Explaining Mathematical Proofs to Computers

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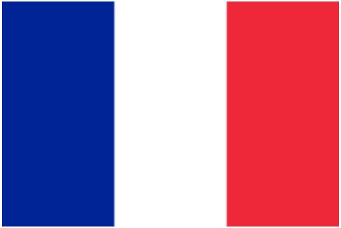
What is `Lean`?



Un assistant de preuve

A theorem prover

```lean
/-
The Hahn-Banach theorem about extending linear functionals to a field `𝕜 = ℝ` or `𝕜 = ℂ`: -/

theorem Hahn_Banach [complete_space V] (p : subspace 𝕜 V)
(f : p →L[𝕜] 𝕜) : ∃ g : V →L[𝕜] 𝕜, (∀ x : p, g x = f x) ∧ ‖g‖ = ‖f‖ :=
begin
  letI : module ℝ V := restrict_scalars.module ℝ 𝕜 V,
  letI : is_scalar_tower ℝ 𝕜 V := restrict_scalars.is_scalar_tower _ _ _,
  letI : semi_normed_space ℝ V := semi_normed_space.restrict_scalars _ 𝕜 _,
  -- Let `fr: p →L[ℝ] ℝ` be the real part of `f`.
  let fr := re_clm.comp (f.restrict_scalars ℝ),
  have fr_apply : ∀ x, fr x = re (f x), by { assume x, refl },
  -- Use the real version to get a norm-preserving extension of `fr`, which
  -- we'll call `g : V →L[ℝ] ℝ`.
  rcases real.exists_extension_norm_eq (p.restrict_scalars ℝ) fr with ⟨g, ⟨hextends, hnormeq⟩⟩,
  -- **Now `g` can be extended to the `V →L[𝕜] 𝕜` we need.**
  refine ⟨g.extend_to_𝕜, _⟩,
  -- It is an extension of `f`.
  have h : ∀ x : p, g.extend_to_𝕜 x = f x,
  { assume x,
```

```
▼ talk.lean:21:5
▼ Tactic state                     wid
1 goal                filter: no filte
𝕜 : Type
_inst_1 : is_R_or_C 𝕜
V : Type u_1
_inst_2 : normed_group V
_inst_3 : normed_space 𝕜 V
_inst_4 : complete_space V
p : subspace 𝕜 V
f : ↥p →L[𝕜] 𝕜
⊢
∃ (g : V →L[𝕜] 𝕜),
  (∀ (x : ↥p), ⇑g ↑x = ⇑f x)
   ‖g‖ = ‖f‖
▶ All Messages (3)
```

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Quasi-projectivity of moduli spaces of polarized varieties

Pages 597-639 from Volume 159 (2004), Issue 2 by *Georg Schumacher, Hajime Tsuji*

Abstract

By means of analytic methods the quasi-projectivity of the moduli space of algebraically polarized varieties with a not necessarily reduced complex structure is proven including the case of nonuniruled polarized varieties.

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To appear in

Non-quasi-projective moduli spaces

Pages 1077-1096 from Volume 164 (2006), Issue 3 by *János Kollár*

Abstract

We show that every smooth toric variety (and many other algebraic spaces as well) can be realized as a moduli space for smooth, projective, polarized varieties. Some of these are not quasi-projective. This contradicts a recent paper (Quasi-projectivity of moduli spaces of polarized varieties, *Ann. of Math.* **159** (2004) 597–639.).

Let’s do some Lean

These two systems (Principia Mathematica and Zermelo–Fraenkel) are so comprehensive that in them all methods of proof used today in mathematics are formalized, that is, reduced to a few axioms and rules of inference. One might therefore conjecture that these axioms and rules of inference are sufficient to decide any mathematical question that can at all be formally expressed in these systems. ”

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(K. Gödel, *On formally undecidable propositions of Principia Mathematica and related systems*, 1931.)

Liquid tensor experiment

Posted[1]*, on December* 5^{th}*, 2020*

I [Peter Scholze] want to propose a challenge: Formalise the proof of the following theorem.

Theorem (Clausen–S.)

Let $0 < p' < p \leq 1$ *be real numbers, let* S *be a profinite set, and let* V *be a* p*-Banach space. Let* $\mathcal{M}_{p'}(S)$ *be the space of* p'*-measures on* S*. Then*

$$\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geq 1$*.*

[1] On the Xena Project blog, see https://xenaproject.wordpress.com/2020/12/05/liquid-tensor-experiment/

Liquid tensor experiment

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Why do I want a formalization?

[1] On the Xena Project blog, see `https://xenaproject.wordpress.com/2020/12/05/liquid-tensor-experiment/`

Liquid tensor experiment

Posted, on December 5^{th}, 2020

- ...
- [...] In the end, we were able to get an argument pinned down on paper, but I think nobody else has dared to look at the details of this, and so I still have some small lingering doubts.
- [...] I think the theorem is of utmost foundational importance, so being 99.9% sure is not enough.
- I have occasionally been able to be very persuasive even with wrong arguments. [1]
- I think this may be my most important theorem to date [...]. Better be sure it's correct...

[1] Fun fact: In the selection exams for the international math olympiad, twice I got full points for a wrong solution. Later, I once had a full proof of the weight-monodromy conjecture that passed the judgment of some top mathematicians, but then it turned out to contain a fatal mistake.

Liquid tensor experiment

Posted[2] on June 5, 2021

Exactly half a year ago I wrote the Liquid Tensor Experiment blog post, challenging the formalization of a difficult foundational theorem from my Analytic Geometry lecture notes on joint work with Dustin Clausen. While this challenge has not been completed yet, I am excited to announce that the Experiment has verified the entire part of the argument that I was unsure about. I find it absolutely insane that interactive proof assistants are now at the level that within a very reasonable time span they can formally verify difficult original research. Congratulations to everyone involved in the formalization!!

[2] Voir `https://xenaproject.wordpress.com/2021/06/05/half-a-year-of-the-liquid-tensor-experiment-amazing-developments/`