:

$$E(X^2)=\text{var}(X)+[E(X)]^2=750{,}000+500^2=1{,}000{,}000$$

and:

$$\text{var}(S)=1{,}000{,}000\mu$$

Hence the standard deviation of the profit is $1{,}000\sqrt{\mu}$.

With reinsurance, the insurer's profit is equal to premiums charged less the reinsurance premium less the net claims paid. Since the premiums are fixed amounts, the variance of the insurer's profit is:

$$\text{var}(S_I)=\mu E(Y^2)$$

where:

$$E(Y^2)=\int_0^{1{,}000} x^2 \frac{3\times 1{,}000^3}{(1{,}000+x)^4}\,dx+\int_{1{,}000}^{\infty} 1{,}000^2\frac{3\times 1{,}000^3}{(1{,}000+x)^4}\,dx$$

$$=3\times 1{,}000^3\int_0^{1{,}000}\frac{x^2}{(1{,}000+x)^4}\,dx+3\times 1{,}000^5\int_{1{,}000}^{\infty}\frac{1}{(1{,}000+x)^4}\,dx$$

The second integral is:

$$\int_{1{,}000}^{\infty}\frac{1}{(1{,}000+x)^4}\,dx=\left[\frac{(1{,}000+x)^{-3}}{-3}\right]_{1{,}000}^{\infty}=\frac{1}{3\times 2{,}000^3}$$

For the first integral, we can set $u=1{,}000+x$ to give:

$$\int_0^{1{,}000}\frac{x^2}{(1{,}000+x)^4}\,dx=\int_{1{,}000}^{2{,}000}\frac{(u-1{,}000)^2}{u^4}\,du=\left[-\frac{1}{u}+\frac{1{,}000}{u^2}-\frac{1{,}000{,}000}{3u^3}\right]_{1{,}000}^{2{,}000}=\frac{1}{24{,}000}$$

Alternatively, we could integrate by parts (twice).

So:

$$E(Y^2)=\frac{3\times 1{,}000^3}{24{,}000}+\frac{3\times 1{,}000^5}{3\times 2{,}000^3}=250{,}000 \quad \text{and} \quad \text{var}(S_I)=250{,}000\mu$$

Hence the standard deviation of the insurer's profit is now $500\sqrt{\mu}$, which is a reduction of 50%.

The standard deviation is reduced by a greater percentage than the mean. This is very often the case for excess of loss reinsurance.

20.3 We have $X \sim U(0,200)$ and:

$$Y = \begin{cases} X & \text{if } X \le M \\ M & \text{if } X > M \end{cases} \qquad Z = \begin{cases} 0 & \text{if } X \le M \\ X - M & \text{if } X > M \end{cases}$$

The expected amount paid by the insurer on any claim is:

$$E(Y) = \int_0^M x \frac{1}{200} dx + \int_M^{200} M \frac{1}{200} dx = 50$$

Solving this:

$$\left[\frac{x^2}{400}\right]_0^M + \left[\frac{Mx}{200}\right]_M^{200} = 50$$

$$\Rightarrow \quad \frac{M^2}{400} + M - \frac{M^2}{200} = 50$$

$$\Rightarrow \quad M^2 - 400M + 20,000 = 0$$

$$\Rightarrow \quad M = \frac{-(-400) \pm \sqrt{(-400)^2 - 4 \times 1 \times 20,000}}{2 \times 1} = 58.579 \text{ or } 341.42$$

Since claims are a maximum of 200, M must be 58.579.

The variance of the aggregate annual claims paid by the insurer is:

$$\text{var}(S) = \lambda E(Y^2) = 20E(Y^2)$$

where:

$$E(Y^2) = \int_0^M x^2 \frac{1}{200} dx + \int_M^{200} M^2 \frac{1}{200} dx = \left[\frac{x^3}{600}\right]_0^M + \left[\frac{M^2 x}{200}\right]_M^{200}$$

$$= \frac{M^3}{600} + M^2 - \frac{M^3}{200}$$

$$= \frac{58.579^3}{600} + 58.579^2 - \frac{58.579^3}{200}$$

$$= 2,761.42$$

Hence:

$$\text{var}(S) = 20 \times 2,761.42 = 55,228$$

20.4 For each age group, the individual claim amounts have a uniform distribution. So the mean and variance of the individual claim distributions are:

$$E(X) = \tfrac{1}{2}(b+a) = 1{,}000x$$

and: $$\text{var}(X) = \tfrac{1}{12}(b-a)^2 = \frac{20{,}000^2}{12}$$

Using the individual risk model, the variance of the aggregate claim amount is:

$$\begin{aligned}\text{var}(S) &= \sum_{i=1}^{n}\left\{q_i\sigma_i^2 + q_i(1-q_i)\mu_i^2\right\} \\ &= 100\left[(0.0004)\times\frac{20{,}000^2}{12} + (0.0004)(0.9996)\times 30{,}000^2\right] \\ &\quad + 200\left[(0.001)\times\frac{20{,}000^2}{12} + (0.001)(0.999)\times 40{,}000^2\right] \\ &= 37.32\text{m} + 326.35\text{m} = 363.67\text{m}\end{aligned}$$

