Modelling the perfect squash serve

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Introduction

I've been playing squash for over 6 years now and it has become my favorite sport. In squash you are supposed to serve the ball from a small box on the side of the court to the other side of the court from where your opponent has to hit the ball. In squash one of the most integral shots is the serve, if you serve properly you can force the opponent to not take the shot or give you an easy shot. One type of serve which is the hardest to hit is described as one which hits the corner of the wall and is really slow, so that it drops on hitting the wall rather than bouncing back, however in serving slowly there is a risk involved with the opponent hitting the ball before it reaches the corner. As I was practicing this serve I observed that the ball was travelling in a parabola after hitting the wall I had also studied the chapter the mensuration very recently and thus realized that I could derive a 3 dimensional vector using which I could do the serve in such a way that only a very skilled player would be able to take it. The passion to develop such a vector started me onto this exploration in which I will investigate the perfect squash serve. This brought me to my research question for this exploration, "How to hit the perfect squash serve?" Before I started, I had to make a few assumptions regarding the system, the first was that the ball and wall had a coefficient of restitution of 1, that is no energy was lost or gained from the bounce, I also had to neglect air resistance.

To better understand the path of the ball we first need to understand the dimensions of the squash court and player behavior. The court is depicted in Figure 1.0.

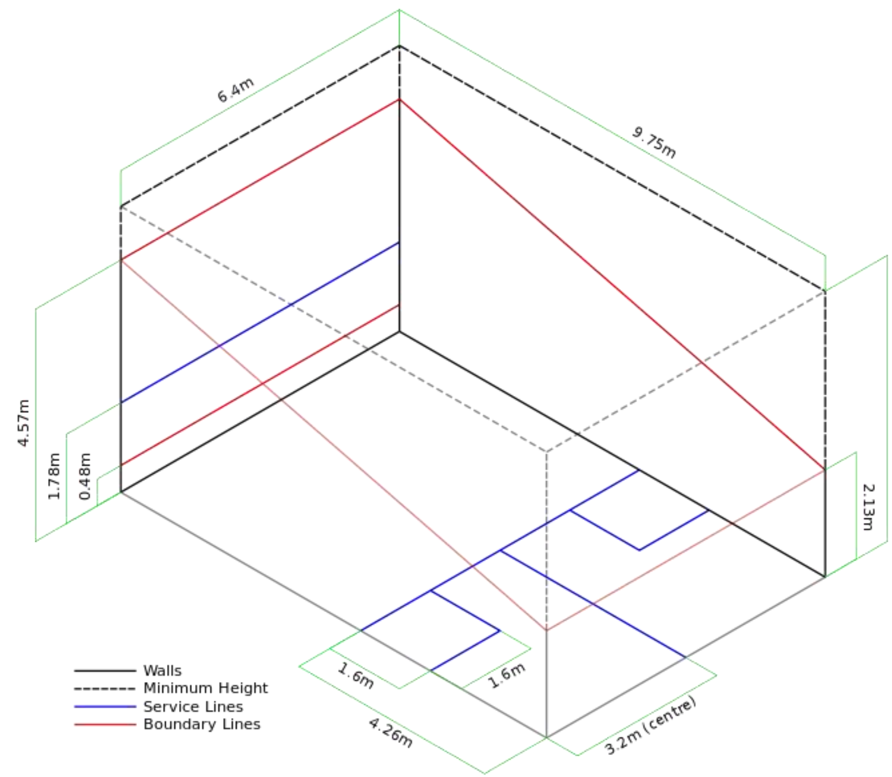


Figure 1.0

According to the rules of the game, the player has to serve from one of the boxes, the ball has to hit the wall at a maximum height of 4.57 meters and a minimum height of 1.78 meters and can hit the back wall at a maximum height of 2.13 meters. The box from which players have to serve is 1.6 meters in width and 1.6 meters in length, the total length of the court from the back wall to the front wall is 9.75 meters and the maximum distance from the wall from which a player can serve is 4.26 meters from the back wall. The court has a total width of 6.4 meters.

When it comes to player psychology immediately after serving players don’t prefer to stand where they serve from and instead move to the center of the court so that they have better command over the court and can reach the ball easily. This is represented through Figure 1.1.

