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**ABSTRACT**

This is summarization of different materials as guideline for leading learning of the course ST2109 “Risk theory and management” offered at the University of Dodoma UDOM for B.Sc. Actuarial Statistics year 2 candidates. The guide is to at best reflect the course outline (as of academic year 2024-2025), prospective professional development considering IFoA curriculum, hands on practice (data driven exercises) and development in the field (relevant research works and methodological development, including applied machine learning)

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# CHAPTER 1: UTILITY THEORY (CM2)

## INTRODUCTION

ST PETER'SBURG PARADOX on expected returns and fair price for a gamble

Utility function $u(w)$ for wealth $w$

Expected Utility Theorem (EUT) $E[u(w)]$

- Function express investors wealth in the future
- Decision aims to maximize expected utility

Calculating

$$E[u(w)] = \sum_{\forall_i} p_i * u(w_i)$$

Maximizing Distribution of Asset

$\alpha, \beta, \gamma$ ... The proportion of asset to be placed in variety

$$\frac{\partial(E[u(w)])}{\partial(\alpha, \beta, \gamma \dots)}$$

## EUT AXIOMS

- **Comparability**

Options $a, b, c$ ... are comparable denoted by

$u(a) > u(b) > u(c)$ ... or $u(b) = u(c)$

- **Transitivity**

If

$u(a) > u(b)$ and $u(b) > u(c)$ Then $u(a) > u(c)$

Likewise, if

$u(a) = u(b)$ and $u(b) = u(c)$ Then $u(a) = u(c)$

- **Independence**

If $u(a) = u(b)$ at known probability $p$

$$p * u(a) + (1 - p) * u(c) = p * u(b) + (1 - p) * u(c)$$

- **Certainty Equivalence**

If $u(a) > u(b) > u(c)$ There is s probability $p$ utility of $a, c$ will be indifferent from $b$ that is

$$p * u(a) + (1 - p) * u(c) = u(b)$$

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## EXPRESION OF ECONOMIC CHARACTERISTICS

**Non-Satiation**

$$u'(w) > 0$$

**Risk Averse** (reject fair gamble)

$$u''(w) < 0$$

**Risk Seeking** (Lover) (accept fair gamble)

$$u''(w) > 0$$

**Risk Neutral** (act indifferent)

$$u''(w) = 0$$

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## CERTAINTY EQUIVALENCE

*Let* $c_w$ *the CE of wealth,* $w$ *on gamble* $x$

$c_x$ *the CE of gamble* $x$

**Additive gamble**

$$u(c_w) = E[u(w + x)]$$

then

$$c_x = c_w - w$$

Example:

Betting Account: TZS 10

Play: Win +1 TZS, Loss -1 TZS

Utility: $u(w) = \ln(w)$

$$\ln(c_w) = 0.5 * (\ln(11) + \ln(9))$$

$$c_w = 9.94987437\,;\, c_x = 9.94987437 - 10 = -0.0501256$$

Investor will be indifferent between guaranteed amount $c_w$ and taking the gamble with potential outcome of 9 or 11

He is willing to pay 0.0501256 to avoid the gamble

**Proportional gamble**

$y$ – sums of wins or lost expressed as proportion of $w$

$$u(c_w) = E[u(w * y)]$$

example if win $+15\%$ ,then $y = 115\%$

if lose $-15\%$ ,then $y = 85\%$

Relative Risk Aversion

$$RRA = \frac{c_x}{w} = \frac{c_w - w}{w}$$

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## ARROW PARATT MEASURES

**Absolute Risk Aversion** (Risk Aversion Coefficient)

$$r(w) = -\frac{u''(w)}{u'(w)}$$

**Relative Risk Aversion**

$$R(w) = -w * \frac{u''(w)}{u'(w)} = -w * r(w)$$

| | $r(w)$ | $R(w)$ |
|---|---|---|
| Increasing | $r'(w) > 0$ | $R'(w) > 0$ |
| Constant | $r'(w) = 0$ | $R'(w) = 0$ |
| Decreasing | $r'(w) < 0$ | $R'(w) < 0$ |

## CLASS OF UTILITY FUNCTIONS

**Examples**

**Quadratic**

$$u(w) = a + bw + cw^2$$

**Logarithmic**

$$u(w) = \ln(w)$$

- It is ***iso-elastic,*** *constant RRA* $(r(w))$
- Allows **myopic decisions**, forget past and decide based on present expected outcomes

**Power**

$$u(w) = \frac{w^{\gamma} - 1}{\gamma}\ ; w > 0$$

- It is ***iso-elastic***
- Generalized form of Hyperbolic Absolute Risk Aversion (HARA), $\gamma -$ risk aversion

