ELSEWEDY UNIVERSITY OF TECHNOLOGY
- POLYTECHNIC OF EGYPT -

Applied Discrete Mathematics

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Applied Discrete Mathematics
Lecture 5

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Sets

A **set** is an **unordered** collection of objects.

The objects in a set are called the *elements*, or *members*, of the set. A set is said to contain its elements.

$$S=\{a,b,c,d\}$$

We write $a \in S$ to denote that $\boldsymbol{a}$ is an element of the set S.

The notation $e \notin S$ denotes that $\boldsymbol{e}$ is not an element of the set S.

Sets

The set O of odd positive integers less than 10 can be expressed by

$$O=\{1,3,5,7,9\}.$$

The set of positive integers less than 100 can be denoted by $\{1,2,3,\ldots,99\}$.

ellipses(…)

The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

Sets

Another way to describe a set is to use **set builder** notation.

The set O of odd positive integers less than 10 can be expressed by $O=\{1,3,5,7,9\}$.

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\},$$

or, specifying the universe as the set of positive integers, as

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}.$$

Sets

These sets, each denoted using a boldface letter, **play an important role in discrete mathematics:**

N = {0, 1, 2, 3, . . .}, the set of **natural numbers**
Z = {. . . ,−2,−1, 0, 1, 2, . . .}, the set of **integers**
$\mathbf{Z}_{+}$ = {1, 2, 3, . . .}, the set of **positive integers**
Q = $\{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}$, and $q = 0\}$, the set of **rational numbers**
R, the set of **real numbers**
$\mathbf{R}_{+}$, the set of **positive real numbers**
C, the set of **complex numbers**.

Sets

Intervals

Recall the notation for **intervals** of real numbers. When a and b are real numbers with $a < b$, we write

$$[a,b]=\{x|a\leq x\leq b\}$$

$$[a,b)=\{x|a\leq x<b\} \qquad [a,b[$$

$$(a,b]=\{x|a<x\leq b\} \qquad]a,b]$$

$$(a,b)=\{x|a<x<b\} \qquad]a,b[$$

Closed interval $[a,b]$

Open interval (a,b)

Sets

If A and B are sets, then $A$and B are equal if and only if

$$\forall x(x \in A \leftrightarrow x \in B).$$

We write $A = B$, if $A$and $B$are equal sets.

- The sets $\{1,3,5\}$and $\{3,5,1\}$are equal, because they have the same elements.

- $\{1,3,3,5,5,5\}$is the same as the set $\{1,3,5\}$ because they have the same elements.

Sets

Empty Set

There is a special set that has no elements. This set is called the empty set, or null set, and is denoted by $\emptyset$.

The empty set can also be denoted by $\{\}$

Sets

Cardinality

The cardinality is the number of distinct elements in S.
The cardinality of S is denoted by $|S|$.

Example1

$S=\{a,b,c,d\}$

$|S|=4$

$A=\{1,2,3,7,3,9\}$

$|\mathrm{S}|=5$

Sets

Infinite

A set is said to be **infinite** if it is not finite.

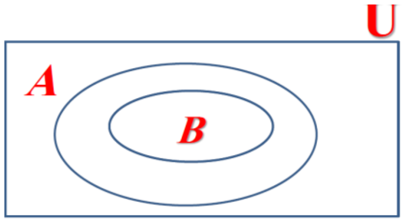
The set of positive integers is infinite.
$Z+=\{1,2,3,...\}$

Subset

The set A is said to be a subset of B if and only if Every element of A is also an element of B.

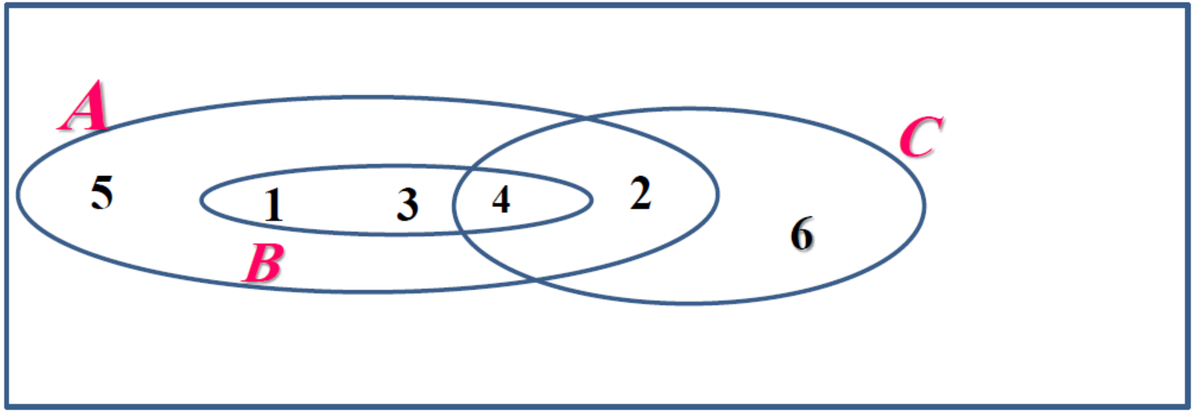
We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

$$A \subseteq B \leftrightarrow \forall x (x \in A \rightarrow x \in B)$$



Sets

Let A = { 1 , 2 , 3,4,5}, B = { 1 , 3 , 4 } and C ={2,4,6}
Then



This means $B \subseteq A$ *but* $C \not\subseteq A$

Proper Subset

The set A is a subset of the set B but that $A \neq B$, we write $A \subset B$

And say that A is a **proper subset** of B.

Sets

Example

For each of the following sets,
Determine whether 3 is an element of that set.

