MS&E 321 Homework 2 Answer

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1.1

For $x \geq 0$, we have

$$\begin{aligned}
&\mathbf{P}\left(S_{\tau(b)} - b \geq x | \tau(b) = n\right) \\
=&\mathbf{P}\left(S_n - b \geq x | \tau(b) = n\right) \\
=&\mathbf{P}\left(S_n - b \geq x | S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right) \\
=&\mathbb{E}\left(\mathbb{1}\left\{S_n - b \geq x\right\} | S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right) \\
=&\mathbb{E}\left(\mathbb{E}\left(\mathbb{1}\left\{S_n - b \geq x\right\} | S_1, S_2, \ldots, S_{n-1}, (S_1, S_2, \ldots, S_{n-1}) < b, S_n \geq b\right) | S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right).
\end{aligned}$$

Note that

$$\mathbb{E}\left(\mathbb{1}\left\{S_n - b \geq x\right\} | S_1, S_2, \ldots, S_{n-1}, (S_1, S_2, \ldots, S_{n-1}) < b, S_n \geq b\right) = e^{-\beta x},$$

so we have

$$\begin{aligned}
&\mathbf{P}\left(S_{\tau(b)} - b \geq x | \tau(b) = n\right) \\
=&\mathbb{E}\left(\mathbb{E}\left(\mathbb{1}\left\{S_n - b \geq x\right\} | S_1, S_2, \ldots, S_{n-1}, (S_1, S_2, \ldots, S_{n-1}) < b, S_n \geq b\right) | S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right) \\
=&\mathbb{E}\left(\mathbb{E}\left(e^{-\beta x} | S_1, S_2, \ldots, S_{n-1}, S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right) | S_1, S_2, \ldots, S_{n-1} < b, S_n \geq b\right) \\
=&e^{-\beta x},
\end{aligned}$$

which means $S_{\tau(b)} - b$ given $\tau(b) = n$ is exponential.

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2.1

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3.1