Alternatively, we could model the aggregate claim amount from each group as a compound binomial random variable. For example, $N \sim Bin(100, 0.0004)$ *for the 100 lives aged exactly 30. We could then use the formula for* $\text{var}(S)$ *from the collective risk model.*

20.5 Under this reinsurance arrangement, we have:

$$S_I = \begin{cases} S & \text{if } S \le 400 \\ 400 & \text{if } S > 400 \end{cases} \qquad S_R = \begin{cases} 0 & \text{if } S \le 400 \\ S - 400 & \text{if } S > 400 \end{cases}$$

where S is the total monthly claim amount.

Since individual claim amounts must be either 200 or 300, the possible values of S_I are 0, 200, 300, and 400 and:

$$E(S_I) = 0 \times P(S_I = 0) + 200 \times P(S_I = 200) + 300 \times P(S_I = 300) + 400 \times P(S_I = 400)$$

The insurer's aggregate claim amount is 0 if there are no claims. So:

$$P(S_I = 0) = P(N = 0) = \frac{e^{-1.5} \times 1.5^0}{0!} = e^{-1.5}$$

The insurer's aggregate claim amount is 200 if there is one claim and the amount of the claim is 200. So:

$$P(S_I = 200) = P(N = 1, X_1 = 200)$$

Since N and X_1 are independent, we have:

$$P(S_I = 200) = P(N = 1)P(X_1 = 200) = \frac{e^{-1.5} \times 1.5^1}{1!} \times 0.65 = 0.975e^{-1.5}$$

Similarly:

$$P(S_I = 300) = P(N = 1, X_1 = 300) = \frac{e^{-1.5} \times 1.5^1}{1!} \times 0.35 = 0.525 e^{-1.5}$$

Finally, the insurer's aggregate claim amount is 400 if the total claim amount is 400 or more. This probability can be calculated by subtraction as follows:

$$P(S_I = 400) = 1 - P(S_I = 0) - P(S_I = 200) - P(S_I = 300)$$

$$= 1 - e^{-1.5} - 0.975 e^{-1.5} - 0.525 e^{-1.5}$$

$$= 1 - 2.5 e^{-1.5}$$

So:

$$E(S_I) = 0 \times e^{-1.5} + 200 \times 0.975 e^{-1.5} + 300 \times 0.525 e^{-1.5} + 400(1 - 2.5 e^{-1.5}) = 255.52$$

We can now calculate $E(S_R)$ using the result:

$$E(S_R) = E(S) - E(S_I)$$

We have:

$$E(S) = \lambda E(X) = 1.5\left[200 \times 0.65 + 300 \times 0.35\right] = 1.5 \times 235 = 352.50$$

Hence:

$$E(S_R) = 352.50 - 255.52 = 96.98$$

20.6 *This is part of Subject 106, April 2003, Question 9.*

(i) ***Mean and variance of T***

Let $T = Y_1 + Y_2 + \cdots + Y_{500}$, where Y_i is the total claim on the i th policy. Then:

$$E[T] = E[Y_1] + E[Y_2] + \cdots + E[Y_{500}]$$

$$\text{var}[T] = \text{var}[Y_1] + \text{var}[Y_2] + \cdots + \text{var}[Y_{500}]$$

Since $E[Y_i] = q_i \mu$ and $\text{var}[Y_i] = q_i \sigma^2 + q_i(1 - q_i)\mu^2$, we have:

$$E[T] = \mu \sum_{i=1}^{500} q_i \qquad [2]$$

$$\text{var}[T] = \sigma^2 \sum_{i=1}^{500} q_i + \mu^2 \sum_{i=1}^{500} q_i(1 - q_i) \qquad [2]$$

since the risks are independent.

(ii) ***Mean and variance of $\tilde{T}$***

Let C be the amount claimed in one year on a single risk. Then, according to the approximation:

$$C = X_1 + X_2 + \cdots + X_N$$

where $N \sim Poi(q_i)$, $E[X] = \mu$ and $\text{var}[X] = \sigma^2$.