$\{1,2,3,4\}$

$\{\{1\},\{2\},\{3\},\{4\}\}$

$\{1,2,\{1,3\}\}$

Sets

Example

For each of the following sets,
Determine whether 3 is an element of that set.

$\{1,2,3,4\}$ True

$\{\{1\},\{2\},\{3\},\{4\}\}$ False

$\{1,2,\{1,3\}\}$ False

Cartesian Products

Let A and B be sets.

The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a,b),where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Cartesian Products - Example

Let $A = \{1,2\}$, and $B = \{a, b, c\}$

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

$$|A \times B| = |A| * |B| = 2 * 3 = 6$$

Find $B \times A$?

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

Sets

The Cartesian product of more than two sets.

Example:

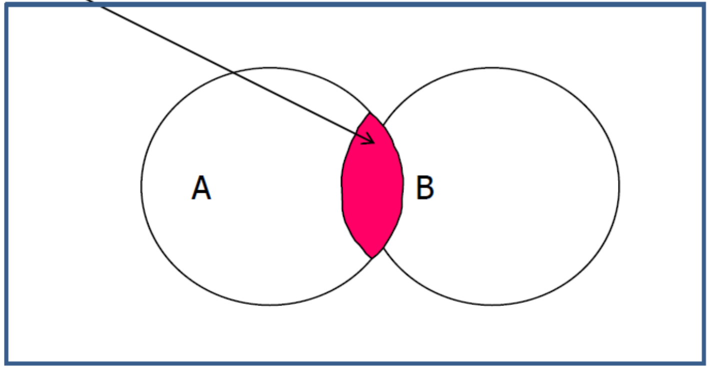
$A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$

$$\boxed{\begin{aligned} A \times B \times C = \{&(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), \\ &(1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}. \end{aligned}}$$

Set Operations

1] The ***Intersection*** of two sets A and B is defined by

$$A \cap B = \{x : x \in A \wedge x \in B\}$$



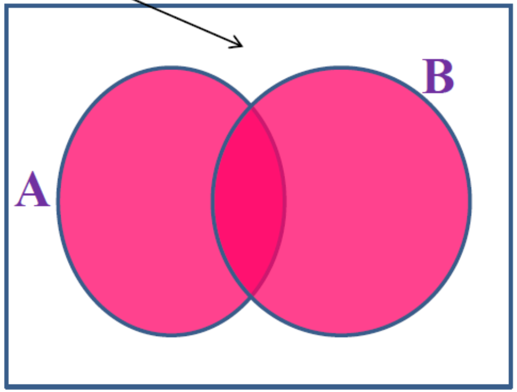
The intersection of the sets {1,3,5} and {1,2,3}

is the set {1,3}

Set Operations

2] The ***Union*** of two sets A and B is defined by

$$A \cup B = \{x : x \in A \vee x \in B\}$$



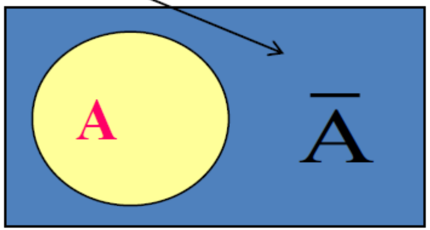
The union of the sets {1,3,5}and {1,2,3} is the set {1,2,3,5}

Sets

Set Operations

3] The **Complement** of the set A is defined by

$$\bar{A} = \{x : x \in U \wedge x \notin A\}$$



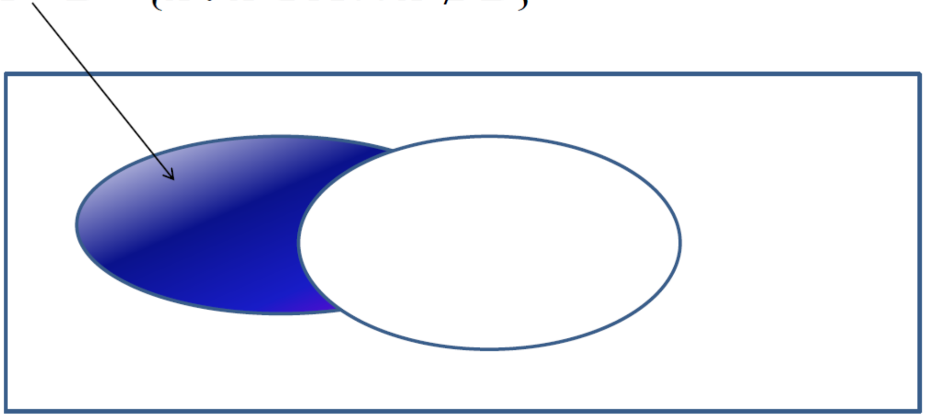
$U=1,2,3,4,5, \quad A=1,3$

$\bar{A}=2,4,5$

Set Operations

4] The **Difference** between the set A and the set B is defined by

$$A - B = \{x : x \in A \land x \notin B\}$$



$A = 1,3,5, \quad B = 1,2,3$

$A - B = 5$

Also it could be calculated by $A - B = A \cap \overline{B}$

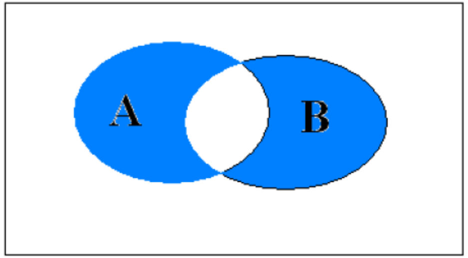
Set Operations

5] Symmetric difference

If A and B are two sets , we define their symmetric difference as the set of all elements that belong to A or to B but not to both A and B, and we denote it by

