Applied Discrete Mathematics

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Applied Discrete Mathematics
Lecture 3

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Quiz Week 4

(Lecs 1, 2 ,3)

Lab (Time slot)

Matrices

Finding ways to describe many situations in mathematics and economics leads to the study of rectangular arrays of numbers.
Consider, for example, the system of linear equations

$$2x+y-3z = 0,$$
$$8x +5y +4z = 0,$$
$$7x - 9y -8z = 0,$$

can be described by the rectangular array

$$\begin{bmatrix} 2 & 1 & -3 \\ 8 & 5 & 4 \\ 7 & -9 & -8 \end{bmatrix}$$

which is called a matrix

Matrices

Definition:

A ***matrix*** is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix. A matrix with the same number of rows as columns is called ***square***.

$$\text{The matrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} \text{is a } 3 \times 2 \text{ matrix.}$$

Matrices

Note that:

1] in symbolically representing matrices, we shall use capital letters such as: A, B, C, and so on

2] The horizontal are rows, and the vertical are columns

3] Since $A = \begin{bmatrix} 10 & 12 & 16 \\ 5 & 9 & 7 \end{bmatrix}$ has two rows and three columns, we say that A has size 2x3 (read 2 by 3), where the number of rows is specified first.

Matrices

4] An m x n matrix A and if the (i, j) - entry of A is denoted by a_{ij}, and then A is displayed as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

Example(1) : The matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

have sizes 2x3 , 2x2 , and 3x1 , respectively

5] If m = n, we say that A is a square matrix of order n

Matrices

Definition(1) : Row matrix

A matrix of size 1 x n is called a row matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \end{bmatrix}$$

Definition(2) : Column matrix

A matrix of size n x 1 is called a column matrix , $A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix}$

Matrices

Example(2) : Construct a three – elements column matrix such that $a_{21} = 6$ and $a_{ij} = 0$ otherwise

Solution : since $a_{11} = a_{31} = 0$, the matrix is $A = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$

Example(3) : If $A = [\ a_{ij}\ ]$ has size 3 x 4 , and $a_{ij} = i + j$, find A ?

Solution : Here , i = 1, 2, 3 and j = 1, 2, 3, 4 and A has (3)(4) = 12 elements .
Since $a_{ij} = i + j$, the element in row i and column j is obtained by adding the numbers i and j . Hence
$a_{11} = 1+1 = 2$, $a_{12} = 1+2 = 3$, $a_{13} = 1+3 = 4$ and so on . Thus

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

Matrices

Definition(3) : Zero or Null matrix

An m x n matrix , each of whose elements is 0 , is called a zero matrix or a null matrix of size m x n .

Example :

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \end{bmatrix}$$

are null matrices of size 2 x 3 , 3 x 2 , and 1 x 2 respectively .

Matrices

Definition(4): Square matrix

An m x n matrix , for which m = n, we say that A is a square matrix of order n, and that the elements a_{11} , a_{22}, …, a_{nn} form the main diagonal of A

(a square matrix $\underset{m \times m}{\mathbf{A}}$ has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

Definition(5): Diagonal matrix :

A square matrix in which each one of the non-diagonal elements is 0 , is called a diagonal matrix .

Thus , in a diagonal matrix $A = [a_{ij}]$, we have $a_{ij} = 0$ for $i \neq j$ and we express it as

$A = \text{diag}(a_{11}, a_{22}, a_{33}, \ldots, a_{nn})$

Example :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is a diagonal matrix and it may be written as diag (1, 2, 3) .

Matrices

Definition(6): Identity matrix

An identity matrix I is a square matrix with 1's on the main diagonal and zeros elsewhere .

If it is important to stress the size of an n x n identity matrix , we shall denote it by I_n ; however, these matrices are usually written simply as I .

, $$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrices

Definition(7): Upper triangular matrix :
A square matrix in which each element below the main diagonal is 0 , is called an upper triangular matrix .