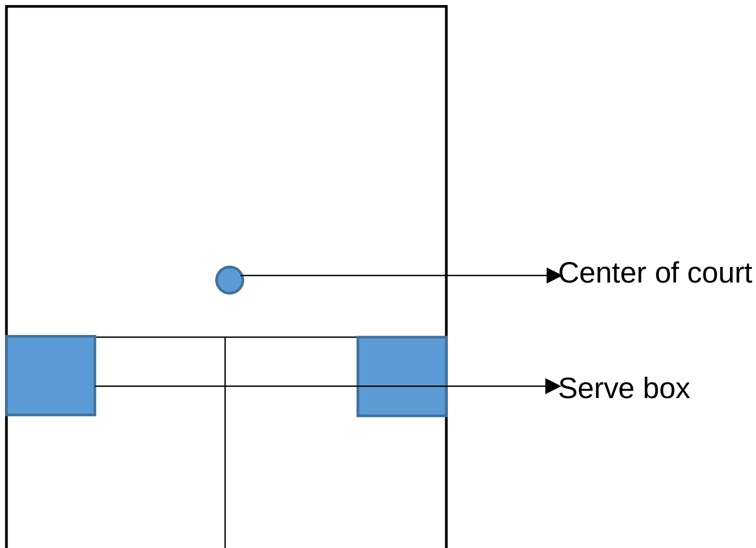


Figure 1.1

Thus we can see from the diagram that a player would prefer from the top right edge of the serve box to minimize his distance to the center of the court. Hence an ideal player would serve at a distance of 1.6 meters from the left wall and a distance of 5.49 meters from the front wall, this will be the starting point for the mathematical analysis of the launching of the ball.

The ball like all real world things travels in a 3D plane. However, the ball travels in different ways in two planes, that is X and Y plane and the Y and Z plane. While in the X and Y plane the ball travels its entire journey in a straight line, in the Y and Z axis it traverses the path in a parabolic manner, for this reason I’ll be evaluating the ball’s motion in these two planes separately and then combine them to get the final vector.

Part 1: 2D plane with ball travelling in X and Y axis

To analyze this the motion of the ball in this plane we'll have to get a bird's eye view of the motion of the ball while travelling in the plane. The serve from a bird's eye view is depicted in Figure 2.0.

