Applied Discrete Mathematics

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- POLYTECHNIC OF EGYPT -

Applied Discrete Mathematics
Lecture 2

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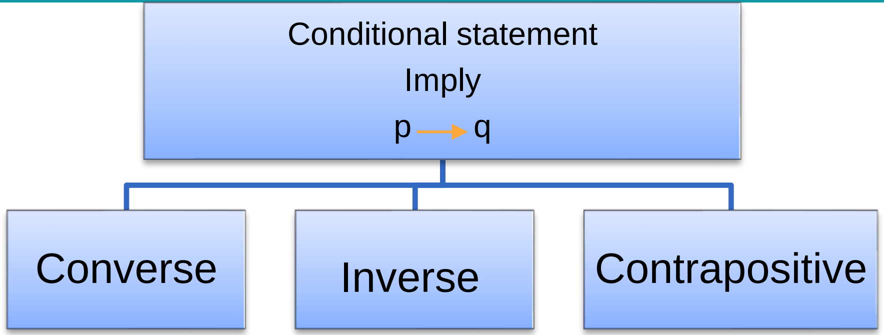
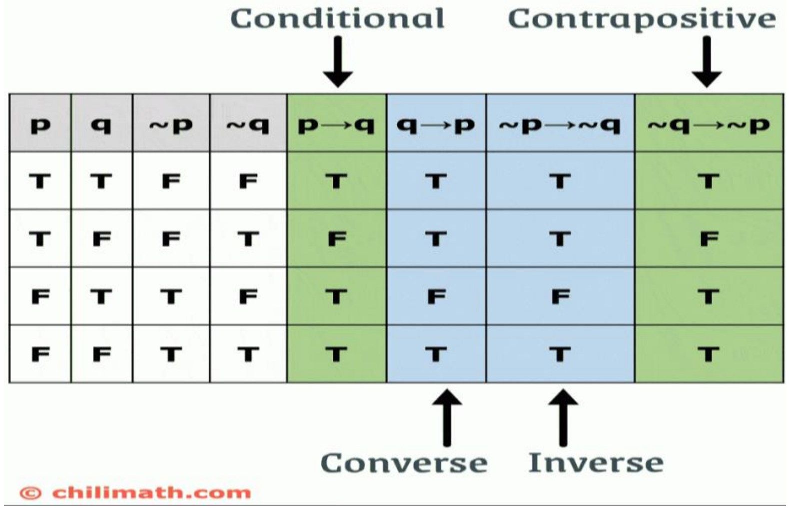
The Foundations: Logic and Proofs



$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$q \rightarrow p \equiv \neg p \rightarrow \neg q$

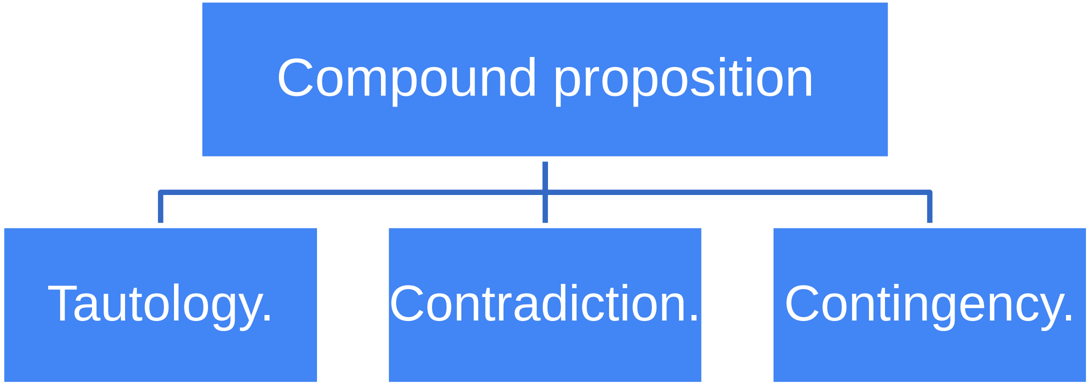


p	q	~p	~q	p→q	q→p	~p→~q	~q→~p
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

The Foundations: Logic and Proofs

Propositional Equivalences:

DEFINITION 1 A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.



The Foundations: Logic and Proofs

Compound proposition

- Tautology
  - All the truth table (T)
- Contradiction.
  - All the truth table (F)
- Contingency.
  - Neither T nor F

Example:

- Show that following conditional statement is a **tautology** by using truth table

$$(p \wedge q) \rightarrow p$$

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$

Example:

- Show that following conditional statement is a **tautology** by using truth table

$$(p \wedge q) \rightarrow p$$

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called **logically equivalent**. We can also define this notion as follows.