Also:

$$\tilde{T} = C_1 + C_2 + \cdots + C_{500}$$

where C_i is the total amount claimed on the i th risk.

Using the formulae for the mean and variance of compound Poisson random variable:

$$E[C_i] = \mu q_i \qquad \text{var}[C_i] = q_i(\mu^2 + \sigma^2)$$ [2]

Since $\tilde{T}$ is the sum of claims for the whole portfolio, we have:

$$E[\tilde{T}] = \mu \sum_{i=1}^{500} q_i \qquad \text{var}[\tilde{T}] = (\sigma^2 + \mu^2) \sum_{i=1}^{500} q_i$$ [2]

The mean is the same but the variance is larger than that obtained in part (i).

(iii) ***MGF***

By definition we have:

$$M_T(t) = E[e^{tT}] = E[e^{t(Y_1 + Y_2 + \cdots + Y_{500})}] = M_{Y_1}(t) M_{Y_2}(t) \ldots M_{Y_{500}}(t)$$ [½]

From the information given in the question, Y_i is either 0 with probability 0.98 or μ with probability 0.02. We can therefore work out the moment generating function of Y_i:

$$M_Y(t) = E[e^{tY}] = e^{t \times 0} \times 0.98 + e^{t\mu} \times 0.02 = 0.98 + 0.02e^{t\mu}$$ [½]

Substituting this into the expression for the moment generating function for T, we get:

$$M_T(t) = (0.98 + 0.02e^{t\mu})^{500}$$ [½]

This is of the form of the moment generating function for a compound binomial distribution with parameters 500 and 0.02, and claim size distribution that is constant. [½]

(iv) ***Compound Poisson***

By definition we have:

$$M_{\tilde{T}}(t) = E[e^{t\tilde{T}}] = E[e^{t(C_1 + C_2 + \cdots + C_{500})}] = M_{C_1}(t) M_{C_2}(t) \ldots M_{C_{500}}(t)$$

From the information given in the question, since C_i has a compound Poisson distribution it has moment generating function:

$$M_{C_i}(t) = \exp\left[q_i(M_X(t)-1)\right] = \exp\left[0.02(M_X(t)-1)\right]$$ [½]

The random variable X takes the value μ with probability 1, so:

$$M_X(t) = e^{t\mu}$$

and:

$$M_{C_i}(t) = \exp\left[0.02(e^{t\mu}-1)\right]$$ [½]

Substituting this into the expression for the moment generating function for $\tilde{T}$, we get:

$$M_{\tilde{T}}(t) = \exp\left[0.02(e^{t\mu}-1)\right]\exp\left[0.02(e^{t\mu}-1)\right]\ldots\exp\left[0.02(e^{t\mu}-1)\right]$$

$$= \exp\left[10(e^{t\mu}-1)\right]$$ [½]

This is of the form of the moment generating function for a compound Poisson distribution with parameter 10, and claim size distribution that is constant. [½]

20.7 (i) ***Total number of claimants***

The aggregate amount paid out by the company is $1{,}000S$, where:

$S = Z_1 + \cdots + Z_N$ is the total number of employees claiming benefit in a year

$$Z = \begin{cases} Y & Y < 3 \\ 3 & Y \geq 3 \end{cases}$$

$$Y = X + 1$$

and:

$$X \sim Poisson(2.2)$$

Now:

$$P(Y=1) = P(X=0) = e^{-2.2}$$ [½]

$$P(Y=2) = P(X=1) = 2.2e^{-2.2}$$ [½]

$$P(Y \geq 3) = 1 - e^{-2.2} - 2.2e^{-2.2} = 1 - 3.2e^{-2.2}$$ [½]

So:

$$E(Z) = 1P(Y=1) + 2P(Y=2) + 3P(Y \geq 3)$$

$$= e^{-2.2} + 4.4e^{-2.2} + 3\left(1 - 3.2e^{-2.2}\right) = 3 - 4.2e^{-2.2} = 2.53463 \qquad [1]$$

and:

$$E(Z^2) = 1^2 P(Y=1) + 2^2 P(Y=2) + 3^2 P(Y \geq 3)$$

$$= e^{-2.2} + 8.8e^{-2.2} + 9\left(1 - 3.2e^{-2.2}\right) = 9 - 19e^{-2.2} = 6.89474 \qquad [1]$$

Hence the variance of Z is:

$$\text{var}(Z) = E\left(Z^2\right) - \left(E(Z)\right)^2 = 6.89474 - 2.53463^2 = 0.47041 \qquad [½]$$

To find $E(N)$ and $\text{var}(N)$, we use the fact that N has a Type 2 negative binomial distribution with parameters $p = 0.8$, $q = 0.2$ and $k = 1$. Using the formulae for the moments given on page 9 the *Tables*, we have:

$$E(N) = \frac{kq}{p} = \frac{0.2}{0.8} = 0.25 \qquad [½]$$

and: $$\text{var}(N) = \frac{kq}{p^2} = \frac{0.2}{0.8^2} = 0.3125 \qquad [½]$$

Alternatively, we could derive the moment generating function of N, and then use MGF formulae to derive the mean and variance of N.