$A \oplus B$

Thus $A \oplus B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$



$A \oplus B$

Sets

Set Operations

Disjoint

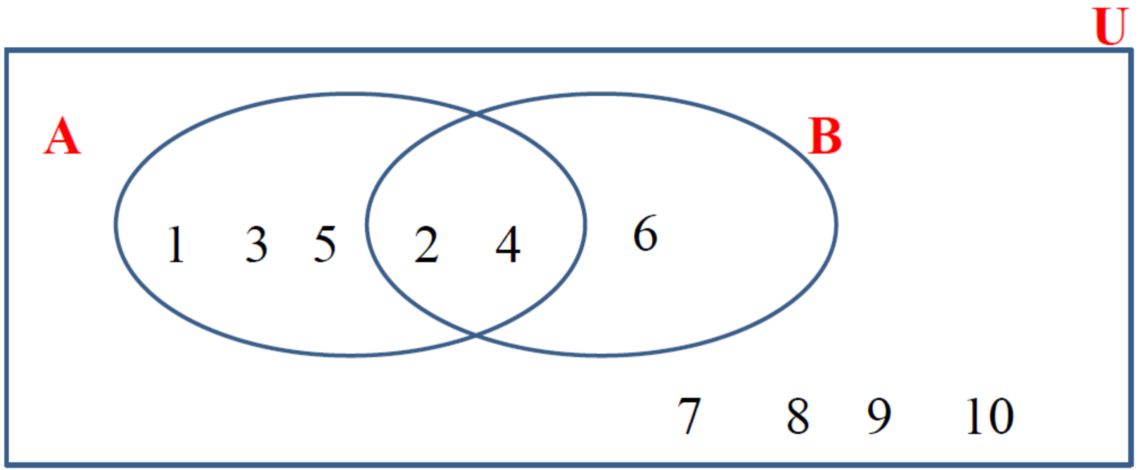
Two sets are called disjoint if their intersection is the empty set.

$$A \cap B = \emptyset$$

Set Operations **Example (3):**

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, \underline{4}, 6\}$ and $U = \{1, 2, 3, \ldots, 10\}$

Find $A \cup B$, $A \cap B$, $A - B$ and $\bar{A}$



Set Operations

Solution :

i) $A \cup B = \{1,2,3,4,5,6\}$

ii) $A \cap B = \{2,4\}$

iii) $A - B = \{1,3,5\}$

iv) $\overline{A} = \{6,7,8,9,10\}$

Set Operations

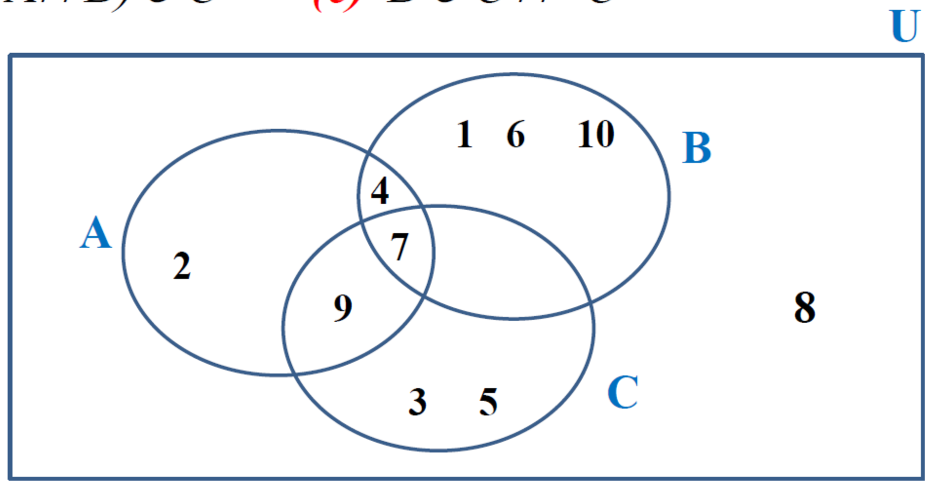
Example(4):

Let $A = \{2,4,7,9\}$, $B = \{1,4,6,7,10\}$, $C = \{3,5,7,9\}$ *and the universal set is* $U = \{1,2,3,\ldots,10\}$

Find

(a) $A \cup B$ **(b)** $A \cap C$ **(c)** $B \cap \overline{C}$

(d) $(A \cap \overline{B}) \cup C$ **(e)** $\overline{B \cup C} \cap C$



Sets

Set Operations

Solution:

(a) $A \cup B = \{1,2,4,6,7,9,10\}$

(b) $A \cap C = \{7, 9\}$

(c) $B \cap \overline{C} = \{1,4, 6, 10\}$

(d) $\overline{B} = \{2, 3 , 5,8, 9\}$ Then $(A \cap \overline{B}) = \{2,9\}$ Then

$(A \cap \overline{B}) \cup C = \{2, 3, 5, 7, 9\}$

(e) $\overline{B \cup C} = \{2, 8\}$ Then

$\overline{B \cup C} \cap C = \emptyset$

Functions

Functions

Function

Let A and B be nonempty sets.

A function f from A to B is an assignment of exactly one element of B to each element of A.
We write $f(a)=b$ if b is the unique element of B assigned by the function f to the element a of A.