DEFINITION 2

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Example1:

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T					
T	F					
F	T					
F	F					

The Foundations: Logic and Proofs

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Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T				
T	F	T				
F	T	T				
F	F	F				

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p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F			
T	F	T	F			
F	T	T	F			
F	F	F	T			

The Foundations: Logic and Proofs

Example1:

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	
T	F	T	F	F	T	
F	T	T	F	T	F	
F	F	F	T	T	T	

Example1:

Show that $\neg(p \lor q)$ and $\neg p \land \neg q$ are logically equivalent.

Truth Tables for $\neg(p \lor q)$ and $\neg p \land \neg q$.

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The Foundations: Logic and Proofs

Example1:

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Example:

A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

The Foundations: Logic and Proofs

Logically Equivalences

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws

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$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

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TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

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Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ by the second De Morgan law

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws

Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$$\begin{aligned}\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law}\\ &\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law}\end{aligned}$$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws

The Foundations: Logic and Proofs

Example 1:

Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent.

$$
\begin{aligned}
\neg(p \lor (\neg p \land q)) &\equiv \neg p \land \neg(\neg p \land q) && \text{by the second De Morgan law} \\
&\equiv \neg p \land [\neg(\neg p) \lor \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \land (p \lor \neg q) && \text{by the double negation law}
\end{aligned}
$$

$\neg(\neg p) \equiv p$	Double negation law

Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law}
\end{aligned}$$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws

Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F}
\end{aligned}
$$

Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction}
\end{aligned}
$$

$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws

The Foundations: Logic and Proofs

Example 1:

Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent.

$$
\begin{aligned}
\neg(p \lor (\neg p \land q)) &\equiv \neg p \land \neg(\neg p \land q) && \text{by the second De Morgan law} \\
&\equiv \neg p \land [\neg(\neg p) \lor \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \land (p \lor \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \land p) \lor (\neg p \land \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \lor (\neg p \land \neg q) && \text{because } \neg p \land p \equiv \mathbf{F} \\
&\equiv (\neg p \land \neg q) \lor \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \land \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity laws

Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$$
\begin{aligned}
\boxed{\neg(p \vee (\neg p \wedge q))} &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\boxed{\equiv \neg p \wedge \neg q} && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

The Foundations: Logic and Proofs

EXAMPLE 7 Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: We will use one of the equivalences in Table 6 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$. (*Note:* we could also easily establish this equivalence using a truth table.) We have the following equivalences.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent. ◀

The Foundations: Logic and Proofs

EXAMPLE 8 Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to $\mathbf{T}$. (*Note:* This could also be done using a truth table.)

$$\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by Example 3} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative laws for disjunction} \\
&\equiv \mathbf{T} \vee \mathbf{T} && \text{by Example 1 and the commutative law for disjunction} \\
&\equiv \mathbf{T} && \text{by the domination law}
\end{aligned}$$

Predicates and Quantifiers

A **predicate** is a statement containing one or more variables . If values are assigned to all the variables in a predicate , the resulting statement is a proposition .

X < 5 is a predicate , where x is a variable denoting any real number .
If we substitute a real number for x , we obtain a proposition ; for example ' 3 < 5 ' and ' 6 < 5 ' are propositions with truth values T and F respectively .

Predicates and Quantifiers

Predicate:

$$\boxed{x}\ \boxed{\text{is greater than 3}}$$

The subject The predicate

- Predicate is a statement has variables.
- Proposition is a statement does not have variables.

We can denote the statement "x *is greater than* 3" by $\boldsymbol{P(x)}$

where $\boldsymbol{P}$ denotes the predicate "*is greater than* 3" and $\boldsymbol{x}$ is the variable.

The statement $\boldsymbol{P(x)}$ is also said to be the value of the **propositional function** $\boldsymbol{P}$ at $\boldsymbol{x}$. Once a value has been assigned to the variable $\boldsymbol{x}$, the statement $\boldsymbol{P(x)}$ becomes a proposition and has a truth value.

You are good person	(Proposition)
Mohamed is married to X	(predicate)
X+10=20	(predicate)
2+3=5	(Proposition)

The Foundations: Logic and Proofs

Example1:

Let $P(x)$ denote the statement "$x > 3$."

