① $y' = y\cdot ctg(x) + \sin(x)$ — Линейное дифф. первого порядка

$$y' = \underbrace{p(x)}_{e^{\int p}}\cdot y + q(x)$$

/* $\int ctg(x)\,dx = // u = \sin x;\ du = \cos x\,dx;\ dx = du/\cos x // = \int \frac{1}{u}du = \ln|u| = \ln|\sin x|$ */

$$y' = y\cdot ctg(x) + \sin(x) \ \Big|\cdot e^{-\int ctg(x)}$$

$\exists\ x \in [0;\pi]$:

$$\left(y\cdot e^{-\int ctg(x)}\right)' = \sin(x)\cdot e^{-\int ctg(x)}$$

$$\left(y\cdot 1/\sin(x)\right)' = 1$$

$$y\cdot 1/\sin(x) = x + A$$

$$y = x\cdot\sin(x) + A\cdot\sin(x)$$

$$y(\pi/2) = \pi/2\cdot 1 + A\cdot 1 = \pi/2 \to A = 0$$

$$y(\pi/6) = \pi/6\cdot 1/2 = \pi/12$$

② $x\cdot y' = y - x\cdot e^{y/x}$ — Однородное дифф. первого порядка

$$y = t\cdot x;\quad y' = t'\cdot x + t\cdot x' = t'x + t$$

$$x(t'x + t) = tx - xe^{tx/x}$$

$$t'x = -e^{t}$$

$$\frac{dt}{dx}\cdot x = -e^{t}$$

$$\int -e^{-t}dt = \int\frac{dx}{x}$$

$$e^{-t} = \ln|x| + C$$

$$e^{-y/x} = \ln|x| + \ln|A| = \ln|Ax|$$

$$-y/x = \ln(\ln|Ax|)$$

$$y = -x\cdot\ln(\ln|Ax|)$$

$$y(1) = -1\cdot\ln(\ln(A)) = -1 \to A = e^{e}$$

$$y(2) = -2\ln(\ln(2e^{e})) =$$

③ $\frac{2x(1-e^{y})\,dx}{(1+x^2)^2} + \frac{e^{y}}{1+x^2}dy = 0$ — Уравнение с разделяемыми переменными.

$$\frac{2x(1+x^2)}{(1+x^2)^2}dx = -\frac{e^{y}}{(1-e^{y})}dy$$

$$\int\frac{2x}{1+x^2}dx = -\int\frac{e^{y}}{1-e^{y}}dy$$

/* $\int\frac{2x}{1+x^2}dx = // u = 1+x^2;\ du = 2x\cdot dx;\ dx = \frac{du}{2x} //$

$= \int\frac{1}{u}du = \ln|u| = \ln|1+x^2| = \ln(1+x^2)$

$\int\frac{e^{y}}{1-e^{y}}dy = // u = 1-e^{y};\ du = -e^{y}dy;\ dy = \frac{du}{-e^{y}} //$

$= -\int\frac{1}{u}du = -\ln|e^{y}-1|$

$$\ln(1+x^2) + C = \ln|e^{y}-1|$$

$$A(1+x^2) = e^{y} - 1$$

$$y = \ln\left(A(1+x^2)+1\right)$$

$$7 = \ln(A\cdot 2+1) \to A = (e^{7}-1)/2$$

$$y_1 = \ln(A\cdot 10+1) = 8.61$$

④ $yy'' = (4y'^4 - y'^2)\cdot e^{y}$; $y(0) = 1$; $y'(0) = -1/2$.

$$z(y) = y';\quad y'' = z'(y)\cdot y' = z'\cdot z$$

$$yz'z = (4z^4 - z^2)\cdot e^{y}$$

— $z = 0$: $y = A$

— $z = 1/2$: $y = 1/2x + B$

— $z = -1/2$: $y = -1/2x + C$

$y'(0) = -1/2$; $y(0) = 1 \Rightarrow C = 1$

$y(-12) = 6 + 1 = 7$

— иначе $\int\frac{z}{4z^4 - z^2}dz = \int\frac{e^{y}}{y} + C$ ...