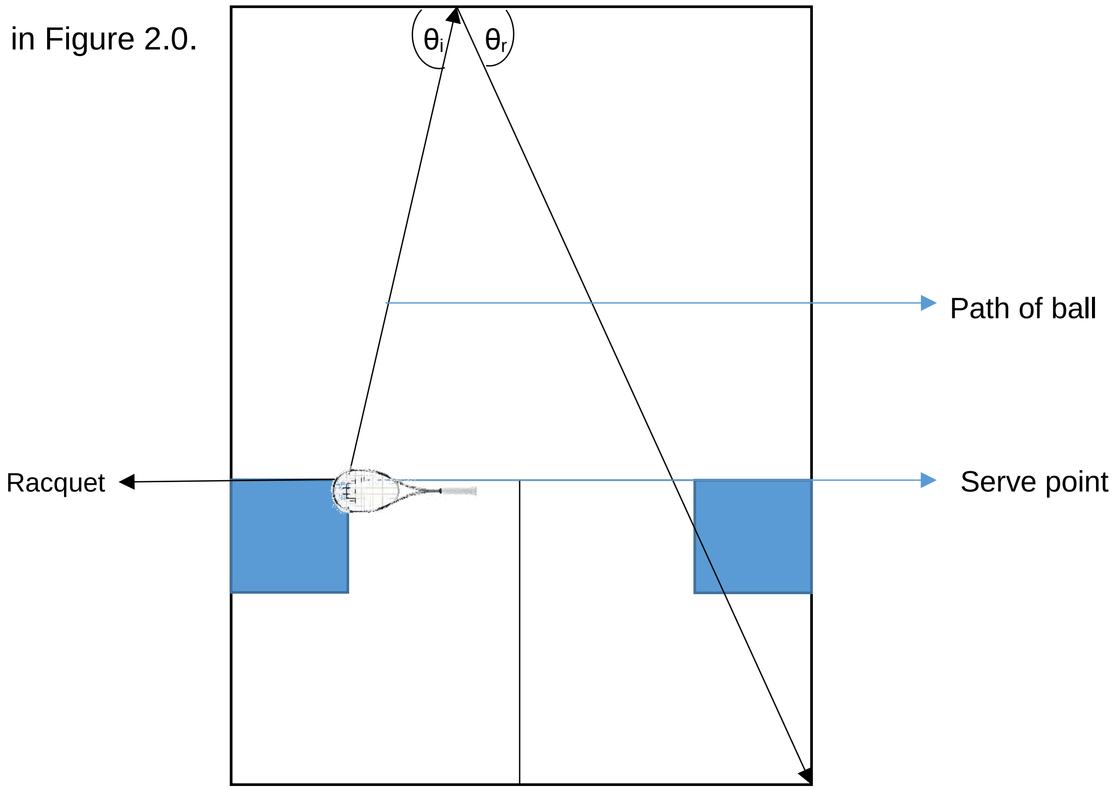


Figure 2.0

Since in a perfect system the velocity of the racquet won't influence the angle of reflection, from physics we know that $\theta_i = \theta_r$. To find the angle at which the racquet hits the ball, we'll have to find the angle of incidence. This can be done by drawing an imaginary triangle from the point of contact. Once we've drawn that triangle we can use trigonometry to find the angle of incidence and hence the angle at which the racquet hits the ball. The imaginary triangle is represented in Figure 2.1 with a dotted line to represent the imaginary line from the point of contact.

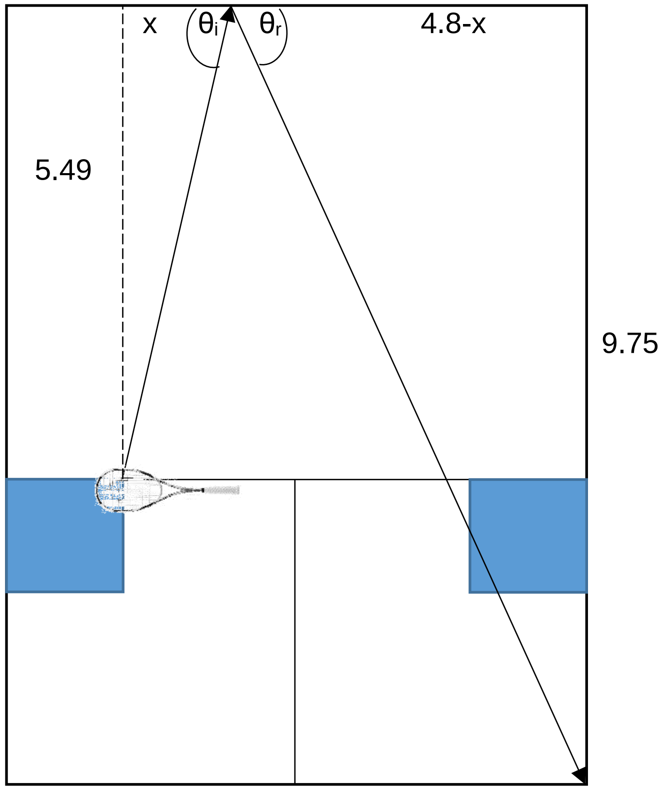


Figure 2.1

The perpendicular distance from the point of contact, to the wall is $9.75 - 4.26 = 5.49$

The ball then travels a perpendicular distance of 9.75 meters to reach the right edge, if the ball hits the wall a distance of x from the perpendicular from the point of contact then it travels 4.8-x meters along the right wall. Now, we know that $\theta_i = \theta_r$, therefore,

$$\tan(\theta_i) = \tan(\theta_r)$$

$$\frac{9.75}{4.8-x} = \frac{5.49}{x}$$

$$26.352 - 5.49x = 9.75x$$

$$26.352 = 15.24x$$

Therefore, x = 1.729, since the perpendicular is 1.6 meters away from the left wall, the player has to hit 3.33 meters to the right from the left wall.

Since, $\tan(\theta_i) = \frac{5.49}{1.729}$, θ_i = 72.52°

In the imaginary triangle, 90°+72.52°+(Angle of contact along perpendicular) = 180 °

Therefore, Angle of contact along perpendicular = 17.48°

Therefore, Angle of contact along x axis = 90-17.48 = 72.53°

Hence, the vector can be represented as $\begin{pmatrix} v\cos(72.52) \\ v\sin(72.52) \\ 0 \end{pmatrix}$

Part 2: 2D Plane with ball travelling in Y and Z axis

To analyze this motion of the ball, we need to look at it from a different view, the view we choose is actually dependent on which plane is constant in the case, which, in this case is the x axis, hence we'll see the serve either from the left or the right side. The motion of the ball in the motion along this plane is shown in Figure 3.0.

