问题：在 120 枚外观相同的硬币中，有一枚是假币，并且已知假 币与真币的重量不同，但不知道假币与真币相比较轻还是较重。可以 通过一架天平来任意比较两组硬币，最坏情况下，能不能只比较 5 次就检测出这枚假币？

答案:

在不知道假币是重还是轻的情况下,根据表格数据可知,题目满足条件A:假币数量大于1,条件H:我们不需要找出假币比真币轻还是比真币重,条件E:没有给额外的真币。

因此,在前人的基础上可知,只能检测5次,就得到假币的总硬币数量上限为:

$$\frac{(3^5-1)}{2}=121(\text{枚})>120(\text{枚})$$

		E (no additional genuine coins)	F/G (we are given additional genuine coins)
A (we know that the counterfeit coin exists) **The problem makes sense only if $n\geq1$**	H (we do not have to find out if the counterfeit coin is heavier or lighter than the genuine ones)	$n\leq(3^k-1)/2$ $n\neq2$	$n\leq(3^k+1)/2$
	I (we have to find out if the counterfeit coin is heavier or lighter than the genuine ones)	$n\leq(3^k-3)/2$ $n\neq1$ $n\neq2$	$n\leq(3^k-1)/2$
B (it is possible that counterfeit coin does not exist)		$n\leq(3^k-3)/2$ or $n=0$ $n\neq1$ $n\neq2$	$n\leq(3^k-1)/2$

Let us turn to variant BDEIJ. For $\mathbf{w}=(w_1,w_2,\ldots,w_k)\in\{R,L,0\}^k$ we denote by $-\mathbf{w}$ the k-tuple with all L replaced by R and vice versa, e.g $-(L,0,R,R,L)=(R,0,L,L,R)$. We will call $-\mathbf{w}$ the inverse of $\mathbf{w}$. Moreover, for $A\subset\{R,L,0\}^k$ we denote by $-A$ the set $\{-\mathbf{w}: \mathbf{w}\in A\}$. Assume that for all k, n satisfying the condition from the table there exists an n-element set $\Gamma^n_k\subset\{R,L,0\}^k$ such that $(0,0,\ldots,0)\notin\Gamma^n_k$, $\Gamma^n_k\cap(-\Gamma^n_k)=\varnothing$ and for all $j\in\{1,2,\ldots,k\}$ the sets $\{(w_1,w_2,\ldots,w_k)\in\Gamma^n_k: w_j=R\}$ and $\{(w_1,w_2,\ldots,w_k)\in\Gamma^n_k: w_j=L\}$ have the same number of elements. We assign to each coin a different element from Γ^n_k. Notice that the sequence of k weightings planned in this way allows us to indicate the counterfeit coin (if exists) and to determine whether it is heavier or lighter. In fact, let us write the results of the k weightings in the form $(w_1,w_2,\ldots,w_k)$, with $w_j=R$, $w_j=L$, $w_j=0$, as before. If all coins are fair we get $(0,0,\ldots,0)$. If not and the counterfeit coin is heavier, we get a k-tuple assigned to it when planning the weightings. If the false coin is lighter, we get the inverse of the same tuple. As $\Gamma^n_k\cap(-\Gamma^n_k)=\varnothing$, we get a different result in each of those situations, which allows to indicate the counterfeit coin and determine whether it is heavier or lighter. It remains to prove that

For any $k\geq0$ and n such that $3\leq n\leq(3^k-3)/2$ or $n=0$ there exists an n-element set $\Gamma^n_k\subset\{L,R,0\}^k$ such that $\Gamma^n_k\cap(-\Gamma^n_k)=\varnothing$ and for all $j\in\{1,2,\ldots,k\}$ the sets $\{(w_1,w_2,\ldots,w_k)\in\Gamma^n_k: w_j=R\}$ and $\{(w_1,w_2,\ldots,w_k)\in\Gamma^n_k: w_j=L\}$ have the same number of elements.

The sets Γ^n_k will be constructed using induction on k. First we consider some special cases. Let $\Gamma^0_k=\varnothing$ for every k.

For every k≥2 we set: Γ^3_k={(L,R,0,…,0), (R,0,0,…,0), (0,L,0,…,0)}.

For every k≥3 we set:

Γ^4_k={(R,R,L,0,…,0), (R,L,P,0,…,0), (L,R,R,0,…,0), (L,L,L,0,…,0)}

Γ^5_k={(0,0,R,0,…,0), (P,0,0,0,…,0), (L,0,R,0,…,0), (0,L,L,0,…,0), (0,R,L,0,…,0)}

Γ^6_k={(L,R,L,0,…,0), (R,0,L,0,…,0), (0,L,L,0,…,0), (L,R,R,0,…,0), (R,0,R,0,…,0), (0,L,R,0,…,0)}

Γ^7_k={(L,R,0,0,…,0), (R,0,0,0,…,0), (0,L,0,0,…,0), (R,R,L,0,…,0), (R,L,R,0,…,0), (L,R,R,0,…,0),(L,L,L,0,…,0)}

Γ^8_k={(L,L,L,0,…,0), (L,R,L,0,…,0), (R,L,L,0,…,0), (R,R,L,0,…,0), (0,R,R,0,…,0), (0,L,R,0,…,0), (R,0,R,0,…,0), (L,0,R,0,…,0)}

The sets $\Gamma_k^{(3^k-3)/2}$ will be defined using induction on k≥1. The set $\Gamma^0_1=\varnothing$ is already constructed. Let k>1. We set

$$\Gamma_k^{(3^k-3)/2} = \Gamma_{k-1}^{(3^{k-1}-3)/2} \times \{L,R,0\} \cup \{(L,L,\dots,L,R),\ (R,R,\dots,R,0),(0,0,\dots,0,L)\}$$

The constructed above sets $\Gamma_k^{(3^k-3)/2}$ satisfy the following conditions:

1° (R,…,R), (L,…,L)$\notin \Gamma_k^{(3^k-3)/2}$ for k≥0,

2° (L,R,0,0,…,0), (R,R,0,0,…,0), (0,L,0,0,…,0), (L,…,L,R,0), (0,…,0,0,L), (0,…,0,L,0)$\in \Gamma_k^{(3^k-3)/2}$ for k≥3,

3° (0,R,0,0,…,0), (0,0,…,0,0,R,0)$\notin \Gamma_k^{(3^k-3)/2}$ for k≥3.