What are the truth values of $P(4)$ and $P(2)$?

Solution

We obtain the statement *P(4)* by setting $x = 4$ in the statement "$x > 3$." Hence, *P(4)*, which is the statement "$4 > 3$," is true. However, *P(2)*, which is the statement "$2 > 3$," is false.

Example2:

Let $Q(x, y)$ denote the statement "$x = y + 3$."

What are the truth values of the propositions

$Q(1, 2)$ and $Q(3, 0)$?

F T

Example3:

1. Let $P(x)$ denote the statement "$x \leq 4$." What are the truth values?
   a) $P(0)$ b) $P(4)$ c) $P(6)$
2. Let $P(x)$ be the statement "the word x contains the letter a." What are the truth values?
   a) $P(\text{orange})$ b) $P(\text{lemon})$
   c) $P(\text{true})$ d) $P(\text{false})$

Example3:

1. Let $P(x)$ denote the statement "$x \leq 4$." What are the truth values?
   a) $P(0)$ T
   b) $P(4)$ T
   c) $P(6)$ F
2. Let $P(x)$ be the statement "the word x contains the letter a." What are the truth values?
   a) $P(\text{orange})$ T
   b) $P(\text{lemon})$ F
   c) $P(\text{true})$ F
   d) $P(\text{false})$ T

Example 4:

Let $P(x,y)$ denote the statement "$x + y \leq 4$." and Q(x,y,z) denote the statement " $x+2y+5z > 20$"
What are these truth values?

- $P(2,3) \wedge Q(2,1,5)$
- $P(2,1) \longrightarrow Q(3,1,3)$

The Foundations: Logic and Proofs

Example:

Let $P(x,y)$ denote the statement "$x + y \leq 4$."   and Q(x,y,z) denote the statement " $x+2y+5z > 20$"

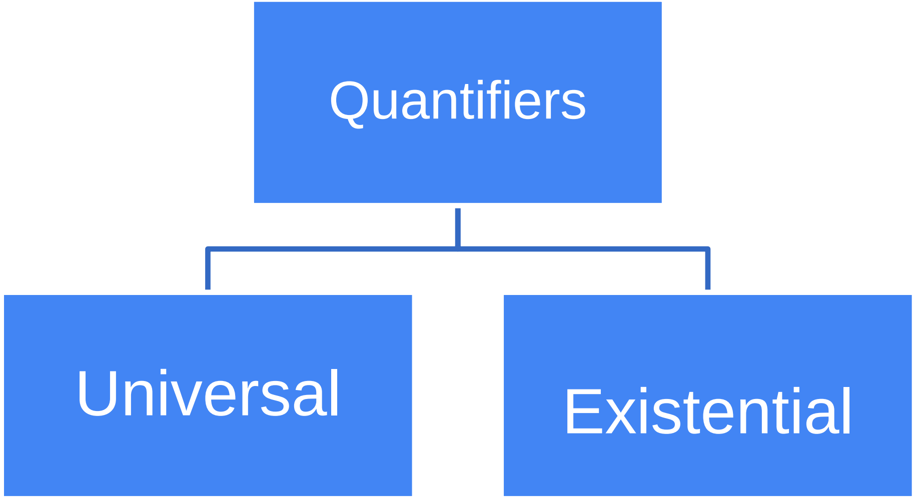
What are these truth values?

- $P(2,3) \wedge Q(2,1,5)$ $F \wedge T$ False
- $P(2,1) \longrightarrow Q(3,1,3)$ $T \rightarrow F$ False

The Foundations: Logic and Proofs

Quantifiers:

Expresses the extent to which a predicate is true over a range of elements.



The Foundations: Logic and Proofs

Quantifiers

Universal — $\forall x\, P(x)$

Existential — $\exists x\, P(x)$

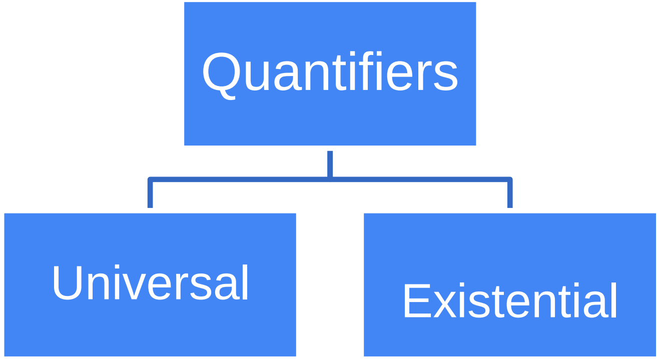
DEFINITION 1

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x P(x)$ as "for all x $P(x)$" or "for every x $P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.