**It** can take variety form of functions

## UTILITY AND INSURANCE

**Finding Maximum Premium** (Individual willing to pay)

For Loss $x$ on wealth $w$ and premium $P$ can be determined

$$E[u(w - x)] = u(w - P)$$

**Finding Minimum Premium** (Insurer has to charge)

$$E[u(w + P - x)] = u(w)$$

## PREMIUM AND RISK ARVERSION

Estimating Premium from the risk's moments and risk aversion

- Provided empirical distribution of risk can be derived then: -

$$P = \frac{1}{r(w)} * \ln\left(M_x\left(r(w)\right)\right)$$

$$P \approx \mu + \frac{1}{2} * r(w - x) * \delta^2$$

## MARGINAL UTILITY

Change in utility with change in quantity consumed

$$MU = \frac{\Delta U(w)}{\Delta Q}$$

- diminishing marginal utility decreasing and vice versa

## LIMITATIONS OF UTILITY THEORY

- Difficulty in determining individual utility function
- Difficulty applicability for corporate (collection of individuals with varied utilities)

## EXAMPLE AND QUESTIONS (TUTORIAL)

**Question 1** (CM2A SEPT 2022, QN 1)

An individual has the following utility function: $u(w) = \frac{w^{\gamma}-1}{\gamma}, (w > 0)$, where w is wealth in \$000s. Their current wealth is \$8,000 and their current utility is 2.1012.

(i) Show that $\gamma = 0.01$ to two decimal places.

(ii) Show that U(w) exhibits declining absolute risk aversion and constant relative risk aversion.

The individual has been offered a ticket to enter a lottery with a 1 in 10,000 chance to win \$1m.

(iii) Calculate, to the nearest \$, the maximum price, P, that the individual would pay for the ticket.

(iv) Discuss why this form of utility function with $\gamma > 1$ would be inconsistent with common utility theory.

**Question 2** (CM2A APR 2022, QN 7)

An investor makes decisions based on the utility function

$$u(w) = w - 6w^2$$

where w is the investor's wealth in millions of dollars ($m).

(i) Demonstrate that the investor has both increasing absolute and relative risk aversion.

The investor has $50,000 to invest over a 1-year period and has no other wealth. They have three options:

a. Invest in a risk-free account. There will be no change in the value of the investment over 1 year.
b. Invest in an asset that will give a 60% return over 1 year with probability 0.2, a 20% return with probability 0.7 and a –40% return with probability 0.1.
c. Invest in an asset that will give a 30% return with probability 0.5 and a 20% return with probability 0.5.

The investor makes no allowance for discounting when making investment decisions. The investor must invest the whole $50,000 in a single option.

(ii) Determine which option the investor should choose to maximise their expected utility at the end of the year.

(iii) Comment on why the investor could not use U(w) to choose from the above options if their initial wealth was $65,000.

**Question 3** (CM2A SEPT 2021, QN 1)

An investor makes decisions using the utility function $u(w) = \ln(w)$ where $w > 0$. The investor is going to invest \$100 now for a period of 1 year, and has identified the following two assets to invest in:

- Asset A is risk-free and will not change in value over the year.
- Asset B will increase in value by 50% over the year with probability 0.6 or decrease in value by 50% over the year with probability 0.4.

The investor does not make any allowance for discounting when making investment decisions. They are going to invest a proportion, $x$, of their wealth in Asset A and the remaining proportion, $(1 - x)$, in Asset B.

(i) Construct a formula, in terms of x, for their expected utility at the end of the year.

(ii) Determine, using your result from part $(i)$, the amount that the investor should invest in each asset to maximize their expected utility.

**Question 4**

Do all examples in core reading (**CM2 chapter 2**) and end chapter questions

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## EXCEL TASK 1 (FOR CLASS, ILLUSTRATION)

An investor with wealth $(w)$ has a log utility

$$u(w) = \ln(w)$$

i) Calculate the following result for $w = \{10, 20, \ldots, 90, 100\}$

a. Utility
b. Absolute risk aversion
c. Relative risk aversion

Investor has the opportunity to invest in risky asset which offers total returns, X, that are uniformly distributed between $-5\%\ and + 2\%$. The remainder of the investor's wealth generates zero return. Let $r$ be the amount invested in this risky asset, then the expected utility to the investor is given by

$$E[u] = E\left[\ln\left(w - r + r(1 + X)\right)\right]$$
$$= E[\ln(w + rX)]$$
$$= \frac{1}{b-a} * \int_a^b \ln(w + rx)\, dx$$
$$= \frac{1}{b-a} * \left[\frac{(w + rx) * \ln(w + rx) - (w + rx)}{r}\right]_a^b$$
$$a = -5\%\,, b = 2\%$$

Investor decides to keep 5 GBP of their wealth in the risky asset, regardless of their total level of wealth.

ii) Calculate the expected Utility of the investor at each wealth level

iii) Determine the certainty equivalent of the wealth $c_w$, and therefore the certainty equivalent of the gamble $c_x$ for each level of wealth

iv) Plot the chart of $c_x$ and describe its nature with reference to the measure of absolute risk aversion

The investor then decides instead to keep 5% of their wealth in risky asset, regardless of their total level of wealth

v) Calculate the expected utility of the investor for each wealth level

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## EXCEL TASK 2 (FOR CLASS, ASSIGNMENT)