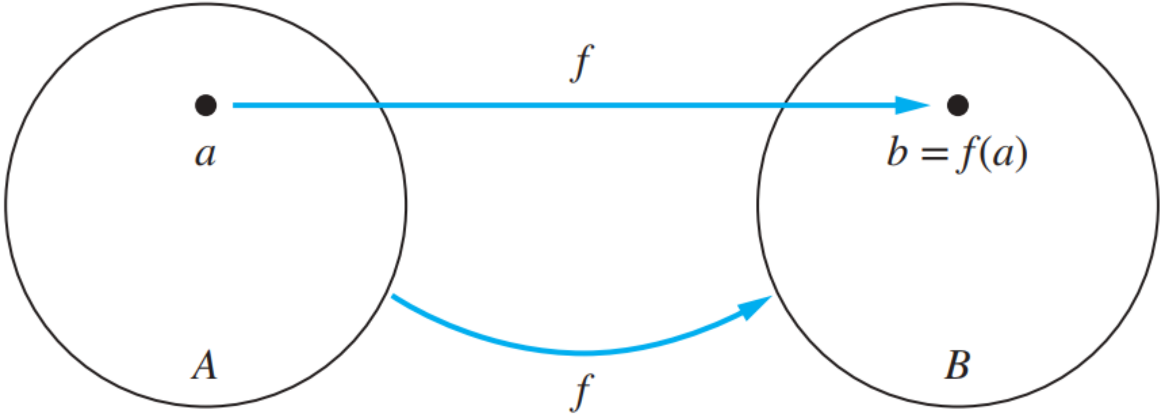
If f is a function from A to B, we write $f: A \rightarrow B$

Functions

The Function $f:A\rightarrow B$

Domain: A
Co-Domain: B
$f(a)=b$

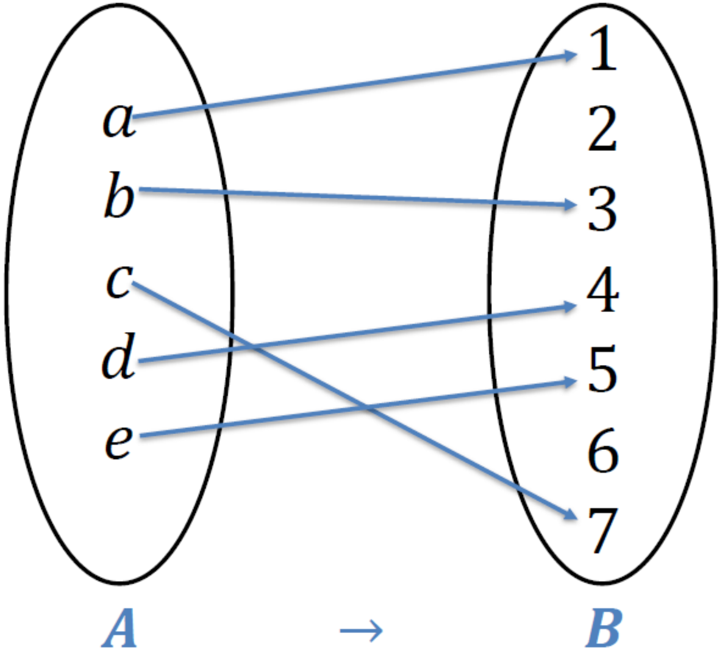
b is the ***image*** of a
a is a ***preimage*** of b



The function f maps A to B.

Functions

The Function $f: A \to B$



$\text{Domain} = \{a, b, c, d, e\}$

$\text{Co-Domain} = \{1,2,3,4,5,6,7\}$

$\text{Range} = \{1,3,4,5,7\}$

The **range**, or image, of f is the ***set of all images*** of elements of A.

Functions

Definition

Let $f1$ and $f2$ be functions from A to **R**.

Then $f1+f2$and $f1f2$are also functions from $A$to **R** defined for all $x \in A$ by:

$(f1+f2)(x)=f1(x)+f2(x)$,

$(f1f2)(x)=f1xf2(x)$.

Functions

Example

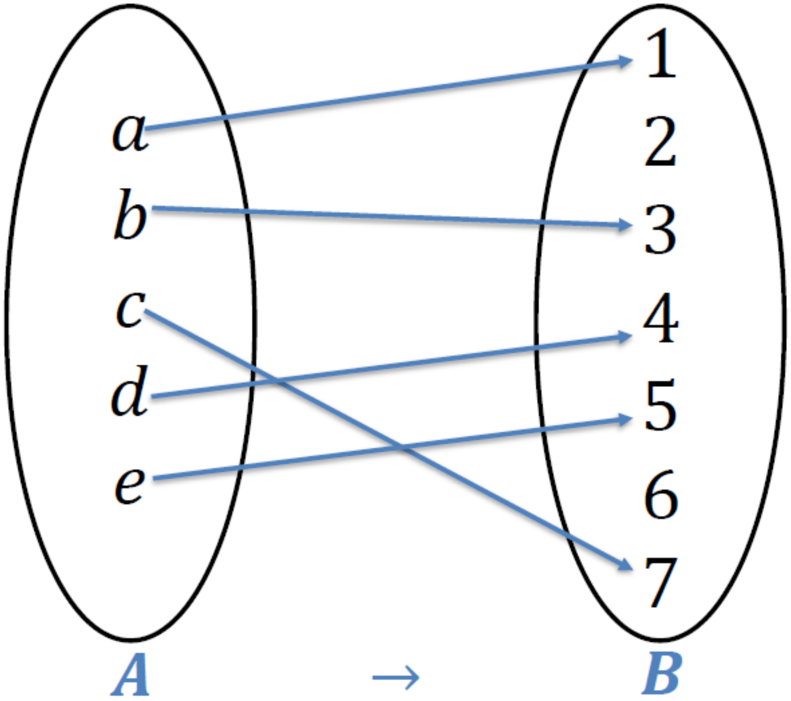
Let f_1 and f_2 be functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x,$$

$$(f_1 f_2)(x) = f_1(x) f_2(x) = x^2(x - x^2) = x^3 - x^4.$$

Functions

One-to-One function (injective)