Example :

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}$$ is an upper triangular matrix

Definition(8): Lower triangular matrix
A square matrix in which each element above the main diagonal is 0 , is called a lower triangular matrix .

Example : $$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$ is a lower triangular matrix

Matrices

Definition(9): Triangular matrix

A triangular matrix is one that is either upper or lower triangular .

Definition(10): Symmetric Matrix.

A square matrix A is said to be symmetric if $A^T = A$.

Operations with Matrices

Definition of Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if they have the same size $(m \times n)$ and

$$a_{ij} = b_{ij}$$

for $1 \le i \le m$ and $1 \le j \le n$.

EXAMPLE 1 Equality of Matrices

Consider the four matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 3 \end{bmatrix}, \qquad \text{and} \qquad D = \begin{bmatrix} 1 & 2 \\ x & 4 \end{bmatrix}.$$

Matrices A and B are **not** equal because they are of different sizes. Similarly, B and C are not equal. Matrices A and D are equal if and only if $x = 3$.

A matrix that has only one column, such as matrix in Example 1, is called a **column matrix** or **column vector.** Similarly, a matrix that has only one row, such as matrix in Example 1, is called a **row matrix** or **row vector.**

Matrices

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times n$, then their **sum** is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different sizes is undefined.

EXAMPLE 2 Addition of Matrices

(a) $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

is undefined.

Matrices

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** of A by c is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

EXAMPLE 3 Scalar Multiplication and Matrix Subtraction

For the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

find (a) $3A$, (b) $-B$, and (c) $3A - B$.

Matrices

EXAMPLE 3 Scalar Multiplication and Matrix Subtraction

For the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

find (a) $3A$, (b) $-B$, and (c) $3A - B$.

SOLUTION

(a) $$3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

(b) $$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

(c) $$3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

Matrices

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the **product** AB is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}.$$

EXAMPLE 4 Finding the Product of Two Matrices

Find the product AB, where

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}.$$

Matrices

First note that the product AB is defined because A has size 3×2 and B has size 2×2. Moreover, the product AB has size 3×2 and will take the form

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

To find c_{11} (the entry in the first row and first column of the product), multiply corresponding entries in the first row of A and the first column of B. That is,

$$c_{11} = (-1)(-3) + (3)(-4) = -9$$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

Matrices

The product is

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}.$$

Be sure you understand that for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix. That is,

$$\underset{m \times n}{A} \qquad \underset{n \times p}{B} \qquad = \qquad \underset{m \times p}{AB}.$$

equal

size of AB

Matrices

$$\underset{m \times \mathbf{n}}{A} \qquad \underset{\mathbf{n} \times p}{B} \qquad = \qquad \underset{m \times p}{C}$$

Must be the same

Size of product

Example (11): If $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$

Find AB . Does BA exist ? Why?

Solution : Since , A is a 3×2 matrix and B is a 2×2 matrix ,

so AB exist and it is 3×2 matrix

Matrices

$$AB = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} (1)(-1)+(-1)(3) & (1)(2)+(-1)(1) \\ (3)(-1)+(4)(3) & (3)(2)+(4)(1) \\ (-1)(-1)+(5)(3) & (-1)(2)+(5)(1) \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 9 & 10 \\ 16 & 3 \end{bmatrix}$$

Then ,

$$AB = \begin{bmatrix} -4 & 1 \\ 9 & 10 \\ 16 & 3 \end{bmatrix}$$

Again , B is a 2 x 2 matrix , and A is a 3 x 2 matrix , so the number of columns in B is not equal to the number of rows in A . Hence , BA does not exist .