### Insurance Premium

The insurance company uses claims experience to determine the minimum premium it is prepared to charge in order to cover future risks. The given data shows the severity of 1,000 such claims. The insurance company and a prospective policy holder have wealth $w_i$ and $w_p$, utility function $u(w_i) = \sqrt{w_i}$ and d $u(w_p) = \ln(w_p)$ and initial wealth 10,000 GBP and 1,000 GBP respectively

i) Calculate the frequency of claims occurring in the ranges with limits 0,10,20..., 90,100,

ii) Using the results from part (i), calculate the likelihood of future claims occurring within each of the claim ranges

Let p be the insurance premium, and let X be the random variable representing future claims severity

iii) Assuming that future claims arise at the mid-point of each of the claim ranges, calculate the utility of the insurer and the policyholder for each of the claim ranges

iv) Combining the results from the previous two parts, calculate the following items

a. The expected utility of the insurer who accepts the risk, $E[u(w_i + p - x)]$

b. The expected utility of the prospective policyholder who fails to take out insurance $E[u(w_p - x]$

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**EXCEL TASK 3 (FOR CLASS, ASSIGNMENT)**

**Constructing Utility Function**

A set of individuals want to insure their properties, an insurance company used indirect questioning based on the likelihood of occurrence of the risk from previous claim experience, sum to be insured (wealth) and premium insured is willing to pay. Use the data in excel provided in 'TASK 4' sheet to determine nature of utility function of the potential insurance company customers.

## ASSIGNMENT 1

## Stopping time of a gamble

Investors are allowed to choose how much they stake on a particular gamble. The outcome of the gamble is determined by the toss of unfair coin in the following way

- 'Heads' – the investor's wealth increases by the amount of their stake
- 'Tails' – the investor's wealth decreases by the amount of their stake

Assume that the probability of heads is 0.55

An investor has realized that by starting with a stake of 1 GBP and repeating until they eventually win (learn St Petersburg paradox, not necessary) can result in a guaranteed profit of 1 GBP if they double their stake each time the coin is tossed. The investor is so convinced of this strategy that they are prepared to borrow money (use boom 😊) in order to execute it.

The same investor has an initial wealth of 49 GBP and quadratic utility function

$$u(w) = w + d * w^2$$

$w$ −level of wealth, $d = -0.005$

i) Calculate the investor's utility if they were eventually to win the gamble

ii) Create a table to calculate the stake that the investor would wager if the coin is tossed ten times each time resulting in 'Tails'

iii) Calculate the investor's wealth and utility before and after each losing coin toss

iv) Determine when the investor needs to borrow money in order to keep playing

v) Calculate the expected utility of the investor each time they have a choice to whether to play or not

# CHAPTER 2: MEASURES OF INVESTMENT RISK (CM2)

## INTRODUCTION

## VARIANCE OF RETURNS

$$var(X) = \int_{-\infty}^{\infty} (\mu - x)^2 * f(x)dx$$

**Example 1**

Investment returns (% pa), X , on a particular asset are modelled using a probability distribution with density function:

$$f(x) = 0.000075 * (100 - (x - 5)^2), -5 \leq X < 15$$

Calculate the mean return and the variance of return.

## SEMI-VARIANCE OF RETURNS

$$\int_{-\infty}^{\mu} (\mu - x)^2 * f(x)dx$$

- Takes into account variability of the mean

**Example 2**

From example 1 calculate semi variance of the returns

## SHORTFALL PROBABILITY

- Probability of returns falling below a certain level

$$\int_{-\infty}^{L} f(x)dx$$

**Example 3**

From example 1calculate shortfall probability when benchmark return is

(i) 0

(ii) 3

## VALUE AT RISK

- Value $t$ for loss going below a certain probability value $\alpha$
- find $t$ given $\alpha$
- Can use inverse normal approximation

$$P(X < t) = \alpha$$

**Example 4**

If company has portfolio with 100M USD, calculate the value at risk with

(i) 95% confidence level

(ii) 50% confidence level

(iii) 99% confidence level

(iv) Comment on the above results

---

## TAIL VALUE AT RISK

$$E[L - X, 0] = \int_{-\infty}^{L} (L - x) * f(x)\, dx$$

**Example 5**

Repeat example 4 by calculating Tail Value at Risk

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## ASSIGNMENT 2

**Objective: Investment Analysis and Forecasting Assignment**

The goal of this assignment is to analyze the investment returns of at least 20 stocks (e.g., "AAPL", "IBM", "MSFT", etc.) by retrieving daily closing prices from Yahoo Finance. You will apply various investment measures such as variance, semi-variance, shortfall probability, and value at risk (VaR). You will also perform projections using a linear model to forecast stock prices and evaluate the ideal stock for investment.

