ELSEWEDY
UNIVERSITY
OF TECHNOLOGY
- POLYTECHNIC OF EGYPT -

Applied Discrete Mathematics

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ELSEWEDY UNIVERSITY OF TECHNOLOGY
- POLYTECHNIC OF EGYPT -

Applied Discrete Mathematics
Lecture 8

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The Basics of Counting.

The Basics of Counting.

Counting is used to determine the complexity of algorithms. Counting is also required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand. Recently, it has played a key role in mathematical biology, especially in sequencing DNA. Furthermore, counting techniques are used extensively when probabilities of events are computed.

Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit.

How many such passwords are there?

The techniques needed to answer this question and a wide variety of other counting problems will be introduced in this section.

The Basics of Counting.

Multiplication (Product) Rule:

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

The total number of ways to complete the operation is $n_1 \times n_2 \times \cdots \times n_k$

The Basics of Counting.

Multiplication (Product) Rule:

EXAMPLE 1 The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution: The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600.

The Basics of Counting.

Example 2:
A new company with just two employees, Sanchez and John, rents a floor of a building with 12 offices.

How many ways are there to assign different offices to these two employees?

Solution: The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are 12 · 11 = 132 ways to assign offices to these two employees.

<table>
<tr><th>Sanchez</th><th>John</th></tr>
<tr><td>First, we have 12 offices
Then, we select 1 from 12 offices</td><td>Second, we have 11 offices
Then, we select 1 from 11 offices</td></tr>
<tr><td>$n_1 = 12$ ways</td><td>$n_2 = 11$ ways</td></tr>
<tr><td colspan="2">Total = 12 × 11 = 132 ways to assign offices to these two employees.</td></tr>
</table>

The Basics of Counting.

Example 3:

How many different bit strings of length seven are there?

Solution:

Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

Bits #	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Value	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1
Ways	$n_1 = 2$	$n_2 = 2$	$n_3 = 2$	$n_4 = 2$	$n_5 = 2$	$n_6 = 2$	$n_7 = 2$
Total $= 2^7 = 128$ different bit strings of length seven.							

The Basics of Counting.

Example 5:
The design for a Website is to consist of *four colors*, *three fonts*, and *three positions for an image*. How many different designs are possible?

Solution: From the product rule, 4 ×3 ×3 = 36 different designs are possible.

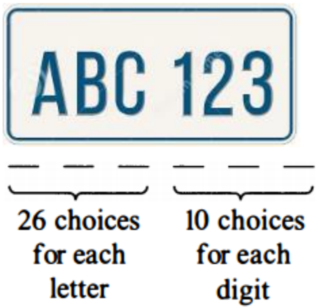
Example 6:
How many bit strings of length 5, start and end with 1's?
Solution:

Bits #	1	2	3	4	5
Value	1	either 0 or 1	either 0 or 1	either 0 or 1	1
Ways	$n_1 = 1$	$n_2 = 2$	$n_3 = 2$	$n_4 = 2$	$n_5 = 1$
Total $= 1 \times 2 \times 2 \times 2 \times 1 = 8$ different bit strings of length 5, start and end with 1's.					

The Basics of Counting.

Example 7:

How many different license plates are available if each plate contains a sequence of *three letters* followed by *three digits*.



Solution:

There are 26 choices for each of the three letters and ten choices for each of the three digits. Hence, by the product rule there are a total of $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates.

The Basics of Counting.

Example 8:

How many functions are there from a set with m elements to a set with n elements?

ways	Domain
n	1
	2
	3
	.
	m

Co-Domain
1
2
3
.
n

ways	Domain
n	1
n	2
	3
	.
	m

Co-Domain
1
2
3
.
n

ways	Domain
n	1
n	2
n	3
.	.
n	m

Co-Domain
1
2
3
.
n

Hence, by the product rule there are $n \cdot n \cdot \ldots \cdot n = n^m$ functions from a set with m elements too new ith n elements.

The Basics of Counting.