The Foundations: Logic and Proofs



$\forall x P(x)$ $\exists x P(x)$

DEFINITION 2

The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

The Foundations: Logic and Proofs

TABLE 1 Quantifiers.

Statement	When True?	When False?
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Universal

Existential

The expression “for all” which is denoted by **∀** is a **quantifier**

The expression “there exists” which is denoted by **∃** is a **quantifier**

For example

If we consider the predicate $P(x) = "x > 5"$

Then : $\forall x\, P(x)$ *is **a false statement***

But: $\exists x\, P(x)$ *is **a true statement***

Example1:

Let $P(x)$ be the statement "$x + 1 > x$."

What is the truth value of the quantification $\forall x P(x)$,

where the domain consists of all real numbers?

The Foundations: Logic and Proofs

Example1:

Let $P(x)$ be the statement "$x + 1 > x$."

What is the truth value of the quantification $\forall x P(x)$,

where the domain consists of all real numbers?

Solution: Because $P(x)$ is true for all real numbers x, the quantification

$$\forall x P(x)$$

is true.

Example2:

Let $Q(x)$ be the statement "$x < 2$."

What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Example2:

Let $Q(x)$ be the statement "$x < 2$."

What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus $\forall x Q(x)$ is false.

Example3:

Let $P(x)$ denote the statement "$x > 3$."

What is the truth value of the quantification $\exists x P(x)$,

where the domain consists of all real numbers?

The Foundations: Logic and Proofs

Example3:

Let $P(x)$ denote the statement "$x > 3$."

What is the truth value of the quantification $\exists x P(x)$,

where the domain consists of all real numbers?

Solution: Because "$x > 3$" is sometimes true—for instance,

when $x = 4$—the existential quantification of $P(x)$,

which is $\exists x P(x)$, is true.

Example4:

What is the truth value of $\exists x P(x)$,

where $P(x)$ is the statement "$x^2 > 10$" and the universe of

discourse consists of the positive integers not exceeding 4?

The Foundations: Logic and Proofs

What is the truth value of $\exists x\, P(x)$,

where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is $\{1, 2, 3, 4\}$, the proposition $\exists x\, P(x)$ is the same as the disjunction

$P(1) \vee P(2) \vee P(3) \vee P(4)$.

Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x\, P(x)$ is true.

Example6:

Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are the truth values?

a) $P(0)$
b) $P(1)$
c) $P(2)$
d) $P(-1)$
e) $\exists x P(x)$
f) $\forall x P(x)$

Example6:

Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are the truth values?

a) $P(0)$ T
b) $P(1)$ T
c) $P(2)$ F
d) $P(-1)$ F
e) $\exists x P(x)$ T
f) $\forall x P(x)$ F

The Foundations: Logic and Proofs

Negating Quantified Expressions

1

“Every student in your class has taken a course in calculus.”

This statement is a universal quantification, namely,

$$\forall x P(x),$$

where $P(x)$ is the statement “x has taken a course in calculus” and the domain consists of the students in your class. The negation of this statement is “It is not the case that every student in your class has taken a course in calculus.” This is equivalent to “There is a student in your class who has not taken a course in calculus.” And this is simply the existential quantification of the negation of the original propositional function, namely,

$$\exists x \neg P(x).$$

The Foundations: Logic and Proofs

1

This example illustrates the following logical equivalence:

$$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x).$$

The Foundations: Logic and Proofs

2

Suppose we wish to negate an existential quantification. For instance, consider the proposition "There is a student in this class who has taken a course in calculus." This is the existential quantification

$$\exists x\, Q(x),$$

where $Q(x)$ is the statement "x has taken a course in calculus." The negation of this statement is the proposition "It is not the case that there is a student in this class who has taken a course in calculus." This is equivalent to "Every student in this class has not taken calculus," which is just the universal quantification of the negation of the original propositional function, or, phrased in the language of quantifiers,

$$\forall x\, \neg Q(x).$$

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This example illustrates the equivalence

$$\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x).$$

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Negating Quantified Expressions

$\neg\forall x P(x) \equiv \exists x \neg P(x)$

$\neg\exists x Q(x) \equiv \forall x \neg Q(x)$

For example
If we consider the predicate $P(x)$ = *"My student x is clever"*
Then

$$\forall x\, P(x) \equiv \text{"All my students are clever"}$$

$$\neg[\forall x\, P(x)] \equiv \exists x\, (\neg P(x)) \equiv \text{"There exists a student who is not clever"}$$