**Part 1: Data Retrieval**

**Choose 20 Stocks**: Select at least 20 publicly traded stocks (e.g., "AAPL", "IBM", "MSFT") from various industries.

**Data Retrieval Period**: Retrieve daily closing prices for each stock for the following years: 2000, 2005, 2010, 2015, 2024

*(Alternatively, you can use one-year intervals starting from 2000 if you prefer).*

**Data Focus**: Focus on **daily closing prices**. These will be the primary data for your analysis.

**Part 2: Investment Return Measures**

For each stock, calculate the following investment return measures based on the **daily closing prices** for each year (2000, 2005, 2010, 2015, 2024):

**Variance of Returns**, **Semi-Variance of Returns**

**Shortfall Probability**

Benchmark return = 0%

**Value at Risk (VaR), Tail Value at Risk (TVaR)**

95% confidence level

**Part 3: Comparative Stock Analysis**

**Compare Investment Returns**: Based on your calculations, compare:

The mean return, Variance and semi-variance, Shortfall probabilities, Value at Risk, Tail Value at Risk.

**Investment Decision**: Based on your analysis, recommend which stock would have been the **most ideal** to invest in considering:

The highest return, The least volatility (variance and semi-variance)

The best risk-adjusted return (e.g., comparison of shortfall probability, VaR, and TVaR), A balance between high returns and low risk

**Part 4: Forecasting with Linear Models (Optional)**

**Linear Model for Price Forecasting**: Use a **simple linear regression model** to forecast the closing prices for the next 5 years (from 2024 onward) for each stock.

**Forecast Evaluation**: Based on your projections, recommend which stock would be the best investment over the next 5 years by comparing growth projections and risk factors.

**Part 5: Reporting and Conclusion**

**Summarize Findings**:

Present your results in a clear format. Include relevant charts (e.g., return distributions, volatility, projections).

**Conclusion**:

Provide a detailed recommendation on which stock is the most ideal for investment based on your calculations and analysis.

**Reflection** (Optional):

Discuss the impact of adjusting constants (confidence level, benchmark return) and how these adjustments affect your results.

Submission Requirements:

A **written report** summarizing your findings. (very brief, not necessary)

**Code** used for data retrieval and analysis. (necessary)

**Charts and visualizations** to support your conclusions. (necessary in code)

# CHAPTER 3: LOSS DISTRIBUTIONS (CS2)

## INTRODUCTION

**Cumulative Distribution Function**

$$F_X(x) = P(X < x)$$

**Probability Density function**

$$f(x) = F'(x)$$

**Moment Generating Function**

$$M_X(t) = E[e^{tX}]$$

**Cumulant generating function**

$$C_X(t) = \ln\left(M_X(t)\right)$$

**Coefficient of skewness**

$$= \frac{skew(X)}{var(x)^{\frac{3}{2}}}$$

## EXPONENTIAL DISTRIBUTION

$$F(x) = 1 - e^{-\lambda x}$$

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$E[X] = \frac{1}{\lambda} \ ; var(X) = \frac{1}{\lambda^2}$$

express, mean variance and skewness

## GAMMA DISTRIBUTION

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}$$

$$E[X] = \frac{\alpha}{\lambda} \ ; var(X) = \frac{\alpha}{\lambda^2}$$

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## CHI-SQUARE DISTRIBUTION

relation to gamma ($\lambda = \frac{1}{2}, \alpha = \frac{\nu}{2}$)

$$2\lambda X \sim X^2_{2\alpha}$$

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## NORMAL DISTRIBUTION

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\frac{1}{2}*\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$M_X(t) = e^{\mu*t+\frac{1}{2}*\sigma^2*t^2}$$

$$E[X] = \mu \ ; var(X) = \sigma^2$$

---

## LOG-NORMAL DISTRIBUTION

$$logX \sim N(\mu, \sigma^2)$$

$$X \sim logN(\mu, \sigma^2)$$

$$E[X] = E[e^{\ln(X)}]$$

$$E[X] = e^{\mu+\frac{1}{2}\sigma^2} \ ; var(X) = e^{\mu+\frac{1}{2}\sigma^2} * (e^{\sigma^2} - 1)$$

---

## TWO-PARAMETER PARETO DISTRIBUTION

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}$$

$$E[X] = \frac{\lambda}{\alpha - 1} \ ; var(X) = \alpha * \frac{\lambda^2}{(\alpha-1)^2(\alpha-2)}$$

---

## BURR DISTRIBUTION

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x^{\gamma}}\right)^{\alpha}$$

$$E[X^r] = \Gamma\left(\alpha - \frac{r}{\gamma}\right) * \Gamma\left(1 + \frac{r}{\gamma}\right) * \frac{\lambda^{\frac{r}{\gamma}}}{\Gamma(\alpha)}$$

---

## THREE PARAMETER PARETO DISTRIBUTION

$$f(x) = \frac{x^{k-1} * \lambda^{\alpha}}{(\lambda + x)^{\alpha+k}} * \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)\Gamma(k)}$$

$$E[X] = \frac{k\lambda}{\alpha - 1} \quad ; var(X) = \alpha * \frac{k(k + \alpha - 1)\lambda^2}{(\alpha - 1)^2(\alpha - 2)}$$

