MS&E 321 Homework 2 Answer

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1

1.1

For $x \geq 0$, we have

$$\begin{aligned}
&\mathbf{P}\left(S_{\tau(b)} - b \geq x | \tau(b) = n\right) \\
=&\mathbf{P}\left(S_n - b \geq x | \tau(b) = n\right) \\
=&\mathbf{P}\left(S_n - b \geq x | S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right) \\
=&\mathbb{E}\left(\mathbb{1}\left\{S_n - b \geq x\right\} | S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right) \\
=&\mathbb{E}\left(\mathbb{E}\left(\mathbb{1}\left\{S_n - b \geq x\right\} | S_1, S_2, \ldots, S_{n-1}, (S_1, S_2, \ldots, S_{n-1}) < b, S_n \geq b\right) | S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right).
\end{aligned}$$

Note that

$$\mathbb{E}\left(\mathbb{1}\left\{S_n - b \geq x\right\} | S_1, S_2, \ldots, S_{n-1}, (S_1, S_2, \ldots, S_{n-1}) < b, S_n \geq b\right) = e^{-\beta x},$$

so we have

$$\begin{aligned}
&\mathbf{P}\left(S_{\tau(b)} - b \geq x | \tau(b) = n\right) \\
=&\mathbb{E}\left(\mathbb{E}\left(\mathbb{1}\left\{S_n - b \geq x\right\} | S_1, S_2, \ldots, S_{n-1}, (S_1, S_2, \ldots, S_{n-1}) < b, S_n \geq b\right) | S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right) \\
=&\mathbb{E}\left(\mathbb{E}\left(e^{-\beta x} | S_1, S_2, \ldots, S_{n-1}, S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right) | S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right) \\
=&e^{-\beta x},
\end{aligned}$$

which means $S_{\tau(b)} - b$ given $\tau(b) = n$ is exponential.

1.2

Define $f(x) = \log \mathbb{E}\left(\exp(xY_1)\right)$, which is continuous on $x \in [0, b)$. Since $f(0) = 0$ and $f(b^-) = +\infty$, there exists $\gamma(\theta)$ such that

$$\log \mathbb{E}\left(\exp(\gamma(\theta)Y_1)\right) = \theta.$$

Consider Wald's Martingale

$$M_n(\gamma(\theta)) = \exp\left(\gamma(\theta)S_n - n\theta\right).$$

Let $\alpha(b,m) = \min\{\tau(b), m\}$. Since $\alpha(b,m)$ is bounded, we have

$$\mathbb{E}\left(M_{\alpha(b,m)}(\gamma(\theta))\right) = \mathbb{E}\left(M_0(\gamma(\theta))\right) = 1.$$

We also have

$$\begin{aligned}
&\mathbb{E}\left(M_{\alpha(b,m)}(\gamma(\theta))\right)\\
&\mathbb{E}\left(\exp\left(\gamma(\theta)S_{\alpha(b,m)} - \alpha(b,m)\theta\right)\right)\\
&\mathbb{E}\left(\exp\left(\gamma(\theta)S_{\tau(b)} - \tau(b)\theta\right)\mathbb{1}\left\{\tau(b) < m\right\}\right) + \mathbb{E}\left(\exp\left(\gamma(\theta)S_m - m\theta\right)\mathbb{1}\left\{\tau(b) \geq m\right\}\right)
\end{aligned}$$

Note that

$$\begin{aligned}
0 &\leq \mathbb{E}\left(\exp\left(\gamma(\theta)S_m - m\theta\right)\mathbb{1}\left\{\tau(b) \geq m\right\}\right)\\
&\leq \mathbb{E}\left(\exp\left(\gamma(\theta)b - m\theta\right)\right),
\end{aligned}$$

which goes to 0 as $m \to \infty$.

So we have

$$\begin{aligned}
1 &= \lim_{m\to\infty} \mathbb{E}\left(M_{\alpha(b,m)}(\gamma(\theta))\right)\\
&= \lim_{m\to\infty} \mathbb{E}\left(\exp\left(\gamma(\theta)S_{\tau(b)} - \tau(b)\theta\right)\mathbb{1}\left\{\tau(b) < m\right\}\right)\\
&= \sum_{i=1}^{\infty}\left(\mathbf{P}\left(\tau(b) = i\right)\mathbb{E}\left(\exp\left(\gamma(\theta)S_{\tau(b)} - \tau(b)\theta\right)|\tau(b) = i\right)\right)\\
&= \sum_{i=1}^{\infty}\left(\mathbf{P}\left(\tau(b) = i\right)e^{-i\theta}\mathbb{E}\left(\exp\left(\gamma(\theta)S_{\tau(b)}\right)|\tau(b) = i\right)\right)\\
&= \sum_{i=1}^{\infty}\left(\mathbf{P}\left(\tau(b) = i\right)e^{-i\theta}e^{\gamma(\theta)b}\frac{\beta}{\beta - \gamma(\theta)}\right)\\
&= e^{\gamma(\theta)b}\frac{\beta}{\beta - \gamma(\theta)} \cdot \sum_{i=1}^{\infty}\left(\mathbf{P}\left(\tau(b) = i\right)e^{-i\theta}\right)\\
&= e^{\gamma(\theta)b}\frac{\beta}{\beta - \gamma(\theta)} \cdot \mathbb{E}\left(\exp\left(-\theta\tau(b)\right)\mathbb{1}\left\{\tau(b) < \infty\right\}\right).
\end{aligned}$$

As a result,

$$\mathbb{E}\left(\exp\left(-\theta\tau(b)\right)\mathbb{1}\left\{\tau(b) < \infty\right\}\right) = \frac{\beta - \gamma(\theta)}{\beta e^{\gamma(\theta)b}}.$$

Furthermore, we have

$$\begin{aligned}
1 &= \lim_{\theta\to 0^+} \mathbb{E}\left(\exp\left(-\theta\tau(b)\right)\mathbb{1}\left\{\tau(b) < \infty\right\}\right)\\
&=\mathbb{E}\left(\mathbb{1}\left\{\tau(b) < \infty\right\}\right)\\
&=\mathbf{P}\left(\tau(b) < \infty\right).
\end{aligned}$$

2

2.1

3

3.1