Matrices

EXAMPLE 5 **Matrix Multiplication**

$$\text{(a)}\ \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}}$$

$$\text{(b)}\ \underset{2\times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}$$

$$\text{(c)}\ \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\text{(d)}\ \underset{1\times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix} 1 \end{bmatrix}}$$

Matrices

Example:

$$\mathbf{A}_{3\times 3} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & -1 \end{bmatrix}_{3\times 3} \qquad \mathbf{M}_{3\times 2} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix}_{3\times 2}$$

$$\mathbf{A}_{3\times 3} \times \mathbf{M}_{3\times 2} = \mathbf{B}_{3\times 2}$$

$$\begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & -1 \end{bmatrix} \times \begin{array}{cc} \boxed{1} & \boxed{2} \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix}} \end{array} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Matrices

$$\mathbf{A}_{3\times3} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & -1 \end{bmatrix}_{3\times3} \qquad \mathbf{M}_{3\times2} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix}_{3\times2}$$

$$\mathbf{A}_{3\times3} \times \mathbf{M}_{3\times2} = \mathbf{B}_{3\times2}$$

$$\boldsymbol{a}_{11} = \mathbf{6}$$
$$= (\mathbf{1} \times \mathbf{1} + \mathbf{1} \times \mathbf{3} + \mathbf{2} \times \mathbf{1})$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & -1 \end{bmatrix} \times \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 6 & 3 \\ 10 & 3 \\ 9 & -2 \end{bmatrix}$$

Matrices

$$\mathbf{A}_{3\times3} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & -1 \end{bmatrix}_{3\times3} \qquad \mathbf{M}_{3\times2} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix}_{3\times2}$$

$$\mathbf{A}_{3\times3} \times \mathbf{M}_{3\times2} = \mathbf{B}_{3\times2}$$

$$\boldsymbol{a_{31} = 9}$$
$$\mathbf{= (1 \times 1 + 3 \times 3 + (-1) \times 1)}$$

$$\begin{array}{c c} \boxed{1} & \\ \boxed{2} & \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & -1 \end{bmatrix} \\ \boxed{3} & \end{array} \times \begin{array}{c} \begin{array}{cc} \boxed{1} & \boxed{2} \end{array} \\ \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 6 & 3 \\ 10 & 3 \\ 9 & -2 \end{bmatrix}$$

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Does $\mathbf{AB} = \mathbf{BA}$?

Matrices

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution: We find that

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}.$$

Hence, $\mathbf{AB} \neq \mathbf{BA}$.

Properties of Matrix Operations

THEOREM 2.1
Properties of Matrix Addition and Scalar Multiplication

If A, B, and C are $m \times n$ matrices and c and d are scalars, then the following properties are true.

1. $A + B = B + A$	Commutative property of addition
2. $A + (B + C) = (A + B) + C$	Associative property of addition
3. $(cd)A = c(dA)$	Associative property of multiplication
4. $1A = A$	Multiplicative identity
5. $c(A + B) = cA + cB$	Distributive property
6. $(c + d)A = cA + dA$	Distributive property

Matrices

THEOREM 2.2 Properties of Zero Matrices

If A is an $m \times n$ matrix and c is a scalar, then the following properties are true.

1. $A + O_{mn} = A$
2. $A + (-A) = O_{mn}$
3. If $cA = O_{mn}$, then $c = 0$ or $A = O_{mn}$.

The matrix O_{mn} is called a **zero matrix,** and it serves as the **additive identity** for the set of all m x n matrices.

$$O_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrices

THEOREM 2.3

Properties of Matrix Multiplication

If A, B, and C are matrices (with sizes such that the given matrix products are defined) and c is a scalar, then the following properties are true.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $c(AB) = (cA)B = A(cB)$

Matrices

EXAMPLE 3 **Matrix Multiplication Is Associative**

Find the matrix product ABC by grouping the factors first as $(AB)C$ and then as $A(BC)$. Show that the same result is obtained from both processes.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

SOLUTION Grouping the factors as $(AB)C$, you have

$$(AB)C = \left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix} \right) \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}.$$

Grouping the factors as $A(BC)$, you obtain the same result.

$$A(BC) = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}$$

Matrices

THEOREM 2.4
Properties of the Identity Matrix

If A is a matrix of size $m \times n$, then the following properties are true.

