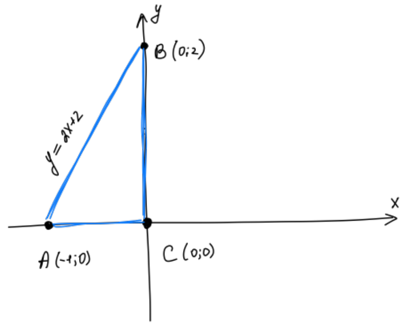
Вариант 17.

Система случайных величин (X,Y) имеет плотность распределения

$$f_{X,Y}(x,y)=\begin{cases} ax^2, & (x,y)\in\Delta_{ABC},\ A(-1,0),\ B(0,2),\ C(0,0),\\ 0, & (x,y)\ \text{вне}\ \Delta_{ABC}.\end{cases}$$

Определить постоянную a. Найти плотности компонент X и Y, коэффициент корреляции $r(X,Y)$, регрессию $m_{Y|X}(x)$, $m_{X|Y}(y)$. Изобразить эти линии графически.

1



$$\iint\limits_{\Delta ABC} ax^2\,dx\,dy = 1 \ :$$

$$\int_{-1}^{0}dx\int_{0}^{2x+2} ax^2\,dy = \int_{-1}^{0} ax^2(2x+2)\,dx =$$

$$= a\int_{-1}^{0} 2x^3+2x^2\,dx =$$

$$= a\cdot\left(\frac{2x^4}{4}+\frac{2x^3}{3}\Big|_{-1}^{0}\right) = a\cdot\left(-1/2+2/3\right) =$$

$$= a/6 \implies a=6$$

$$f_x(x)=\int_0^{2x+2} 6x^2\,dy = 6x^2(2x+2) = 12x^2(x+1) \quad \Big| \ x\in(-1;0)$$

$$f_y(y)=\int_{\frac{y-2}{2}}^{0} 6x^2\,dx = 2x^3\Big|_{\frac{y-2}{2}}^{0} = \frac{(2-y)^3}{4} \quad \Big| \ y\in(0;2)$$

$$k(X,Y)=E(X\cdot Y) - EX\cdot EY$$

$$E(X\cdot Y) = \iint\limits_{\Delta ABC} xy\ f_{x,y}(x,y)\,dx\,dy = \int_{-1}^{0}6x^3\int_0^{2x+2} y\,dy = 6\int_{-1}^{0}\frac{x^3(2x+2)^2}{2} =$$

$$= 12\int_{-1}^{0}x^3(x+1)^2 = 12\left(\int_{-1}^{0}x^5\,dx + \int_{-1}^{0}2x^4\,dx + \int_{-1}^{0}x^3\,dx\right) =$$

$$= 12\left(\frac{x^6}{6}\Big|_{-1}^{0} + \frac{2x^5}{5}\Big|_{-1}^{0} + \frac{x^4}{4}\Big|_{-1}^{0}\right) = 12\left(-1/6+1/5-1/4\right) =$$

$$= -2-3+24/5 = \frac{-25+24}{5} = -1/5$$

$$EX=\int_{-1}^{0}x\cdot 12x^2(x+1)\,dx = \left(\frac{12x^5}{5}+3x^4\right)\Big|_{-1}^{0} = -3/5$$

$$EY=\int_0^2 y\cdot\frac{(2-y)^3}{4}\,dy = 1/4\left(\int_0^2 8y\,dy + \int_0^2 -12y^2\,dy + \int_0^2 6y^3\,dy + \int_0^2 -y^4\,dy\right) =$$

$$= 1/4\left(4y^2\Big|_0^2 - 4y^3\Big|_0^2 + 3/2\,y^4\Big|_0^2 - \frac{y^5}{5}\Big|_0^2\right) =$$

$$= 1/4\left(16-32+24-32/5\right) = 2 - 8/5 = 2/5$$

$$k(X,Y) = -1/5 + 6/25 = 1/25$$

$$r(X,Y) = \frac{k(X,Y)}{\sqrt{DX\cdot DY}}$$

$$DX=\int_{-1}^{0}x^2\cdot 12x^2(x+1)\,dx - 9/25 = 2x^6\Big|_{-1}^{0} + \frac{12x^5}{5}\Big|_{-1}^{0} - 9/25 =$$