Example 9:

How many functions are there from a set with **3** elements to a set with **4** elements?

Solution:

ways	Domain
4	1
4	2
4	3

$\longrightarrow$

Co-Domain
1
2
3
4

Hence, by the product rule there are $4 \cdot 4 \cdot 4 = 4^3$ functions from a set with 3 elements to one with 4 elements.

The Basics of Counting.

Example 10:

Counting One-to-One Functions:

How many one-to-one functions are there from a set with m elements to a set with n elements? (where: $m \leq n$)

Solution:

ways	Domain
n	1
$(n-1)$	2
$(n-2)$	3
.	.
$(n-(m-1))$	m

Co-Domain
1
2
.
.
n

By the product rule, there are $n(n-1)(n-2)\ldots(n-m+1)$ one-to-one functions from a set with m elements to one with n elements.

The Basics of Counting.

How many one-to-one functions are there from a set with **4** elements to a set with **6** elements?

Solution:

ways	Domain
6	1
5	2
4	3
3	4

$\longrightarrow$

Co-Domain
1
2
3
4
5
6

$(n-(m-1))$

$(6-(4-1))$

By the product rule, there are $6 \times 5 \times 4 \times 3 = 360$ one-to-one functions from a set with 4 elements to one with 6 elements.

The Basics of Counting.

The Sum Rule:

THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

The Basics of Counting.

The sum rule in terms of sets :

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

$$\text{if } A_1, A_2, \ldots, A_m \quad \text{disjoint}$$

$$\text{if } A_1, A_2 \;\; NOT \text{ disjoint}$$

$$\therefore \; |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

The Basics of Counting.

Sum Rule – Example 1:

Suppose that **either** a member of the mathematics major or a student who is a physics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are **37** members of the mathematics majors and **83** physics majors and **no one is both** a mathematics and a physics major?

Solution:

By the sum rule it follows that there are $37 + 83 = 120$ possible ways.

The Basics of Counting.

Sum Rule – Example 2:

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution:

By the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

The Basics of Counting.

Sum Rule – Example 3:

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$k := 0$
for $i_1 := 1$ **to** n_1
  $k := k + 1$
for $i_2 := 1$ **to** n_2
  $k := k + 1$
.
.
.
for $i_m := 1$ **to** n_m
  $k := k + 1$

The loop is traversed	**The initial value of *k* is zero.**
n_1 times	**Because we only traverse one loop at a time, the sum rule shows that the final value of k, which is the number of ways to traverse one of the m loops is**
n_2 times	
n_m times	$\boldsymbol{n_1 + n_2 + \cdots + n_m}$

The Basics of Counting.

Counting Problems – Example 2:

In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a) the bride must be in the picture?
b) both the bride and groom must be in the picture?
c) exactly one of the bride and the groom is in the picture?

The Basics of Counting.

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

a) the bride must be in the picture?

1	2	3	4	5	6
bride 	Select from 9 people	Select from 8 people	Select from 7 people	Select from 6 people	Select from 5 people

The Basics of Counting.

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

a) the bride must be in the picture?

1	2	3	4	5	6
bride 	Select from 9 people	Select from 8 people	Select from 7 people	Select from 6 people	Select from 5 people
1	9	8	7	6	5

By the product rule, there are
$1 \times 9 \times 8 \times 7 \times 6 \times 5 = 15{,}120$ ways that the bride must be in the picture.

The Basics of Counting.

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

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1	2	3	4	5	6
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The Basics of Counting.

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The Basics of Counting.

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

a) the bride must be in the picture?

1	2	3	4	5	6
Select from 9 people	Select from 8 people	bride 	Select from 7 people	Select from 6 people	Select from 5 people
9	8	1	7	6	5

By the product rule, there are
$6 \times (1 \times 9 \times 8 \times 7 \times 6 \times 5) = 6 \times 15{,}120$ ways that the bride must be in the picture.

The Basics of Counting.

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

b) both the bride and groom must be in the picture?