$f(a) = 1$

$f(b) = 3$

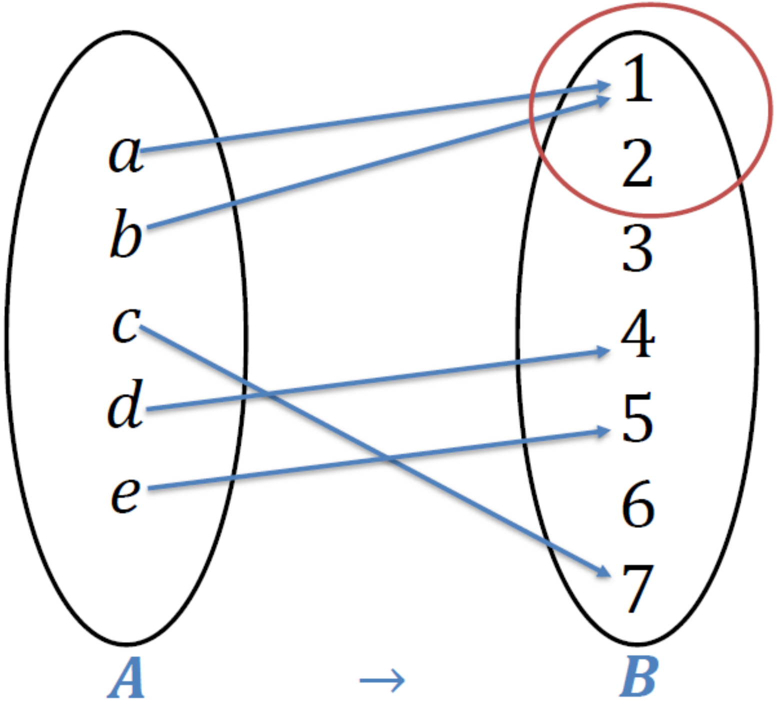
$f(c) = 7$

$f(d) = 4$

$f(e) = 5$

Functions

NOT ***One-to-One*** **function (Not injective)**



$$\boxed{\begin{aligned} f(a) &= 1 \\ f(b) &= 1 \end{aligned}}$$

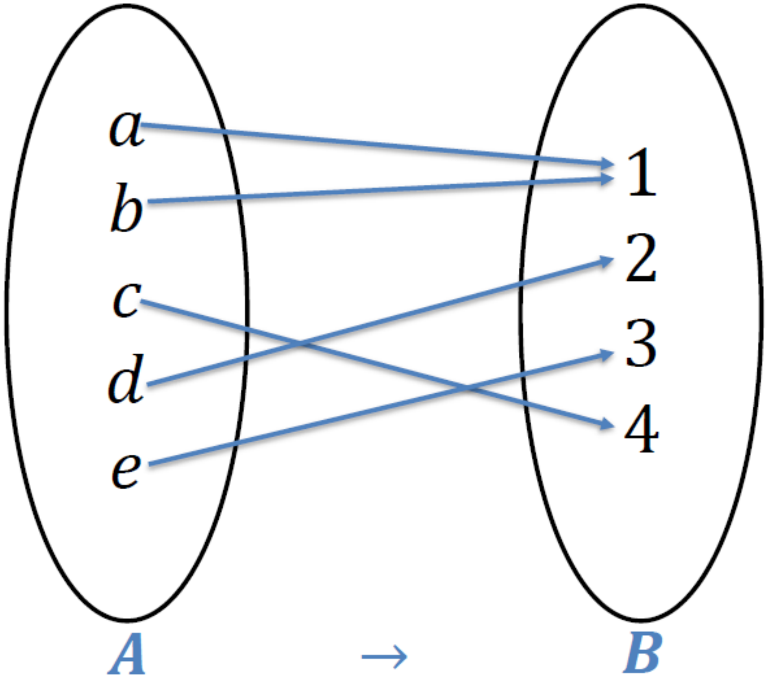
$$f(c) = 7$$

$$f(d) = 4$$

$$f(e) = 5$$

Functions

onto function (surjective)



$f(a) = 1$

$f(b) = 1$

$f(c) = 4$

$f(d) = 2$

$f(e) = 3$

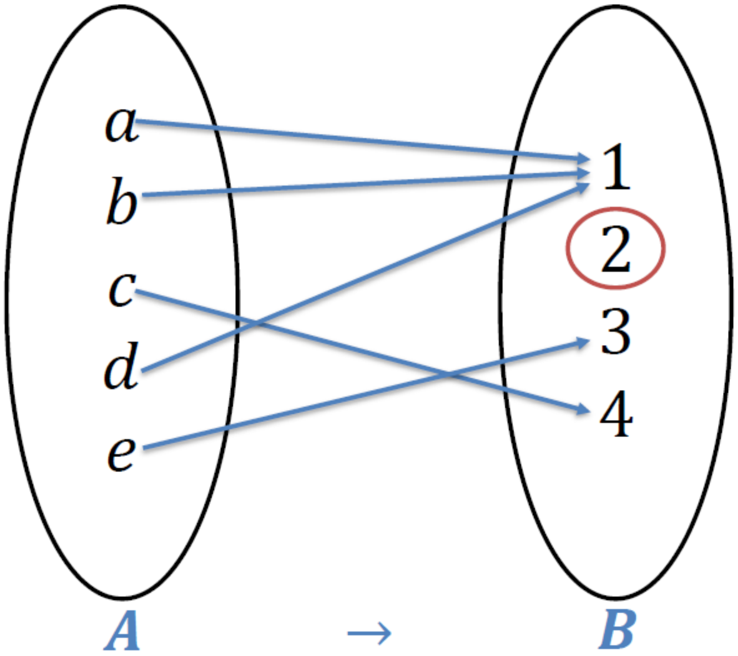
A function f from A to B is called **onto**, or **surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