$$= -2 + 12/5 - 9/25 = 1/25$$

$$DY=\int_0^2 y^2\cdot\frac{(2-y)^3}{4} - 4/25 = 1/4\left(\int_0^2 8y^2\,dy + \int_0^2 -12y^3\,dy + \int_0^2 6y^4\,dy + \int_0^2 -y^5\,dy\right) - 4/25 =$$

$$= 1/4\left(\frac{8y^3}{3}\Big|_0^2 - 3y^4\Big|_0^2 + \frac{6y^5}{5}\Big|_0^2 - \frac{y^6}{6}\Big|_0^2\right) - 4/25 =$$

$$= 1/4\left(\frac{64}{3} - 48 + \frac{192}{5} - \frac{32}{3}\right) - 4/25 = 1/4\cdot 4/15 - 4/25 = 8/75$$

$$r(X,Y) = \frac{1/25}{\sqrt{1/25\cdot 8/75}} = \frac{\sqrt{6}}{4} = 0.6124$$

$$m_{Y/X}(x) = \int y\cdot f_{y/x}(x)\,dy$$

$$f_{y/x}(x) = \frac{f_{x,y}(x,y)}{f_x(x)} = \frac{6x^2}{12x^2(x+1)} = \frac{1}{2(x+1)} \qquad x\in(-1;0),\ y\in(0;2x+2)$$

$$m_{Y/X}(x) = \int_0^{2x+2} y\cdot\frac{1}{2(x+1)}\,dy = \frac{y^2}{4(x+1)}\Big|_0^{2x+2} = (x+1)$$

$$m_{X/Y}(Y) = \int x\cdot f_{x/y}(y)\,dx$$

$$f_{x/y}(y) = \frac{f_{x,y}(x,y)}{f_y(y)} = \frac{6x^2}{\frac{(2-y)^3}{4}} = \frac{24x^2}{(2-y)^3} \qquad y\in(0;2),\ x\in\left(\frac{y-2}{2};0\right)$$

$$m_{X/Y}(Y) = \int_{\frac{y-2}{2}}^{0} x\cdot\frac{24x^2}{(2-y)^3}\,dx = \frac{6x^4}{(2-y)^3}\Big|_{\frac{y-2}{2}}^{0} = \frac{3(y-2)}{8}$$

Проверка:

$$E(E(Y|X)) = E(Y) \quad ?$$

$$E(E(Y|X)) = \int m_{Y|X}(x) \cdot f_X(x)\, dx$$

$$E(E(Y|X)) = \int_{-1}^{0} (x-1) \cdot 12x^2(x+1)\, dx = \frac{12x^5}{5}\Big|_{-1}^{0} + 6x^4\Big|_{-1}^{0} + 4x^3\Big|_{-1}^{0} =$$

$$= 2/5 = EY \text{ — верно}$$

$$E(E(X|Y)) = EX \quad ?$$

$$E(E(X|Y)) = \int m_{X|Y}(y) \cdot f_Y(y)\, dy$$

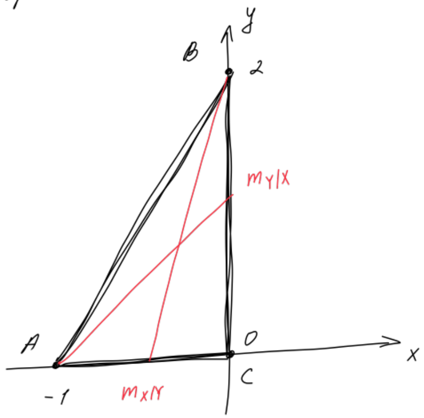
$$E(E(X|Y)) = \int_0^2 \frac{(2-y)^3}{4} \cdot \frac{3(y-2)}{8}\, dy = 3/32 \int_0^2 (2-y)^3(y-2)\, dy =$$

$$= 3/32 \left(\int_0^2 -y^4\, dy + \int_0^2 8y^3 dy + \int_0^2 -24y^2\, dy + \int_0^2 32y\, dy - \int_0^2 16\, dy \right) =$$

$$= 3/32 \left(\frac{-y^5}{5}\Big|_0^2 + 2y^4\Big|_0^2 - 8y^3\Big|_0^2 + 16y^2\Big|_0^2 - 16y\Big|_0^2 \right) =$$

$$= 3/32 \left(-\frac{32}{5} + 32 - 32 + 64 - 64 \right) = -3/5 = EY \text{ — верно}$$

График:



17. Пусть X, и Y независимые случайные величины, равномерно распределенные на промежутках [-3, -1] и [1,2] соответственно. Найти функцию распределения и плотность распределения случайной величины $\xi = X + Y$, а также $E(\xi)$, $D(\xi)$.