1. $AI_n = A$
2. $I_mA = A$

As a special case of this theorem, note that if A is a *square* matrix of order n, then

$$AI_n = I_nA = A.$$

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1×1 $\quad 2 \times 2$ $\quad 3 \times 3$

EXAMPLE 6 Multiplication by an Identity Matrix

(a) $$\begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

Matrices

Powers of square matrices (A^r)

When $\mathbf{A}$ is an $n \times n$ matrix, we have

$$\mathbf{A}^0 = \mathbf{I}_n, \qquad \mathbf{A}^r = \underbrace{\mathbf{A}\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{r \text{ times}}.$$

EXAMPLE 7 Repeated Multiplication of a Square Matrix

Find A^3 for the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$.

SOLUTION

$$A^3 = \left(\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -6 \end{bmatrix}$$

Matrices

The Transpose of a Matrix

The **transpose** of a matrix is formed by writing its columns as rows. For instance, if A is the $m \times n$ matrix shown by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix},$$

Size: $m \times n$

then the transpose, denoted by A^T, is the $n \times m$ matrix below

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

Size: $n \times m$

Transpose of A (A^t)

Interchanging the rows and columns of **A**

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

A $\qquad$ $\mathbf{A^t}$

Matrices

EXAMPLE 8 **The Transpose of a Matrix**

Find the transpose of each matrix.

(a) $A = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$

SOLUTION (a) $A^T = \begin{bmatrix} 2 & 8 \end{bmatrix}$ (b) $B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ (c) $C^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $D^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$

Matrices

THEOREM 2.6 Properties of Transposes

If A and B are matrices (with sizes such that the given matrix operations are defined) and c is a scalar, then the following properties are true.

1. $(A^T)^T = A$ — Transpose of a transpose
2. $(A + B)^T = A^T + B^T$ — Transpose of a sum
3. $(cA)^T = c(A^T)$ — Transpose of a scalar multiple
4. $(AB)^T = B^T A^T$ — Transpose of a product

EXAMPLE 9 Finding the Transpose of a Product

Show that $(AB)^T$ and B^TA^T are equal.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

SOLUTION

$$AB = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$(AB)^T = B^TA^T$$

Matrices

EXAMPLE 10 **The Product of a Matrix and Its Transpose**

For the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}$$

find the product AA^T and show that it is symmetric.

SOLUTION Because

$$AA^T = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 4 & 2 \\ -5 & 2 & 5 \end{bmatrix}$$

it follows that $AA^T = (AA^T)^T$, so AA^T is symmetric.

The Inverse of a Matrix

Definition of the Inverse of a Matrix

An $n \times n$ matrix A is **invertible** (or **nonsingular**) if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is called the (multiplicative) **inverse** of A. A matrix that does not have an inverse is called **noninvertible** (or **singular**).

Nonsquare matrices do not have inverses

The inverse of A is denoted by A^{-1}.

EXAMPLE 1 The Inverse of a Matrix

Show that B is the inverse of A, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

SOLUTION Using the definition of an inverse matrix, you can show that B is the inverse of A by showing that $AB = I = BA$, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finding the Inverse of a Matrix by Gauss-Jordan Elimination

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \vdots I]$. Note that you separate the matrices A and I by a dotted line. This process is called **adjoining** matrix I to matrix A.
2. If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \vdots I]$. The result will be the matrix $[I \vdots A^{-1}]$. If this is not possible, then A is noninvertible (or singular).
3. Check your work by multiplying AA^{-1} and $A^{-1}A$ to see that $AA^{-1} = I = A^{-1}A$.