Co-Domain = {1,2,3,4}

Range = {1,2,3,4}

Functions

NOT *onto* function (Not surjective)



$f(a) = 1$

$f(b) = 1$

$f(c) = 4$

$f(d) = 1$

$f(e) = 3$

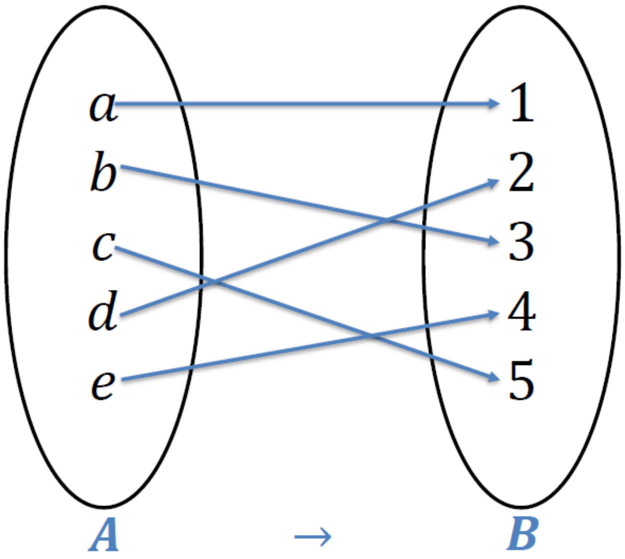
Co-Domain = {1,2,3,4}

Range = {1,3,4}

Functions

One-to-one correspondence (bijection)

$|A| = |B|$



$f(a) = 1$

$f(b) = 3$

$f(c) = 5$

$f(d) = 2$

$f(e) = 4$

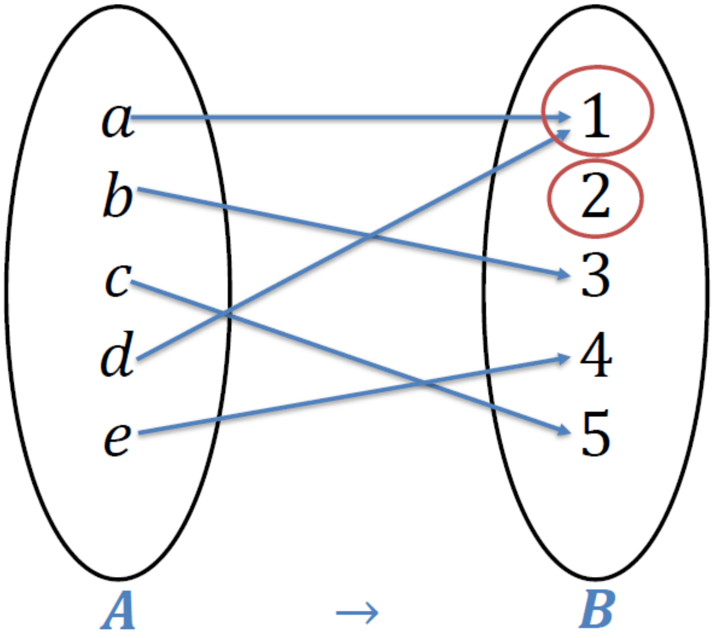
Co-Domain = $\{1,2,3,4,5\}$

Range = $\{1,2,3,4,5\}$

The function f is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto.

Functions

NOT ***One-to-one correspondence*** **(Not bijection)**



$f(a) = 1$

$f(b) = 3$ **NOT one-to-one**

$f(c) = 5$ **NOT onto**

$f(d) = 1$

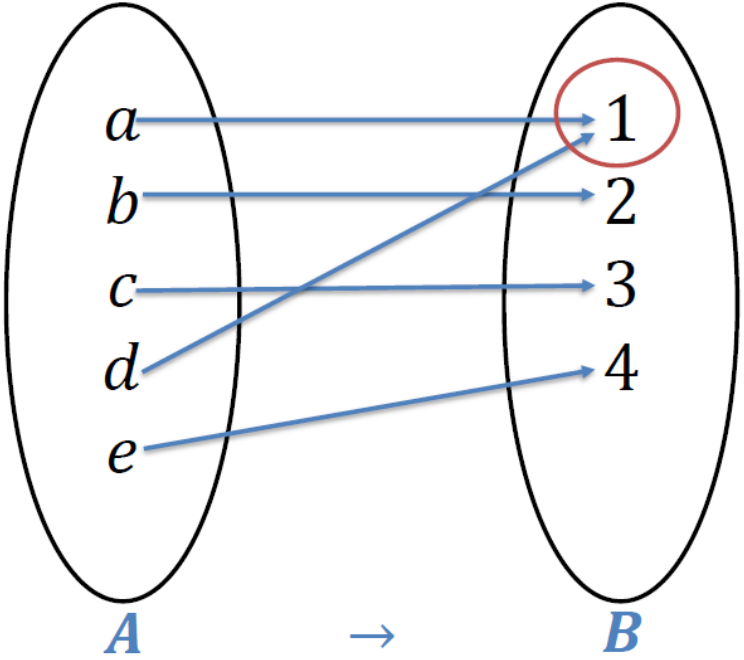
$f(e) = 4$

Co-Domain = {1,2,3,4,5}

Range = {1,3,4,5}

Functions

NOT *One-to-one correspondence* (Not bijection)