Weibull distribution

$F(x) = 1 - e^{-cx^{\gamma}}$ for $X > 0$

---

## ESTIMATION OF PARAMETERS BY METHOD OF MOMENTS

$$E[x_i^j] = \frac{1}{n} * \sum_{i=1}^{n} x_i^j$$

example

$$E[x] = \frac{1}{n} * \sum_{i=1}^{n} x_i$$

$$E[x_i^2] = \frac{1}{n} * \sum_{i=1}^{n} x_i^2$$

you can check for variance and other higher order moments

---

## ESTIMATION OF PARAMETER BY MAXIMUM LIKELIHOOD ESTIMATION (MLE)

- Check the product limit

$$L = \prod_{i=1}^{n} f(x_i)$$

apply log function

$$l = \log(L)$$

take partial derivative with respect to estimated parameter

- in r you can use fitdistr library

---

## ASSESSING GOODNESS OF FIT

Checking for goodness of fit of the distribution

- decide which distribution best fits the data

$\sum\left(\frac{(O-E)^2}{E}\right)$ – focused to minimize this value

---

## QUESTIONS

**Question 1**

Suppose that $X \sim \Gamma(\alpha, \beta)$Derive formulae for the skewness and coefficient of skewness of $X$ .

**Question 2**

Claims arising from a particular group of policies are believed to follow a Pareto distribution with parameters $\alpha$ and $\lambda$ . A random sample of 20 claims gives values such that $\sum x = 1{,}508$ and = $\sum x^2 257{,}212$. Estimate α and λ using the method of moments.

**Question 3**

A loss amount random variable has

$$M(t) = 0.4 * (1 - 20t)^{-2} + 0.6 * (1 - 30t)^{-3}$$

Calculate the expected loss amount.

**Question 4**

Find the maximum likelihood estimate of parameter $\alpha$ of the Burr distribution

**Question 5**

Individual claim amounts on a portfolio of motor insurance policies follow a gamma distribution with parameters $\alpha$ and $\lambda$ . It is known that $\lambda = 0.8$ for all drivers, but the value of $\alpha$ varies across the population. Given that $\alpha \sim \Gamma(200, 0.5)$, calculate the mean and variance of a randomly chosen claim amount.

---

## ASSIGNMENT 3

You are provided with car claim data from one of general insurance company.

(i) You are required to state which distribution best fits the data by evaluating all distributions discussed in this section.

(ii) You should do the (i) above for

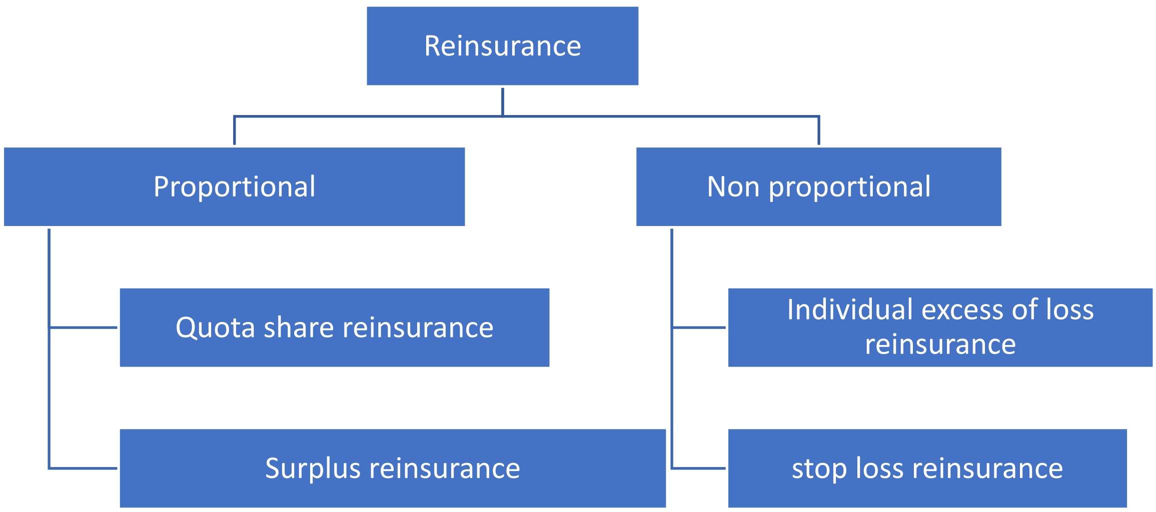
   a. Injury claim amount
   b. vehicle claim amount
   c. property claim amount
   d. Total claim amount

(iii) All columns have the respective names in the dataset

# CHAPTER 4: REINSURANCE (CS2)

**(extreme value theory and copulas to be done quick)**

## INTRODUCTION




## PROPORTIONAL REINSURANCE(CONCEPTS)

Insurer and reinsurer share cost for all claims ($75\%\ by\ 25\%$), despite of claim size