1	2	3	4	5	6
bride 	groom 	Select from 8 people	Select from 7 people	Select from 6 people	Select from 5 people
1	1	8	7	6	5

By the product rule, there are
$6 \times 5 \times (1 \times 1 \times 8 \times 7 \times 6 \times 5) = 30 \times 1{,}680$ ways that both the bride and groom must be in the picture.

The Basics of Counting.

Counting Problems – Example 2:

Group of 10 people

We pick 6 people

c) exactly one of the bride and the groom is in the picture?

The bride must be in the picture	A	$\|A\| = 6 \times 15{,}120 = 90{,}720$
The groom must be in the picture	B	$\|B\| = 6 \times 15{,}120 = 90{,}720$
Both the bride and groom must be in the picture	$A \cap B$	$\|A \cap B\| = 30 \times 1{,}680 = 50{,}400$
The bride in the picture and the groom is **not** in the picture	$A - (A \cap B)$	$= 90{,}720 - 50{,}400 = 40{,}320$
The groom in the picture and the bride is **not** in the picture	$B - (A \cap B)$	$= 90{,}720 - 50{,}400 = 40{,}320$
Exactly one of the bride and the groom is in the picture	Using the sum rule	$= 40{,}320 + 40{,}320 = 80{,}640$

The Basics of Counting.

The Subtraction Rule:

THE SUBTRACTION RULE If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The Basics of Counting.

The principle of inclusion–exclusion:

The subtraction rule is also known as the ***principle of inclusion–exclusion***, especially when it is used to count the number of elements in the union of two sets.

$$\text{if } A_1, A_2 \;\; NOT \text{ disjoint}$$
$$\therefore\ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

The Basics of Counting.

Subtraction Rule –Example 1:

How many bit strings of length **five** *either* start with a 1bit *or* end with the two bits 00?

The Basics of Counting.

Subtraction Rule –Example 1:

How many bit strings of length **five** *either* start with a 1bit *or* end with the two bits 00?

1	--	--	--	--

or

--	--	--	**0**	**0**

1	0	1	0	1
1	0	0	0	0
1	1	1	0	0

1	1	1	**0**	**0**
1	0	0	**0**	**0**
0	1	1	**0**	**0**

The Basics of Counting.

Subtraction Rule –Example 1:

How many bit strings of length **five** *either* start with a 1bit *or* end with the two bits 00?

1	--	--	--	--	A
--	--	--	0	0	B
1	--	--	0	0	$A \cap B$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The Basics of Counting.

Subtraction Rule –Example 1:

How many bit strings of length **five** *either* start with a 1bit *or* end with the two bits 00?

1	--	--	--	--

A

Solution:

Bits #	**1**	**2**	**3**	**4**	**5**
Value	1	either 0 or 1	either 0 or 1	either 0 or 1	either 0 or 1
Ways	$n_1 = 1$	$n_2 = 2$	$n_3 = 2$	$n_4 = 2$	$n_5 = 2$
Total $= 1 \times 2 \times 2 \times 2 \times 2 = 2^4$ different bit strings of length 5, start with 1.					

The Basics of Counting.

Subtraction Rule –Example 1:

How many bit strings of length **five** *either* start with a 1bit *or* end with the two bits 00?

--	--	--	0	0

B

Solution:

Bits #	1	2	3	4	5
Value	either 0 or 1	either 0 or 1	either 0 or 1	0	0
Ways	$n_1 = 2$	$n_2 = 2$	$n_3 = 2$	$n_4 = 1$	$n_5 = 1$

Total $= 2 \times 2 \times 2 \times 1 \times 1 = 2^3$ different bit strings of length 5, end with the two bits 00.

The Basics of Counting.

Subtraction Rule –Example 1:

How many bit strings of length **five** *either* start with a 1bit *or* end with the two bits 00?

1	--	--	0	0

$A \cap B$

Solution:

Bits #	1	2	3	4	5
Value	1	either 0 or 1	either 0 or 1	0	0
Ways	$n_1 = 1$	$n_2 = 2$	$n_3 = 2$	$n_4 = 1$	$n_5 = 1$

Total $= 1 \times 2 \times 2 \times 1 \times 1 = 2^2$ different bit strings of length 5, start with a 1 bit *AND* end with the two bits 00.