$f(a) = 1$

$f(b) = 2$ **Onto**

$f(c) = 3$ **NOT one-to-one**

$f(d) = 1$

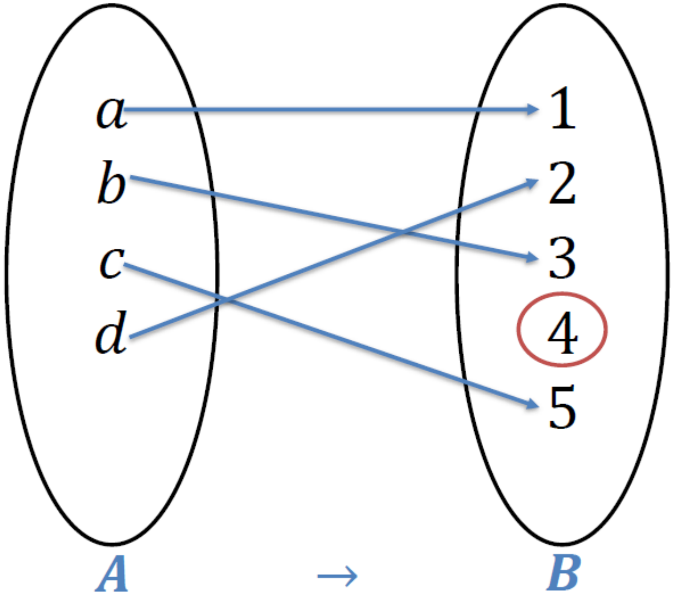
$f(e) = 4$

Co-Domain = {1,2,3,4}

Range = {1,2,3,4}

Functions

NOT *One-to-one correspondence* (Not bijection)