- Quota share reinsurance – proportion are same for all risks
- Surplus reinsurance – proportions may vary for risks

Non-proportional reinsurance

reinsurer pays claim if falls in certain limits $[L, U]$

- lower limit $L$ retention limit
- Upper limit $U$ can be infinity

- **Individual excess of loss reinsurance (XOL)**
  - reinsurer pays if claim exceed a certain limit
  - consider individual policy
- **Stop loss reinsurance**
  - reinsurer pays if total claim exceeds a certain limit

- considers a group of policies
- may be expressed as percentage of gross premium

Reinsurance arrangement
insurer's gross claim amount = total claim resulted from policy ($X$)
insurer's gross premium amount = total premium paid by policy ($P_X$)
insurer's net claim amount ($Y$)= ($X - claim\ paid\ by\ reinsurance\ (Z)$)
insurer's net premium amount = ($P_X - reinsurance\ premium\ (P_Z)$)

Example
For a given reinsurance arrangement, express the random variables Y and Z in terms of X .
For example, suppose that a reinsurer has agreed to make the following payments in respect of individual claims incurred by a direct insurer:

- nothing, if the claim is less than £5,000
- the full amount reduced by £5,000, if the claim is between £5,000 and £10,000
- half the full amount, if the claim is between £10,000 and £20,000
- £10,000, if the claim exceeds £20,000.

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## EXCESS OF LOSS REINSURANCE

given retention level $M$

| Claim condition | Insurer payment | Reinsurer payment |
|---|---|---|
| $X \leq M$ | $Y = X$ | 0 |
| $X > M$ | $Y = M$ | $Y = X - M$ |

this reduces both mean and variance of insurer liability

| | without reinsurance | with reinsurance |
|---|---|---|
| mean | $E[X] = \int_0^\infty x * f(x)\, dx$ | $E[X] = \int_0^M x * f(x)\, dx + M * P(X > M)$ |

| | | |
|---|---|---|
| | $E[X^2] = \int_0^\infty x^2 * f(x)\,dx$ | $E[X^2] = \int_0^M x^2 * f(x)\,dx + M^2 * P(X > M)$ |

Moment generating functions of net insurer payments

$$M_Y(t) = E[e^{tY}] = \int_0^M e^{tx} * f(x)dx + e^{tM} * P(X > M)$$

$$E[Z] = \int_M^\infty (x - M)f(x)dx = E[X] - E[Y]$$

$$E[Z^2] = \int_M^\infty (x - M)^2 * f(x)dx$$

Example

Suppose that claim amounts are uniformly distributed over the interval (0,500). The insurer effects individual excess of loss reinsurance with a retention limit of 375.

Calculate the expected amounts paid by the insurer and the reinsurer in respect of a single claim.

---

## CONDITIONAL CLAIM DISTRIBUTION

- Reinsurer observes claims that are greater than M
- Thus, distribution of his claims is truncated from original call it W

$$W = X - M | X > M$$

$$w = x - M$$

$$P(W < w) = P(X < w + M | X > M) = \frac{P(M < X < w + M)}{P(X > M)}$$

$$= \int_M^{w+M} \frac{f(x)}{1 - F(M)} dx = \frac{F(w + M) - F(M)}{1 - F(M)}$$

$$F(W) = \frac{F(w + M) - F(M)}{1 - F(M)}$$

$$f(w) = \frac{f(w + M)}{1 - F(M)}$$

Reinsurer conditional claim distribution is given as

$$f_W(w) = \frac{f_X(w + M)}{1 - F_X(M)}$$

**Example**

Determine conditional claim distribution for

(i) $X \sim \exp(\lambda)$

(ii) $X \sim \Gamma(\alpha, \beta)$

(iii) $X \sim Pa(\alpha, \lambda)$

---

## PROPORTIONAL REINSURANCE

| | Insurer | Reinsurer |
|---|---|---|
| Loss to random variable | $Y = \alpha * X$ | $Z = (1 - \alpha) * X$ |
| Expected value | $E[Y] = \alpha * E[X]$ | $E[Z] = (1 - \alpha) * E[X]$ |
| Variance | $Var[Y] = \alpha^2 * Var[X]$ | $Var[Z] = (1 - \alpha)^2 * Var[X]$ |

**Example**

Claims from a particular portfolio have a generalised Pareto distribution with parameters $\alpha = 6$ , $\lambda = 200$ and $k = 4$ . A proportional reinsurance arrangement is in force with a retained proportion of 80%.

Calculate the mean and variance of the amount paid by the insurer and the reinsurer in respect of a single claim.

**Example**

Claims from a particular portfolio have an exponential distribution with mean 1,000. The insurer takes out proportional reinsurance with a retained proportion of 0.9.

Determine the distribution of the insurer's net claim amount random variable.