Example1:

What are the negations of the statements

$$\forall x(x^2 > x)$$

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What are the negations of the statements

$$\forall x(x^2 > x)$$

$$\neg\forall x(x^2 > x) \equiv \exists x\neg(x^2 > x)$$

$$\exists x\ (x^2 \leq x)$$

Example2:

What are the negations of the statements

$$\exists x(x^2 = 2)$$

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What are the negations of the statements

$$\exists x(x^2 = 2)$$

$$\neg\exists x(x^2 = 2) \equiv \forall x \neg(x^2 = 2)$$

$$\forall x\ (x^2 \neq 2)$$

Example

Write down the negative of the following proposition

"For every number x *there is a number* y *such that* $x < y$ "

Solution

The given proposition could be written on the form

$$\forall x\, \exists y\, (x < y)$$

Then we have

$$\neg[\forall x\, \exists\, y\, (\, x < y\,)] \equiv \exists x\, \forall y\ \neg(x < y) \equiv \exists x\, \forall y\ (x \geq y)$$

This means "There exists a number x *such that for every number* y *we have,* $x \geq y$ "

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Example

Write down the negative of the following proposition

"Every student in the faculty of computer science is genius"

Solution

Let *x denote any student*

Let *P (x) denote the predicate " x is a student in the faculty of computer science"*

Let *Q (x) denote the predicate " x is a genius student"*

The given proposition could be written on the form

"For all *x* , *if x a student in the faculty of computer science then x is genius"*

Or $\forall x\ [P(x) \rightarrow Q(x)]$

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Then we have

$$\begin{aligned} \neg [\forall x\, (P(x) \to Q(x))] &\equiv \exists x\, \neg(P(x) \to Q(x)) \\ &\equiv \exists x\, \neg(\neg P(x) \vee Q(x)) \\ &\equiv \exists x\, (\neg\neg P(x) \wedge \neg Q(x)) \\ &\equiv \exists x\, (P(x) \wedge \neg Q(x)) \end{aligned}$$

This means

“There exists a student who is in the faculty of computer science and he is not genius“

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1. $\neg\forall x(A \vee B)$
2. $\left(\overline{A \wedge (B \vee C)}\right)$
3. $\neg\exists x(A \wedge (B \vee C))$
4. $\left(\overline{A \wedge (A \vee C)}\right)$
5. $\left(\overline{A \wedge (A \vee C)}\right)$
6. $\neg\forall x\big((A \wedge (A \vee C))\big)$

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	Answer
1. $\neg\forall x(A \vee B)$	1. $\neg\forall x(A \vee B) = \exists x(\bar{A} \wedge \bar{B})$
2. $\left(\overline{A^\wedge(B \vee C))}\right.$	*2.* $\left(\overline{A^\wedge(B \vee C)}\right) = \bar{A} \vee (\bar{B} \vee C) = \bar{A} \vee (\bar{B}^\wedge\bar{C}) = (\bar{A} \vee \bar{B})^\wedge(\bar{A} \vee \bar{C})$
3. $\neg\exists x(A^\wedge(B \vee C))$	3. $\neg\exists x(A^\wedge(B \vee C)) = \forall x\neg(A^\wedge(B \vee C)) = \forall x(\bar{A} \vee \bar{B})^\wedge(\bar{A} \vee \bar{C})$
4. $\left(\overline{A^\wedge(A \vee C)}\right)$	*4.* $\left(\overline{A^\wedge(A \vee C)}\right) = \bar{A} \vee (\overline{A \vee C}) = \bar{A} \vee (\bar{A} \wedge \bar{C}) = \bar{A}$
	• Another Solution: $(\mathrm{A}^\wedge(\mathrm{A} \vee \mathrm{C}) = \mathrm{A} \rightarrow$ So: $\left(\overline{A^\wedge(A \vee C}\right)$ is $\bar{A}$
5. $\left(\overline{A^\wedge(A \vee C)}\right)$	5. $\neg\forall x\big((A^\wedge(A \vee C))\big)$
6. $\neg\forall x\big((A^\wedge(AVC))\big)$	6. $\neg\forall x\big((A^\wedge(A \vee C))\big) = \exists x\neg(\mathrm{A}) = \exists x\bar{A}$

Thank you !