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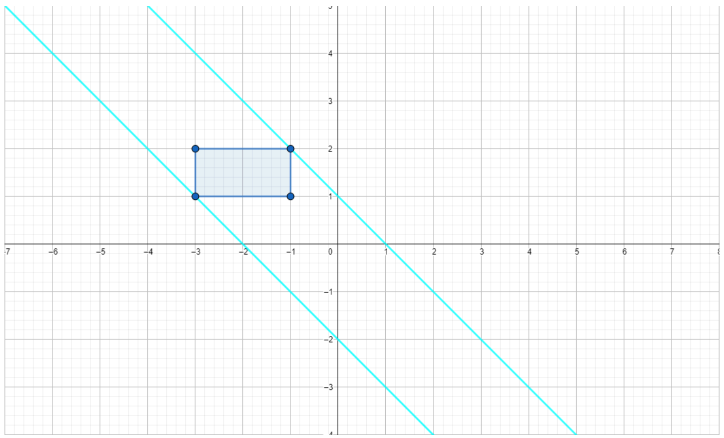
$$F_x(x) = \begin{cases} 0, & x < -3 \\ 1/2, & -3 \le x \le -1 \\ 0, & x > -1 \end{cases} \qquad F_y(y) = \begin{cases} 0, & y < 1 \\ 1/2, & 1 \le y \le 2 \\ 0, & y > 2 \end{cases}$$

$$\varepsilon = x + y$$

$$F_\varepsilon(\varepsilon) = \iint F_x(x) \cdot F_y(y)\, dx\, dy = \tfrac{1}{2} S(\varepsilon)$$

$$F_\varepsilon(\varepsilon) = \begin{cases} 0, & \varepsilon \le -2 \\ \dfrac{(\varepsilon+2)^2}{4}, & -2 < \varepsilon \le -1 \\ \dfrac{2\varepsilon+3}{4}, & -1 < \varepsilon \le 0 \\ 1 - \dfrac{(1-\varepsilon)^2}{4}, & 0 < \varepsilon \le 1 \\ 1, & \varepsilon > 1 \end{cases} \qquad f_\varepsilon(\varepsilon) = \begin{cases} 0, & \varepsilon \le -2 \\ \dfrac{\varepsilon+2}{2}, & -2 < \varepsilon \le -1 \\ 1/2, & -1 < \varepsilon \le 0 \\ \dfrac{1-\varepsilon}{2}, & 0 < \varepsilon \le 1 \\ 0, & \varepsilon > 1 \end{cases}$$



$$E(\varepsilon) = \int_{-\infty}^{+\infty} \varepsilon \cdot f_\varepsilon(\varepsilon)\, d\varepsilon = \int_{-2}^{-1} \frac{\varepsilon(\varepsilon+2)}{2}\, d\varepsilon + \int_{1}^{0} \varepsilon/2\, d\varepsilon + \int_{0}^{1} \frac{\varepsilon(1-\varepsilon)}{2}\, d\varepsilon = -1/2$$

$$E(\varepsilon^2) = \int_{-\infty}^{+\infty} \varepsilon^2 \cdot f_\varepsilon(\varepsilon)\, d\varepsilon = \int_{-2}^{-1} \frac{\varepsilon^2(\varepsilon+2)}{2}\, d\varepsilon + \int_{1}^{0} \varepsilon^2/2\, d\varepsilon + \int_{0}^{1} \frac{\varepsilon^2(1-\varepsilon)}{2}\, d\varepsilon = 2/3$$

$$D(\varepsilon) = E(\varepsilon^2) - E^2(\varepsilon) = 5/12$$