## TRUNCATED NORMAL DISTRIBUTION

truncated section

$$\int_L^U xf(x)\,dx = \int_L^U x * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{-\frac{1}{2}*\left(\frac{x-\mu}{\sigma}\right)^2}\,dx$$

$$= \mu * (\Phi(U') - \Phi(L')) - \mu * (\phi(U') - \phi(L'))$$

$U' = \frac{U-\mu}{\sigma}$ ; $L' = \frac{L-\mu}{\sigma}$

**Example**

Claims from a particular portfolio are normally distributed with mean 800 and standard deviation 100. An individual excess of loss arrangement with retention limit is 860 is in place. Calculate the insurer's mean claim payment net of reinsurance.

## TRUNCATED LOG-NORMAL DISTRIBUTION

$$X \sim logN(\mu, \sigma^2)$$

$$\int_L^U x^k * f(x)dx = e^{\mu*k+\frac{1}{2}*\sigma^2*k^2}(\Phi(U_k) - \Phi(L_k))$$

$$L_k = \frac{lnL - \mu}{\sigma} - k\sigma$$

$$U_k = \frac{lnU - \mu}{\sigma} - k\sigma$$

**Example**

An insurer is considering taking out one of the following reinsurance treaties:

Treaty 1: Proportional reinsurance with a retained proportion of 0.75

Treaty 2: Individual excess of loss cover with a retention limit of £25,000

The claims distribution is lognormal with parameters 8.5 μ = and 2 σ = 0.64 .

Calculate the insurer's expected net claim payments in the following cases:

(a) without either treaty

(b) with Treaty 1 only

(c) with Treaty 2 only.

---

## IMPACT OF INFLATION

- In long run claims are likely to increase because of inflation
- So, the claim distribution needs also to be adjusted

For proportion reinsurance is simple

$$Y = \alpha X$$

considering inflation $k$

$$Y = \alpha k X$$

For excess of loss reinsurance

the retention limit remains the same but random loss changes

$Y = kX \quad \text{if} \quad kX \le M$

$Y = M \quad \text{if} \quad kX > M$

$$Y = \begin{Bmatrix} kX \; if \; X \le \dfrac{M}{k} \\ M \; if \; X > \dfrac{M}{k} \end{Bmatrix}$$

---

## ESTIMATING MODEL'S PARAMETERS

You can use MLE

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## POLICY EXCESS/ DEDUCTIBLES

Policy Excess/deductible insurance

- Common in motor, health insurance

Insurer only pays when a claim exceeds a certain (prior stated) limit

## QUESTIONS

**Question 1**

Losses from a group of travel insurance policies are assumed to follow a Pareto distribution with parameters 4.5 $\alpha =$ and $\lambda = 3{,}000$ . Next year losses are expected to increase by 3%, and the insurer has decided to introduce a policy excess of 100 per claim. Calculate the probability that a loss next year is borne entirely by the policyholder.

**Question 2**

(i) Loss amounts from a particular type of insurance have a Pareto distribution with parameters $\alpha$ and $\lambda$ . If the company applies a policy excess, E, derive the distribution function of claim amounts paid by the insurer. [3]

(ii) Assuming that 4 $\alpha =$ and $\lambda = 15$, calculate the mean claim amount paid by the insurer:

(a) with no policy excess ( $E = 0$ ),

(b) with an excess of 10 ( $E = 10$ ). [2]

(iii) Using your answers to (ii), comment on the effect of introducing a policy excess. [2]

## ASSIGNMENT 4

# CHAPTER 5: INDIVIDUAL AND COLLECTIVE RISK MODELS (CS2)

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## INTRUDUCTION

For risk to be insurable

- Policyholder must have insurable interest
- Should be measurable/quantifiable
- They should be independent events
- Low probability of occurrence
- There should be ultimate liability
- Moral hazard should be eliminated

Characteristic among general insurance

- Cover is over a fixed time period
- Claims are not of fixed amount
- Claim does not end the policy
- Claims may occur at any time of cover

Long-tail, short-tail claims -> Claims that takes long, short time to settle respectively

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## MODEL DESCRIPTION

Models for short term insurance contracts

Nature

- Short fixed time, typical 1 year
- insurer receives premium
- pays claims that arise during the period
- insured may renew policy (ie pay premium)
- Unlike long life policies where premium may be charged and invested

Consider

$N -$ number of claims arise

$X_i$ – individual claim size

Aggregate/collective claim

$$S = \sum_{i=1}^{N} X_i$$

$S$ is said to have compound distribution

Factors affecting claim amount and number of claims may be different
And so, the distributions may also be independent

Assumptions

- Claim amount and number of claims are independent
- Individual claim amounts are independent
- Distribution of individual claim amount does not change on short period
- Moments/distributions are known (they can be derived from available data)
- Claims are settled immediately