$f(a) = 1$

$f(b) = 3$

$f(c) = 5$

$f(d) = 2$

One-to-one

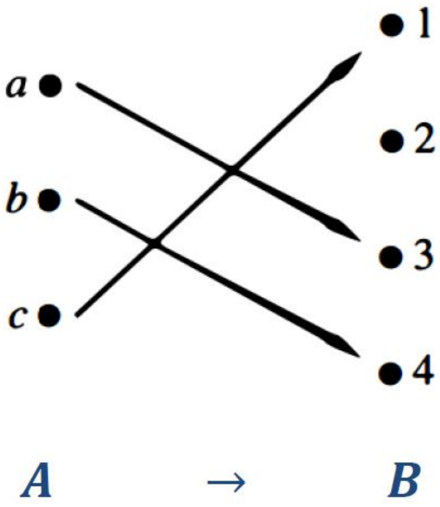
NOT onto

Co-Domain = $\{1,2,3,4,5\}$

Range = $\{1,2,3,5\}$

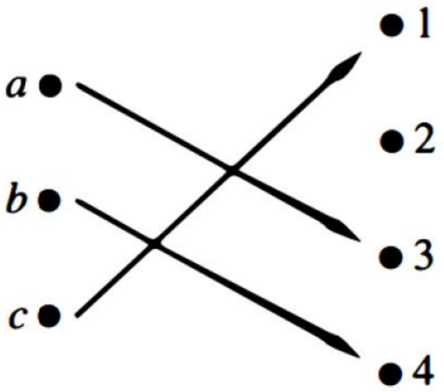
Functions

Examples



Functions

Examples



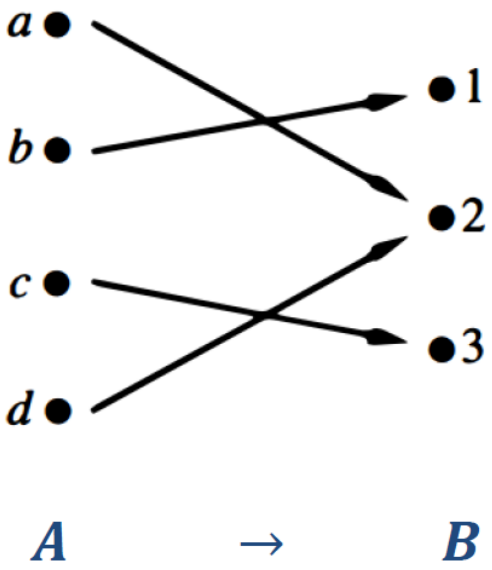
$A \quad \rightarrow \quad B$

One-to-one

NOT onto

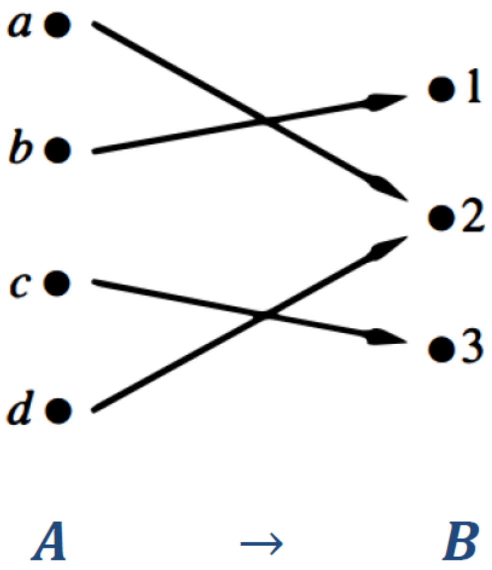
Functions

Examples



Functions

Examples

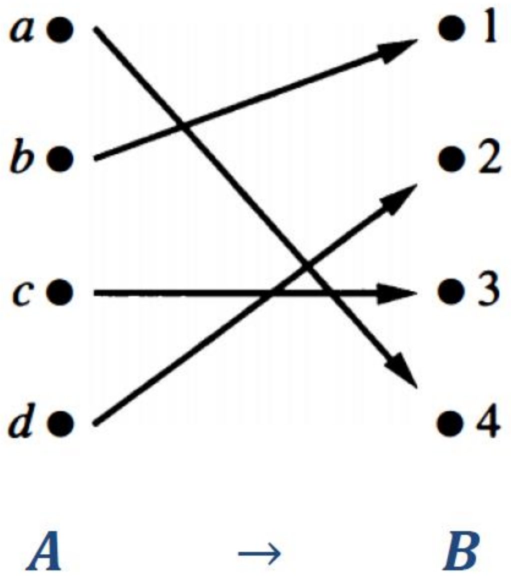


NOT One-to-one

Onto

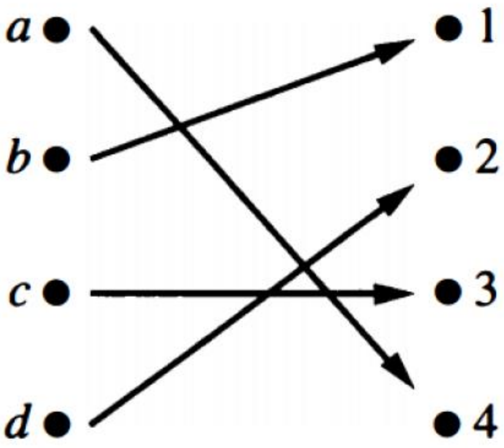
Functions

Examples



Functions

Examples



$A \rightarrow B$

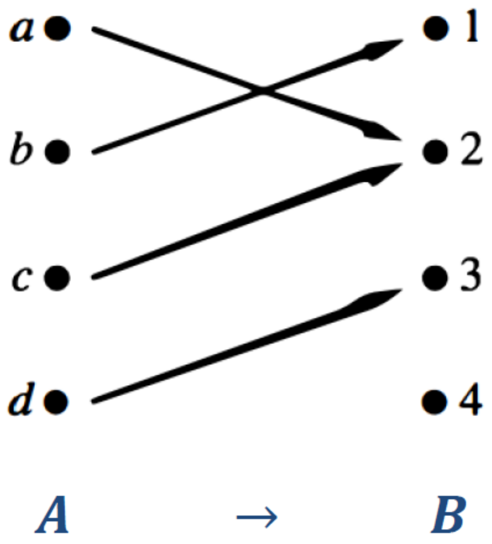
One-to-one

Onto

∴ bijection

Functions

Examples

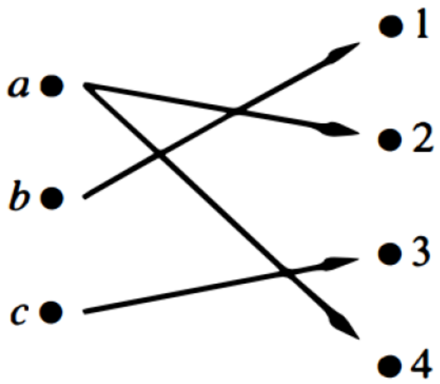


NOT One-to-one

NOT Onto

Functions

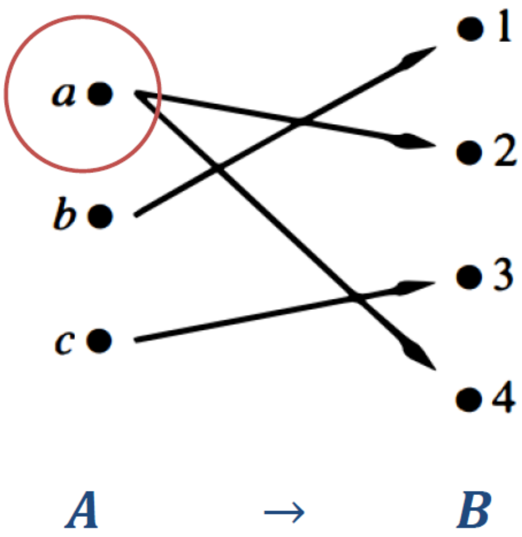
Examples



$A \rightarrow B$

Functions

Examples



NOT a function from A to B

Summations

Next, we introduce **summation notation.**
We begin by describing the notation used to express the sum of the terms

$$a_m, a_{m+1}, \ldots, a_n$$

from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \qquad \textstyle\sum_{1 \leq j \leq n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j)

to represent

Here, the variable j is called the **index of summation.**

$$a_m + a_{m+1} + \cdots + a_n.$$

Summations

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k$$

Here, the index of summation runs through all integers starting with its **lower limit** *m* and ending with its **upper limit** *n*. A large uppercase Greek letter sigma, $\sum$, is used to denote summation.

Summations

Express the sum of the first 100 terms of the sequence $\{a_n\}$, where $a_n = 1/n$ for $n = 1, 2, 3, \ldots$

Answer

$$\sum_{n=1}^{100} 1/n$$

Summations

What is the value of $\sum_{j=1}^{5} j^2$?

Answer

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$
$$= 1 + 4 + 9 + 16 + 25$$
$$= 55.$$

Summations

What is the value of $\sum_{s \in \{0,2,4\}} s$?

Answer:

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$

Summations

Double Summation

Find

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6 + 12 + 18 + 24 = 60.$$

The Foundations: Logic and Proofs

Thank you !

 $\wedge$  $\longrightarrow$