The Basics of Counting.

Subtraction Rule –Example 1:

How many bit strings of length **five** *either* start with a 1bit *or* end with the two bits 00?

Solution:

The total number of bit strings of length five either start with a 1 bit or end with the two bits 00 is:

$$= 2^4 + 2^3 - 2^2 = 16 + 8 - 4 = 20$$

The Pigeonhole Principle.

THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it.

The Pigeonhole Principle.

THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

This illustrates a general principle called the **pigeonhole principle**. For instance, suppose that a flock of 13 pigeons flies into a set of 12 pigeonholes to roost.

The Pigeonhole Principle.

Example 1:

Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example 2:

In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

The Pigeonhole Principle.

Example 3:

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution:

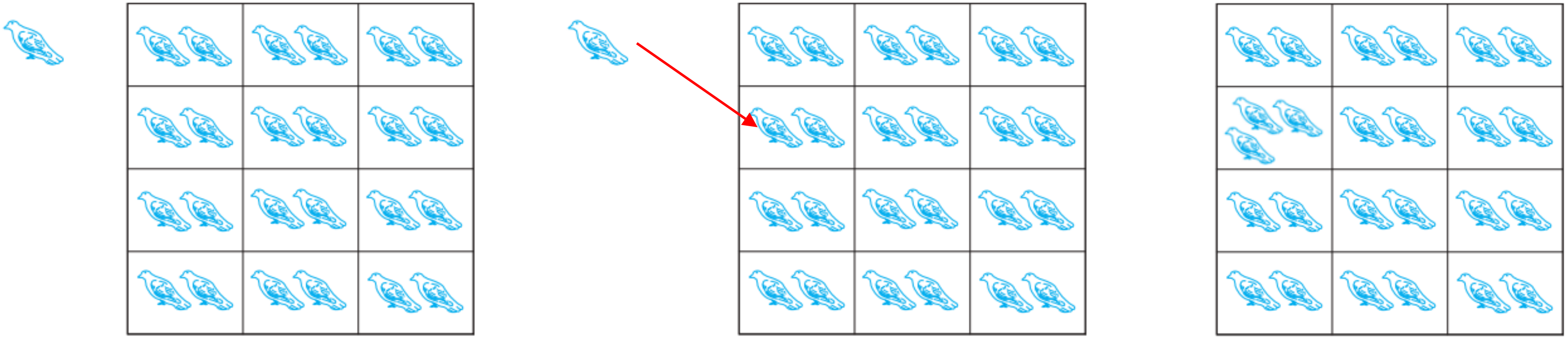
There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

The Pigeonhole Principle.

The Generalized Pigeonhole Principle

The pigeonhole principle states that there must be at least two objects in the same box when there are more objects than boxes. However, even more can be said when the number of objects exceeds a multiple of the number of boxes.

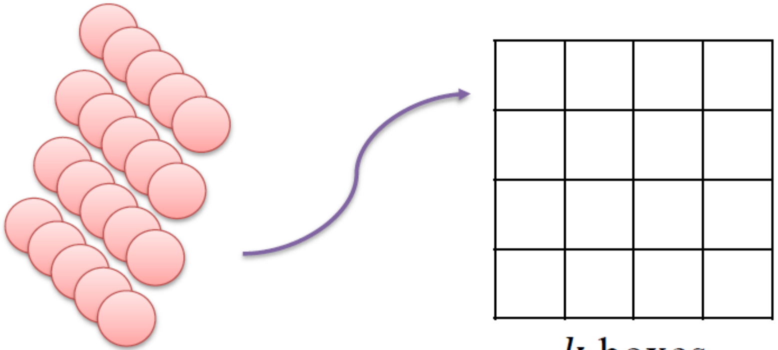
For instance, suppose that a flock of 25 pigeons flies into a set of 12 pigeonholes to roost.