---

## DISTRIBUTION OF AGGREGATE CLAIM

Notations
$G(x) -$ distribution for $S$
$F(x) -$ distribution for $X_i$

$$E[X_i] = m_1$$
$$var[X_i] = m_2 - m_1^2$$

Distribution function
we can construct
Convolution

$$f(z) = \sum_x f_X(x) * f_Y(z - x)$$
$$f(z) = \int f_X(x) * f_Y(z - x)dx$$

Moments of compound distribution
from conditional expectation, variance

$$E[S] = E[E[S|N]]$$
$$var[S] = var\big[E[S|N]\big] + E[var[S|N]]$$

**Expectation**

$$E[S] = E\big[E[S|N]\big]$$
$$= E\left[E\left[\sum_{i=1}^{N} X_i\right]\right]$$
$$= E\left[\left[\sum_{i=1}^{N} E[X_i]\right]\right] = E[Nm_1] = m_1 * E[N]$$
$$E[S] = E[X] * E[N] = m_1 * E[N]$$

**variance**

$$var[S] = var\big[E[S|N]\big] + E\big[var[S|N]\big]$$
$$= var(N * m_1) + E\left[var\left[\sum_{i=1}^{N} X_i\right]\right]$$
$$= m_1^2 * var(N) + E[N * (m_2 - m_1^2)]$$
$$= m_1^2 * var(N) + (m_2 - m_1^2) * E[N]$$

Moment Generating function

$$M_S(t) = E[e^{tS}] = E\left[e^{t*\sum_{i=1}^{N} X_i}\right] = E\left[e^{t*(X_1+X_2\ldots+X_N)}\right]$$
$$E\left[e^{t*(X_1+X_2\ldots+X_N)}\right] = E[e^{t*X_1} * e^{t*X_2} * \ldots * e^{t*X_N}] = [M_X(t)]^N$$

$$M_S(t) = E[M_X(t)]^N = E\left[e^{\ln\left([M_X(t)]^N\right)}\right] = E\left[e^{N\ln([M_X(t)])}\right]$$

$$M_S(t) = M_N\big(\ln\big(M_X(t)\big)\big)$$

**Compound poisson distribution**

If

$N \sim Poi(\lambda)$ then

$$S \sim Poi(\lambda)$$

from

MGF

$$M_S(t) = M_N(\ln(M_X(t)) = E[e^{N*\ln(M_X(t))}]$$
$$= E\left[e^{N*\ln(M_X(t))}\right] = e^{\lambda*\left(e^{\ln(M_X(t))}-1\right)} = e^{\lambda*(M_X(t)-1)}$$

You can derive the moments from MGF or CGF

$$E[S] = E[N] * m_1 = \lambda * m_1$$
$$var[S] = var[N] * m_1^2 + (m_2 - m_1^2) * E[N] = \lambda * m_1^2 + \lambda * m_2 - \lambda * m_1^2$$
$$= \lambda * m_2$$

Compound binomial distribution

$$N \sim Bin(n, p)$$

$E[N] = np$ ; $var[N] = np(1-p)$

$$M_N(t) = (pe^t + 1 - p)^n$$

$$E[S] = E[N] * m_1 = n * p * m_1$$
$$var[S] = n * p * (1-p) * m_1^2 + (m_2 - m_1^2) * n * p = npm_2 - np^2m_1^2$$

---

## CASE OF REINSURANCE

Aggregate claim and reinsurance arrangements

$Y_i$ - amount paid by insurer for ith claim

$Z_i$ – amount paid by reinsurer for ith claim

$$S_i = Y_1 + Y_2 + \cdots + Y_N$$
$$S_R = Z_1 + Z_2 + \cdots + Z_N$$

From previous section then you can just consider

$$E[S_i] = E[N] * E[Y]$$
$$var[S_i] = var[N] * (E[Y])^2 + (E[Y^2] - (E[Y])^2) * E[N]$$
$$M_{S_i}(t) = M_N(\ln(M_Y(t)))$$

---

## PROPORTIONAL REINSURANCE

$$\boldsymbol{Y = \alpha X}$$
$$\boldsymbol{Z = (1-\alpha)X}$$

$$\boldsymbol{S_i = \sum_{i=1}^{N} \alpha X_i = \alpha S}$$

$$\boldsymbol{S_R = \sum_{i=1}^{N} (1-\alpha) * X_i = (1-\alpha) * S}$$

$$\boldsymbol{M_Y(t) = E[e^{tY}] = E[e^{t\alpha X}]}$$

$$\boldsymbol{M_N(t) = E[e^{tN}]}$$

$$\boldsymbol{M_{S_i}(t) = M_N(ln(M_Y(t)) = E\left[e^{t*\ln(M_Y(t))}\right]}$$

---

## INDIVIDUAL EXCESS OF LOSS REINSURANCE

For insurer

$$Y = \begin{Bmatrix} X \ for \ X \leq M \\ M \ for \ X > M \end{Bmatrix}$$

For reinsurer

$$Y = \begin{Bmatrix} 0 \ for \ X \leq M \\ X - M \ for \ X > M \end{Bmatrix}$$

# CHAPTER 6: RUIN THEORY (CM2)

## INTRODUCTION

# CHAPTER 7: PREMIUM PRINCIPLES (BOOK)

## INTRODUCTION

# CHAPTER 8: MACHINE LEARNING FOR RISK MODELLING

## INTRODUCTION