These conditions can be easily proved by induction. The conditions 1°, 2°, 3° imply that we may define the sets $\Gamma_k^{(3^k-5)/2}$ and $\Gamma_k^{(3^k-7)/2}$ by the following formulas:

$$\Gamma_k^{(3^k-5)/2} = \Gamma_k^{(3^k-3)} \setminus \{(L,\dots,L,R,0),(0,\dots,0,0,L),(0,\dots,0,L,0)\} \cup \{(L,\dots,L,L,L),(0,\dots,0,R,0)\}$$
$$\Gamma_k^{(3^k-7)/2} = \Gamma_k^{(3^k-3)} \setminus \{(L,R,0,\dots,0),(R,R,0,\dots,0),(0,L,0,\dots,0)\} \cup \{(0,R,0,\dots,0)\}.$$

Now we may pass to the main part of the construction. We use induction on k. The sets Γ^n_k, where k=0, k=1 and k=2, are already constructed. Let k≥3 and $0\le n\le(3^k-3)/2$, n≠1,2.

If n<9 or $n=(3^k-3)/2$ or $n=(3^k-5)/2$ or $n=(3^k-7)/2$ then sets Γ^n_k have already been determined, so we can assume $9\le n\le(3^k-9)/2$.

Number n can be represented as $n=n_1+2n_2$, $3\le n_1,n_2\le(3^{k-1}-3)/2$, where n_1 and n_2 may be set as:

n_1= n/3	and	n_2=n/3	if n≡0 (mod 3),
n_1=(n-1)/3 + 1	and	n_2=(n-1)/3	if n≡1 (mod 3),
n_1=(n-2)/3	and	n_2=(n-2)/3 +1	if n≡2 (mod 3).

Finally we define $\Gamma^n_k = \Gamma^{n_2}_{k-1} \times \{L,R\} \cup \Gamma^{n_1}_{k-1} \times \{0\}$ ■

以40枚为例,0,L,R分别代表三进制的三个数字代号,可以计算出

$$40(\text{枚}) \le \frac{(3^4-1)}{2} = 40(\text{枚}),\text{即}4\text{次测量的最大上限}$$
$$\Gamma_4^{39} = \Gamma_3^{12}(L,R,0) \cup ((L,L,L,R),(R,R,R,0),(0,0,0,L))$$
$$\Gamma_3^{12} = \Gamma_2^3 \times (L,R,0) \cup ((L,L,R),(R,R,0),(0,0,L))$$
$$\Gamma_2^3 = (L,R),(R,0),(0,L)$$

$$\begin{aligned}\Gamma_4^{39} = &(L,R,L,L),(R,0,L,L),(0,L,L,L),(L,R,R,L),(R,0,R,L),\\ &(0,L,R,L),(L,R,0,L),(R,0,0,L),(0,L,0,L),(L,L,R,L),(R,R,0,L),\\ &(0,0,L,L),(L,R,L,R),(R,0,L,R),(0,L,L,R),(L,R,R,R),(R,0,R,R),\\ &(0,L,R,R),(L,R,0,R),(R,0,0,R),(0,L,0,R),(L,L,R,R),(R,R,0,R),\\ &(0,0,L,R),(L,R,L,0),(R,0,L,0),(0,L,L,0),(L,R,R,0),(R,0,R,0),\\ &(0,L,R,0),(L,R,0,0),(R,0,0,0),(0,L,0,0),(L,L,R,0),(R,R,0,0),\\ &(0,0,L,0),(L,L,L,R),(R,R,R,0),(0,0,0,L)\end{aligned} \quad .$$

给每枚硬币标上序号,1,2,....39,我们精确地映射 Γ_4^{39} 的每一个元素,例如,我们将(L,R,L,L)赋值给硬币1,(R,0,L,L)赋值给硬币2之类的。(0,0,0,L)赋值给硬币39。我们指定(L,L,L,L)为硬币40，最后，(R,R,R,R)为其中一枚真币。通过这种方式，所有的权重都对应如下(0对应于额外的硬币):。

	左侧天平	右侧天平
The 1st weighting	1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40	0, 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38
The 2nd weighting	3, 6, 9, 10, 15, 18, 21, 22, 27, 30, 33, 34, 37, 40	0, 1, 4, 7, 11, 13, 16, 19, 23, 25, 28, 31, 35, 38
The 3rd weighting	1, 2, 3, 12, 13, 14, 15, 24, 25, 26, 27, 36, 37, 40	0, 4, 5, 6, 10, 16, 17, 18, 22, 28, 29, 30, 34, 38
The 4th weighting	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 39, 40	0, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 37

我们将加权结果编码为四组(w1、wz、w3、w4),便知道四次之后称量出来的假币(如果它是较重的)或其逆(如果它是轻于真币)。[1]

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1. Kołodziejczyk, Marcel. (2008). Two-pan balance and generalized counterfeit coin problem. ↩