The Pigeonhole Principle.

The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects. Where N and k are positive integers.



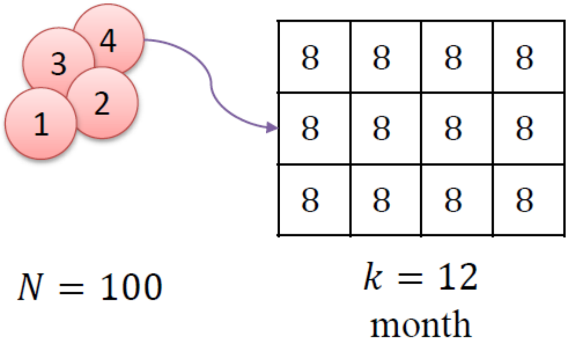
N objects

k boxes

The Pigeonhole Principle.

Example 1:

Among 100 people there are **at least** $\lceil 100/12 \rceil = 9$ who were born in the same month.



The Pigeonhole Principle.

Example 2:

Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

The Pigeonhole Principle.

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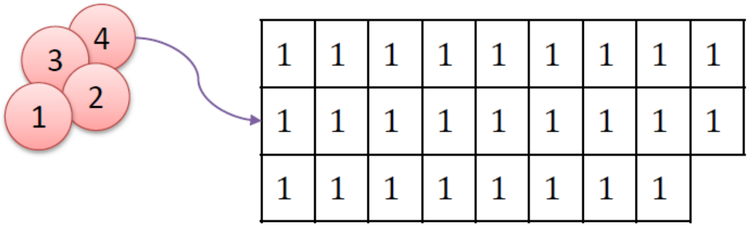


$N = 30$ $k = 26$ letter

at least $\lceil 30/26 \rceil = 2$ names that begin with the same letter.

The Pigeonhole Principle.

Example 3:

Show that among any group of five (not necessarily consecutive) integers, there are at least two with the same remainder when divided by 4.

The Pigeonhole Principle.

Example 3:

Show that among any group of five (not necessarily consecutive) integers, there are at least two with the same remainder when divided by 4.

0	1	2	3

possible remainders

$N = 5$

$k = 4$

Solution:

Because there are four possible remainders when an integer is divided by 4, the pigeonhole principle implies that given five integers, at least two have the same remainder.

at least $\lceil 5/4 \rceil = 2$ integers with the same remainder.

The Pigeonhole Principle.

Rule:

A common type of problem asks for the minimum number of objects such that at least r of these objects must be in one of k boxes when these objects are distributed among the boxes.

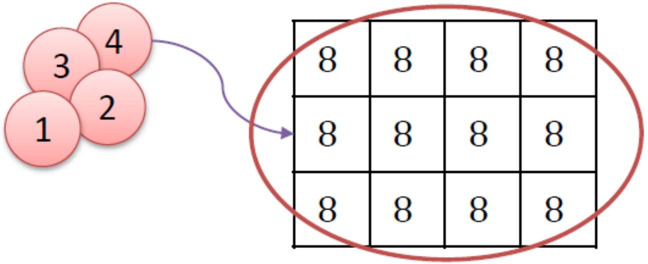
When we have N objects, the generalized pigeonhole principle tells us there must be at least r objects in one of the boxes. The smallest integer $N = k(r - 1) + 1$, is the smallest integer satisfying the inequality $\lceil N / k \rceil \geq r$.

The Pigeonhole Principle.

Recall Example 1:

Among 100 people there are **at least** $\lceil 100/12 \rceil = 9$ who were born in the same month.