EXAMPLE 3 Finding the Inverse of a Matrix

Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

Matrices

SOLUTION Begin by adjoining the identity matrix to A to form the matrix

$$[A \ \vdots \ I] = \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right].$$

Now, using elementary row operations, rewrite this matrix in the form $[I \ \vdots \ A^{-1}]$, as follows.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \qquad R_2 + (-1)R_1 \rightarrow R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \qquad R_3 + (6)R_1 \rightarrow R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \qquad R_3 + (4)R_2 \rightarrow R_3$$

Matrices

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad (-1)R_3 \to R_3$$

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad R_2 + R_3 \to R_2$$

$$\left[\begin{array}{rrr:rrr} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad R_1 + R_2 \to R_1$$

The matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

Try confirming this by showing that $AA^{-1} = I = A^{-1}A$.

EXAMPLE 4 A Singular Matrix

Show that the matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

Matrices

SOLUTION Adjoin the identity matrix to A to form

$$[A \,\vdots\, I] = \left[\begin{array}{rrr:rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

and apply Gauss-Jordan elimination as follows.

$$\left[\begin{array}{rrr:rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \qquad R_2 + (-3)R_1 \rightarrow R_2$$

$$\left[\begin{array}{rrr:rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right] \qquad R_3 + (2)R_1 \rightarrow R_3$$

Now, notice that adding the second row to the third row produces a row of zeros on the left side of the matrix.

$$\left[\begin{array}{rrr:rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \qquad R_3 + R_2 \rightarrow R_3$$

Because the "A portion" of the matrix has a row of zeros, you can conclude that it is not possible to rewrite the matrix $[A \,\vdots\, I]$ in the form $[I \,\vdots\, A^{-1}]$. This means that A has no inverse, or is noninvertible (or singular).

Matrices

THEOREM 2.8
Properties of Inverse Matrices

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then A^{-1}, A^k, cA, and A^T are invertible and the following are true.

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = A^{-1}A^{-1} \cdot \cdot \cdot A^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}, c \neq 0$
4. $(A^T)^{-1} = (A^{-1})^T$

Matrices

THEOREM 2.9 The Inverse of a Product

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

EXAMPLE 7 Finding the Inverse of a Matrix Product

Find $(AB)^{-1}$ for the matrices

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

Matrices

using the fact that A^{-1} and B^{-1} are represented by

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}.$$

SOLUTION Using Theorem 2.9 produces

$$(AB)^{-1} = B^{-1}A^{-1} = \overset{B^{-1}}{\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 8 & -5 & -2 \\ -8 & 4 & 3 \\ 5 & -2 & -\frac{7}{3} \end{bmatrix}.$$

Matrices

Inverse using **Determinant of a Matrix**

2x2 and 3x3

Report: Example of each case.

Boolean Matrix Operations

Zero–One Matrices

A matrix all of whose entries are either **0** or **1**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

join and *meet* (Zero–One Matrices)

meet

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

join

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example (1/3)

Find the join and meet of the zero–one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Example (1/3)

Find the join and meet of the zero–one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: We find that the join of **A** and **B** is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Example (1/3)

Find the join and meet of the zero–one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution:

The meet of **A** and **B** is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Matrices

Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero–one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero–one matrix. Then the *Boolean product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ matrix with (i, j)th entry c_{ij} where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Matrices

EXAMPLE 8 Find the Boolean product of $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: The Boolean product $\mathbf{A} \odot \mathbf{B}$ is given by

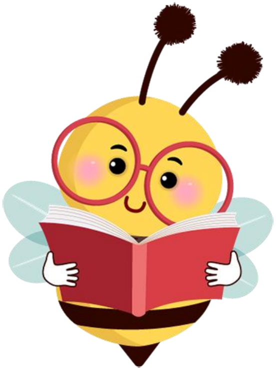
$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The Foundations: Logic and Proofs

Thank you !



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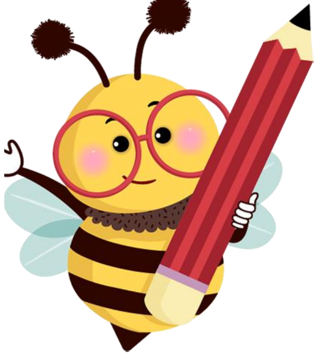

$\longrightarrow$