⑤

$y^2(y'y''' - 2y''^2) = y'^4$, $y(0) = 4$, $y'(0) = 2$, $y''(0) = -1$

$z(y) := y'$, $y'' = z'z$, $y''' = z''z^2 + z'^2z \rightsquigarrow y^2(z''z^3 + z'^2z^2 - 2z'^2z^2) = z^4 \rightsquigarrow$

$\rightsquigarrow y^2(z''z^3 - z'^2z^2) = z^4$ — однородное отн-но z, z', z'' => замена $u = \frac{z'}{z}$ ($\ast\ z \neq 0$) :

$z' = uz$, $z'' = u'z + uz' = z(u' + u^2)$; $F(y, z, z', z'') = 0 = F(y, 1, u, u' + u^2)$;

$y^2[(u' + u^2) - u^2] = 1 \rightsquigarrow y^2u' = 1 \rightsquigarrow u' = \frac{1}{y^2} \rightsquigarrow u = -\frac{1}{y} + C \rightsquigarrow \int\frac{dz}{z} = -\int\frac{1}{y}dy + C\int dy + B \rightsquigarrow$

$\rightsquigarrow \ln|z| = -\ln|y| + Cy + B \rightsquigarrow z = \frac{B}{y}e^{Cy}$ ($\ast\ z, y > 0$, $B > 0$)

$y'(0) = 2 \rightsquigarrow z(y(0)) = z(4) = 2 \rightsquigarrow 2 = \frac{B}{4}e^{4C}$; $y''(0) = -1 \rightsquigarrow -1 = 2\cdot\left[-\frac{B}{y^2}e^{Cy} + \frac{BC}{y}e^{Cy}\right]_{y=4} \rightsquigarrow$

$\rightsquigarrow -\frac{1}{2} = \left(-\frac{B}{16} + \frac{BC}{4}\right)\cdot\frac{8}{B} \rightsquigarrow -\frac{1}{2} = -\frac{1}{2} + 2C \rightsquigarrow C = 0$, $B = 8$;

$z = \frac{8}{y} = y' \rightsquigarrow y = t$, $y' = \frac{8}{t}$;

$dy = dt$, $y'_x = \frac{8}{t}$, $dx = x'_t\,dt \Rightarrow 1 = \frac{8}{t}x'_t \rightsquigarrow x'_t = \frac{t}{8} \rightsquigarrow x = \frac{1}{16}t^2 + C$;

$$\begin{cases} x = \frac{1}{16}t^2 + C \\ y = t \end{cases}$$

$y(0) = 4 \rightsquigarrow t = 4 \rightsquigarrow 0 = \frac{1}{16}\cdot 16 + C \rightsquigarrow C = -1$;

$24 = \frac{1}{16}t^2 - 1 \rightsquigarrow t^2 = 25\cdot 16 \rightsquigarrow t = 5\cdot 4 = 20 \rightsquigarrow y = 20$.

⑥ $x^4y''' + 2x^3y'' = 1$; $y(1) = -1/2$; $y'(1) = 3$; $y''(1) = -2$.

$z = y''$: $x^4\cdot z' + 2x^3z = 1$

$z' = \frac{-2z}{x} + 1/x^4$ — однородное. $\Big|\cdot e^{-\int +2/x} = e^{2\ln(x)} = x^2$

$$x^2\cdot z' = -2z\cdot x + 1/x^2$$

$$(z\cdot x^2)' = 1/x^2$$

$z\cdot x^2 = -1/x + C$; $z = -1/x^3 + C/x^2$

$$z(1) = -1 + C = -2 \to C = -1$$

$$y' = \int -1/x^3\,dx + \int 1/x^2\,dx = +1/2\cdot 1/x^2 + 1/x + B;$$

$$y'(1) = 1/2 + 1 + B = 3 \to B = 3/2$$

$$y = \int 1/2x^2\,dx + \int 1/x\,dx + \int 3/2\,dx = -1/2x + \ln(x) + 3/2x + A$$

$$y(1) = -1/2 + \ln(1) + 3/2 + A = -1/2 \to A = -3/2$$

$$y(2) = -1/4 + \ln(2) + 3 - 3/2 = 1.94$$