$r = 9$



$8 * 12 = 96$

$N = 100$

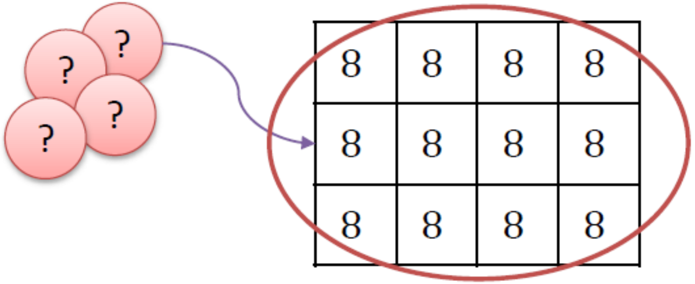
$k = 12$
month

The Pigeonhole Principle.

Example 4:

What is the minimum number of students that there are **at least** 9 who were born in the same month.

$r = 9$



$8 * 12 = 96$

$\min N = ?$

$k = 12$

month

The Pigeonhole Principle.

Example 4:

What is the minimum number of students that there are **at least** 9 who were born in the same month.

$r = 9$

$$\begin{aligned} \min N &= k(r-1)+1 \\ &= 12(9-1)+1 \\ &= 12 * 8 + 1 \\ &= 97 \end{aligned}$$

$+$

1

$8 * 12 = 96$

8	8	8	8
8	8	8	8
8	8	8	8

$\min N = 97$

$k = 12$
month

The Pigeonhole Principle.

Example 5:

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

$r = 6$

$k = 5$

The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$.

$$\min N = k(r-1)+1$$
$$\min N = 5(6-1)+1 = 5*5+1 = 26$$

The Pigeonhole Principle.

Example 6:

What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

$r = 100$ $k = 50$

Solution:

The smallest integer N such that $\lceil N/50 \rceil = 100$.

$$\min N = k(r-1)+1$$
$$\min N = 50(100-1)+1 = 50 * 99 + 1 = 4951$$

The Pigeonhole Principle.

Example 7:

A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?

The Pigeonhole Principle.

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A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

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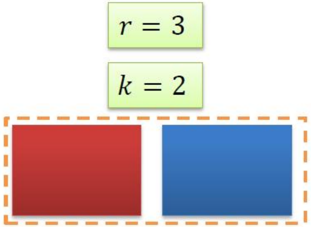
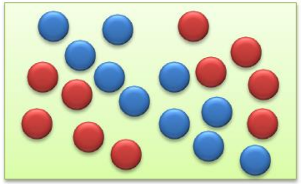




$$\min N = k(r-1)+1 = (2)(3-1)+1 = 5$$

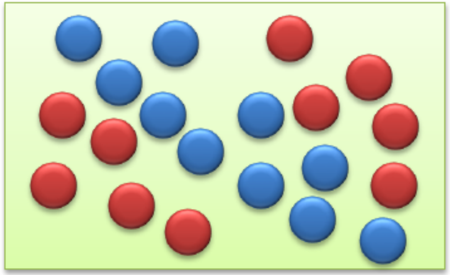
She must select at least 5 balls to be sure of having at least three balls of the same color.

The Pigeonhole Principle.

Example 7:

A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

b) How many balls must she select to be sure of having at least three blue balls?



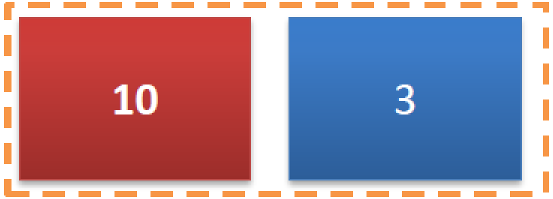
The Pigeonhole Principle.

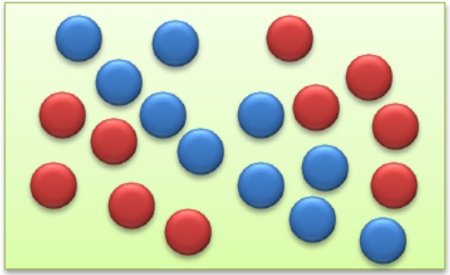
Example 7:

A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

b) How many balls must she select to be sure of having at least three blue balls?

$$N = 10 + 3 = 13$$





She must select at least 13 balls to be sure of having at least three blue balls.

Thank you !