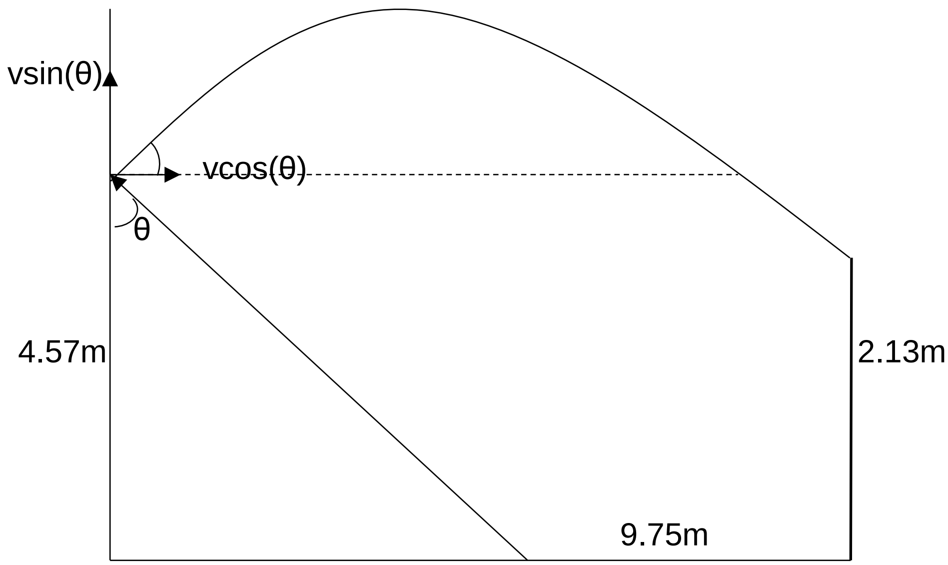


Figure 3.0

Since we are ignoring air resistance, the ball will travel in a perfect parabola after reflecting off the wall, when it collides off the wall it travels like a projectile launched at an angle θ with a velocity v.

For the modelling of this part of the motion we'll assume that the time at which the ball reflects off the wall is 0.

V can be divided into two components, $vcos(\theta)j, vsin(\theta)k$.

Therefore, $|v| = \mathrm{vcos}(\theta)j + \mathrm{vsin}(\theta)k$.

We know Force = ma, and therefore $ma = -mgk$

Therefore, $a = -gk$

$\frac{d^2r}{dt^2} = -gk$

Now we know that $|v| = \mathrm{v}\cos(\theta)j + \mathrm{v}\sin(\theta)k$ and t = 0, r = 0

Therefore, we integrate this twice to get the formula for the range of the ball at time t.

$$\frac{dr}{dt} = -\mathrm{gtk} + \{\mathrm{v}\cos(\theta)j + \mathrm{v}\sin(\theta)k\}t$$

Therefore, $r = \frac{-g}{2}t^2k + vtcos(\theta)j + vtsin(\theta)k$

$$r = vtcos(\theta)j + \left(vtsin(\theta) - \frac{g}{2}t^2\right)k$$

When the ball reaches the same height from which it was launched the vertical displacement becomes 0.

Therefore, $vtsin(\theta) - \frac{g}{2}t^2 = 0$

Therefore, time at which it reaches the same height = $\frac{2vsin(\theta)}{g}$

The distance that the ball travels due to this is thus given by the horizontal component when at time t, therefore, $r = vcos(\theta)\left(\frac{2vsin(\theta)}{g}\right)$

Therefore, $r = \frac{v^2sin(2\theta)}{g}$

When the ball attains its maximum height, the vertical velocity is zero, Therefore, $\frac{dy}{dt} = v_0 \sin(\theta) - gt = 0, therefore, t = \frac{v_0 \sin(\theta)}{g}$

Therefore, maximum height, $y_{max} = v_0 \sin(\theta) \times \left(\frac{v_0 \sin(\theta)}{g}\right) - \frac{1}{2} g \left(\frac{v_0 \sin(\theta)}{g}^2\right) = \frac{(v_0 \sin(\theta))^2}{2g}$

Now that we have the main equations for the path of the ball we can evaluate its motion and find the angle at which it should be launched so that the velocity can be minimized while fulfilling the conditions of the perfect squash serve.