The mean and variance of S are:

$$E(S) = E(Z)E(N) = 2.53463 \times 0.25 = 0.63366 \qquad [1]$$

and:

$$\text{var}(S) = \left(E(Z)\right)^2 \text{var}(N) + \text{var}(Z)E(N) = 2.53463^2 \times 0.3125 + 0.47041 \times 0.25 = 2.12521 \qquad [1]$$

(ii) ***Mean and variance of the aggregate amount***

So the mean and variance of $1{,}000S$ are:

$$E(1{,}000S) = 1{,}000E(S) = 634 \qquad [½]$$

and:

$$\text{var}(1{,}000S) = 1{,}000^2 \text{var}(S) = 2{,}125{,}000 \qquad [½]$$

Alternatively, we could define $S = Z_1 + Z_2 + \cdots + Z_N$*, where:*

$$Z = \begin{cases} 1{,}000Y & \text{if } Y \le 3 \\ 3{,}000 & \text{if } Y > 3 \end{cases}$$

This would give us the aggregate claim amount directly.

20.8 (i) ***Mean, variance and coefficient of skewness***

We have $N_1 \sim Poisson(400 \times \frac{1}{20}) \equiv Poisson(20)$, $X_1 = £3{,}000$, $N_2 \sim Poisson(250 \times \frac{1}{10}) \equiv Poisson(25)$ and:

$$X_2 = \begin{cases} £2{,}000 & \text{with probability } 0.4 \\ £3{,}000 & \text{with probability } 0.6 \end{cases}$$

Working in £000s, we find that:

$$E(X_2) = (2 \times 0.4) + (3 \times 0.6) = 2.6$$

$$E\left(X_2^2\right) = \left(2^2 \times 0.4\right) + \left(3^2 \times 0.6\right) = 7$$

$$E\left(X_2^3\right) = \left(2^3 \times 0.4\right) + \left(3^3 \times 0.6\right) = 19.4 \qquad [1]$$

Let S_i denote the annual aggregate claims from category i. Using the assumption that the policies are independent and the result that, for a compound Poisson random variable T, the k th central moment of T is given by $\lambda E(X^k)$, we obtain:

$$E(S) = E(S_1) + E(S_2) = (20 \times 3) + (25 \times 2.6) = 125 = £125{,}000 \qquad [1]$$

$$\begin{aligned} \text{var}(S) &= \text{var}(S_1) + \text{var}(S_2) \\ &= \lambda_1 E(X_1^2) + \lambda_2 E(X_2^2) \\ &= (20 \times 9) + (25 \times 7) \\ &= 355 \\ &= £^2 355{,}000{,}000 \qquad [1] \end{aligned}$$

$$\begin{aligned} \text{skew}(S) &= \text{skew}(S_1) + \text{skew}(S_2) \\ &= \lambda_1 E(X_1^3) + \lambda_2 E(X_2^3) \\ &= (20 \times 27) + (25 \times 19.4) \\ &= 1{,}025 \\ &= £^3 (1{,}025 \times 10^9) \end{aligned}$$

So the coefficient of skewness is:

$$\frac{\text{skew}(S)}{\left(\text{var}(S)\right)^{3/2}} = \frac{1,025}{355^{3/2}} = 0.15324 \qquad [1]$$

(ii) ***Calculate Y using a normal approximation***

Assuming that $S \sim N(125, 355)$, we have:

$$P(S > Y) = 0.1 \Rightarrow P\left(N(0,1) > \frac{Y-125}{\sqrt{355}}\right) = 0.1$$

$$\Rightarrow \frac{Y-125}{\sqrt{355}} = 1.2816 \qquad \text{from page 162 } \textit{Tables}$$

$$\Rightarrow Y = 149.147$$

So S exceeds £149,000 with a probability of approximately 0.1. [3]

(iii) ***Probability that reinsurer is involved***

The expected value and variance of S are now the same as those of S_1. Working in £000s and assuming that $S \sim N(60, 180)$, we obtain:

$$P(S > 100) = P\left(N(0,1) > \frac{100-60}{\sqrt{180}}\right) \approx 1 - \Phi(2.98) = 1 - 0.9986 = 0.0014 \qquad [3]$$