When I first approached this problem I thought of simply using the derived equations and then optimizing the result to find the minimum v.

We know that the horizontal distance the ball covers when it reaches the height from which it was launched is given by, $\frac{v^2 \sin(2\theta)}{g}$, after this point the ball still travels in the same parabolic path under the influence of gravity and its initial horizontal velocity, however since the formula only tells us the motion till the point the ball reaches the height it was launched at I had use other relations to get the final equation for the motion of the ball.

At the point that the ball reaches the same height it was launched from, its velocity is vsin(θ) the ball has to cover the distance which is the difference between the two heights in a time t and the horizontal also has to cover a distance in that amount of time. The sum of the range and the distance covered in the horizontal distance in the second part should be equal to the length of the court.

$2 \times a \times 2.44 = v^2 - u^2$

Therefore, $v = \sqrt{v^2 sin^2(\theta) + 47.824}$

We know $v = u + at$

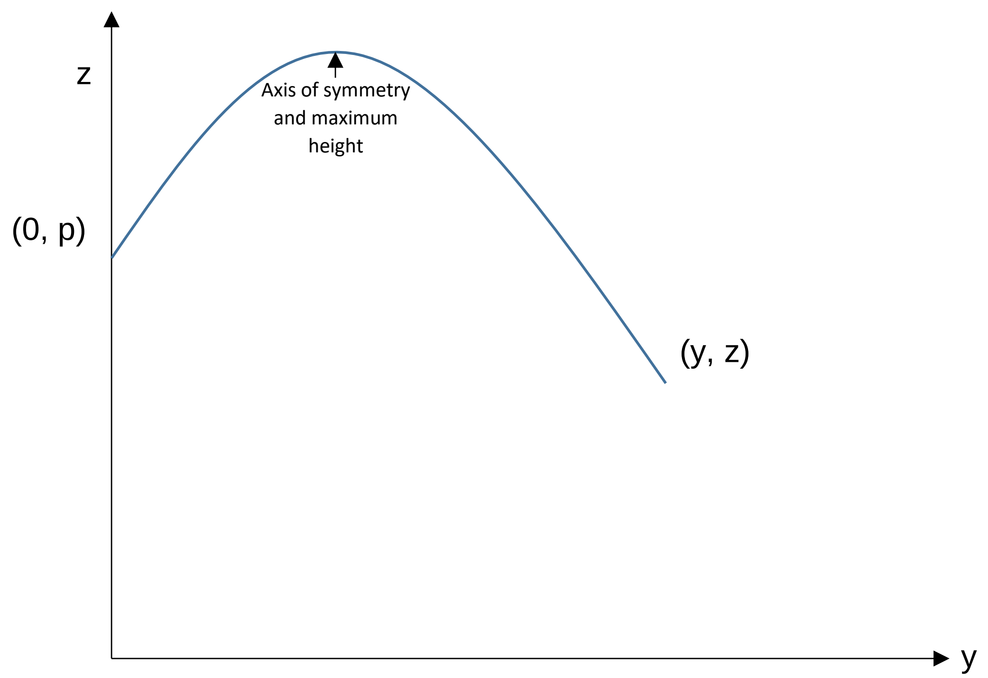
$$t = \frac{\sqrt{v^2 sin^2(\theta) + 47.824} - v sin(\theta)}{9.8}$$

Thus, $\frac{v^2 \sin(2\theta)}{9.8} + \frac{v sin(\theta)\{\sqrt{v^2 sin^2(\theta)+47.824} - v sin(\theta)\}}{9.8} = 9.75$

$$v^2 \sin(2\theta) + \ v sin(\theta)\left\{\sqrt{v^2 sin^2(\theta) + 47.824} - v sin(\theta)\right\} = 95.5$$

To optimize this equation, I would have to make v the subject of the equation and then differentiate it with respect to θ however, I realized that doing this would be very complicated and I decided to take a different approach to solving this problem.

I decided to take the wall as the z axis of a graph and the ground as the y axis, with the help of this I tried to derive the equation for the parabola in which the ball was travelling.



The general equation for a parabola travelling this path is given by $z = \ ay^2 + by + c$

The z intercept gives us the value for c, therefore, c in this case is the height at which the ball hits the wall.

For the next part I'll be assuming the projectile is starting from origin, hence $c = 0$

On deriving the general equation, I can just input, c = p

The y co-ordinate of the axis of symmetry is given by $\frac{-b}{2a}$, from the equations we derived earlier, we also know that the y co-ordinate is given by, $\frac{v^2\sin(2\theta)}{2g}$

Therefore, $\frac{-b}{2a} = \frac{v^2\sin(2\theta)}{2g}$

$$\frac{-bg}{v^2\sin(2\theta)} = a - (1)$$

We also know that at this point the z co-ordinate is given by $ax^2 + bx$, since $c = 0$ and from the derived equations we know that it is also given by, $\frac{v^2\sin^2(\theta)}{2g}$

On equating these two equations we get, $\frac{v^2\sin^2(\theta)}{2g} = \frac{-bg}{v^2\sin(2\theta)}\left(\frac{v^2\sin(2\theta)}{2g}\right)^2 + b\left(\frac{v^2\sin(2\theta)}{2g}\right)$

Therefore, $\frac{v^2\sin^2(\theta)}{2g} = \left(\frac{-b}{v^2\sin(2\theta)}\frac{v^4 sin^2(2\theta)}{4g}\right) + b\left(\frac{v^2\sin(2\theta)}{2g}\right)$

$$\frac{v^2\sin^2(\theta)}{2g} = \left(\frac{-bv^2 sin(2\theta)}{4g}\right) + b\left(\frac{v^2\sin(2\theta)}{2g}\right)$$

$$\frac{v^2\sin^2(\theta)}{2g} = \frac{2bv^2\sin(2\theta) - bv^2\sin(2\theta)}{4g}$$

$$\frac{v^2\sin^2(\theta)}{2g} = \frac{bv^2\sin(2\theta)}{4g}$$

$$\frac{4\sin^2(\theta)}{2\sin(2\theta)} = b$$

$$b = \tan(\theta) - (2)$$

On putting (2) in (1), we get,

$a = \frac{-\tan(\theta)g}{v^2\sin(2\theta)}$

$$a = \frac{-\sin(\theta)g}{v^2 \cos(\theta)\, 2\sin(\theta) \cos(\theta)}$$

$$a = \frac{-g}{2v^2 cos^2(\theta)}$$

By putting the values of a and b in the original equation, we get

$$y = \frac{-gx^2}{2v^2 cos^2(\theta)} + \tan(\theta)x + c$$

Therefore, adjusting the equation for the given case, we get,

$p - z = xtan(\theta) - \frac{gy^2}{2v^2 cos^2(\theta)}$ where p is the height of the point of contact on the wall, y is the total length of the court and z is the height of the back wall.

Therefore, $\frac{gy^2}{2v^2 cos^2(\theta)} = ytan(\theta) + z - p$

Therefore, $v = \sqrt{\frac{gy^2}{2cos^2(\theta)\{ytan(\theta)+z-p\}}}$

Now, to minimize v, we need to minimize the right hand side of the equation, since the numerator is constant, we only need to maximize the denominator, once we maximize the denominator with respect to θ we can get the value of v and hence the vector in the Y – Z plane.

Now, I could have inputted the value of y, z and p to make the simply differentiate it, however, I realized that I can use partial differentiation so that the equation can be used for other applications as well, for example finding the maximum range for a given θ.

Hence, I will find $\frac{\partial v(\theta,y,z,p)}{\partial(\theta,y,z,p)} = \ 2cos^2(\theta)\{ytan(\theta) + z - p\}$

For the first case, I will find $\frac{\partial v(\theta,y,z,p)}{\partial(p)} = \ 2cos^2(\theta)\{ytan(\theta) + z - p\}$

$$\frac{\partial v(\theta,y,z,p)}{\partial(p)} = ysin(2\theta) + 2z\,cos^2(\theta) - 2pcos^2(\theta)$$

$$\frac{\partial v(\theta,y,z,p)}{\partial(p)} = -2cos^2(\theta)$$

This formula can help us optimize v with respect to the height which the projectile has to reach after a displacement of y.

Next I will evaluate, $\frac{\partial v(\theta,y,z,p)}{\partial(z)} = \ 2cos^2(\theta)\{ytan(\theta) - z + p\}$

$$\frac{\partial v(\theta,y,z,p)}{\partial(z)} = 2cos^2(\theta)$$

This formula can help us optimize v with respect to the height from which the projectile is launched.

We can partially differentiate, the function with y, $\frac{\partial v(\theta,y,z,p)}{\partial(y)} = \ 2cos^2(\theta)\{ytan(\theta) + z - p\}$

$$\frac{\partial v(\theta,y,z,p)}{\partial(y)} = sin(2\theta)$$

This formula can help us optimize v for the distance the projectile has to travel after being launched.

Finally, I differentiated the function with respect to θ to optimize the velocity such that it is minimum for the value of θ we get. $\frac{\partial v(\theta,y,z,p)}{\partial(\theta)} = \ 2cos^2(\theta)\{ytan(\theta) + z - p\}$

$$\frac{\partial v(\theta,y,z,p)}{\partial(\theta)} = 39\cos(2\theta) - 9.76\sin(\theta)\cos(\theta)$$

We can equate this equation to zero to maximize the function in order to minimize v

Therefore, $9.76\sin(\theta)\cos(\theta) = 39\cos(2\theta)$

$$4.88 = \frac{39\cos(2\theta)}{2\sin(\theta)\cos(\theta)}$$

$$\frac{4.88}{39} = \cot(2\theta)$$

$$cot^{-1}(0.12513) = 2\theta$$

$$2\theta = 82.8676°$$

$$\theta = 41.4338°$$

Having found out θ which the denominator is maximum, we can find out the velocity that the ball must be travelling at to complete the flight path.

$$v = \sqrt{\frac{gy^2}{2cos^2(\theta)\{ytan(\theta) + z - p\}}}$$

$$v = \sqrt{\frac{9.8 \times 9.75^2}{2cos^2(41.4338)\{9.75tan(41.4338) + 2.44\}}}$$

$$v = \sqrt{\frac{931.6125}{2 \times 0.5621\{9.75 \times 0.8827 + 2.44\}}}$$

$$v = \sqrt{\frac{931.6125}{7.5805}}$$

$$v = \sqrt{122.8959} = 11.0858$$

For this velocity the ball reflects off the wall at an angle of 41.4438°, therefore it hits the wall at an angle of 41.4438° which means that the ball was launched at an angle of,

90 – 41.4438 = 48.5562°

Therefore, the launch vector for this plane is given by, $\begin{pmatrix} 0 \\ v\cos(48.5562) \\ v\sin(48.5562) \end{pmatrix}$

Since we know v, the vector becomes $\begin{pmatrix} 0 \\ 7.3375 \\ 8.3099 \end{pmatrix}$

Part 3: Final Vector

From part 1, the vector for the launch in the X-Y plane is given by $\begin{pmatrix} vcos(72.52) \\ vsin(72.52) \\ 0 \end{pmatrix}$

Plugging the values, we got in the previous part into the vector, we get, $\begin{pmatrix} 3.3298 \\ 10.5738 \\ 0 \end{pmatrix}$

We now know the vectors in the two planes in which the ball is launched, thus their resultant will give us the final vector in which the ball should be launched to achieve the perfect serve.

The resultant is the sum of the two vectors and is hence,

$$\begin{pmatrix} 3.3298 \\ 10.5738 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 7.3375 \\ 8.3099 \end{pmatrix} = \begin{pmatrix} 3.3298 \\ 17.9113 \\ 8.3099 \end{pmatrix}$$

Thus, the launch from the racquet can be represented in the form of a diagram as,

[1]

[1] http://www.fortfrancestimes.com/sites/default/files/story-photos/squash-serve_0.jpg

Conclusion and Reflection

Doing this exploration was a really enlightening and exciting work. I had a lot of fun going into the mathematics behind my favorite sport and finding a fool-proof way to serve perfectly each time. This exploration offered me unique insights into mathematics including the introduction to the concept of partial differentiation. Although there were a few challenges along the way including not being able to perform an actual demonstration to get real world values to base my calculations on. I also could have factored in air resistance if I could click a multiple exposure shot of the ball travelling in the air after reflecting off the wall and then modelling its equation, however I couldn't do the same due to the lack of technology to do so. I would like to thank my teacher Mr Jitender Rekhi for his continued support during the process of this exploration and hopefully the vector I have derived can help players across the globe.