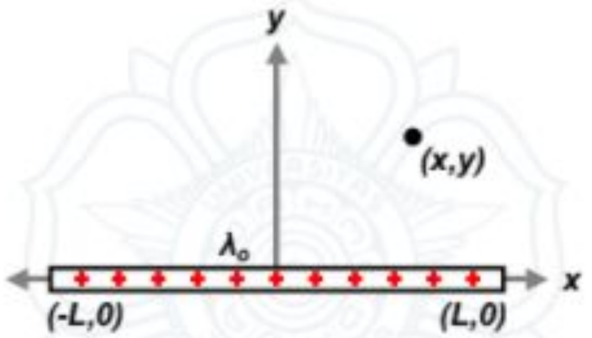
Pembahasan HW #3
Gauss’s Law

Sudiro

Problem #1 : Electric Field due to a Long Charged Rod



Consider a rod with the length $2L$ as shown in the figure above. The charge in this rod is uniformly distributed so that the charge density per unit length is given by :

$$\lambda(x) = \lambda_o$$

Suppose that the electric field at point (x, y) due to this rod is given by :

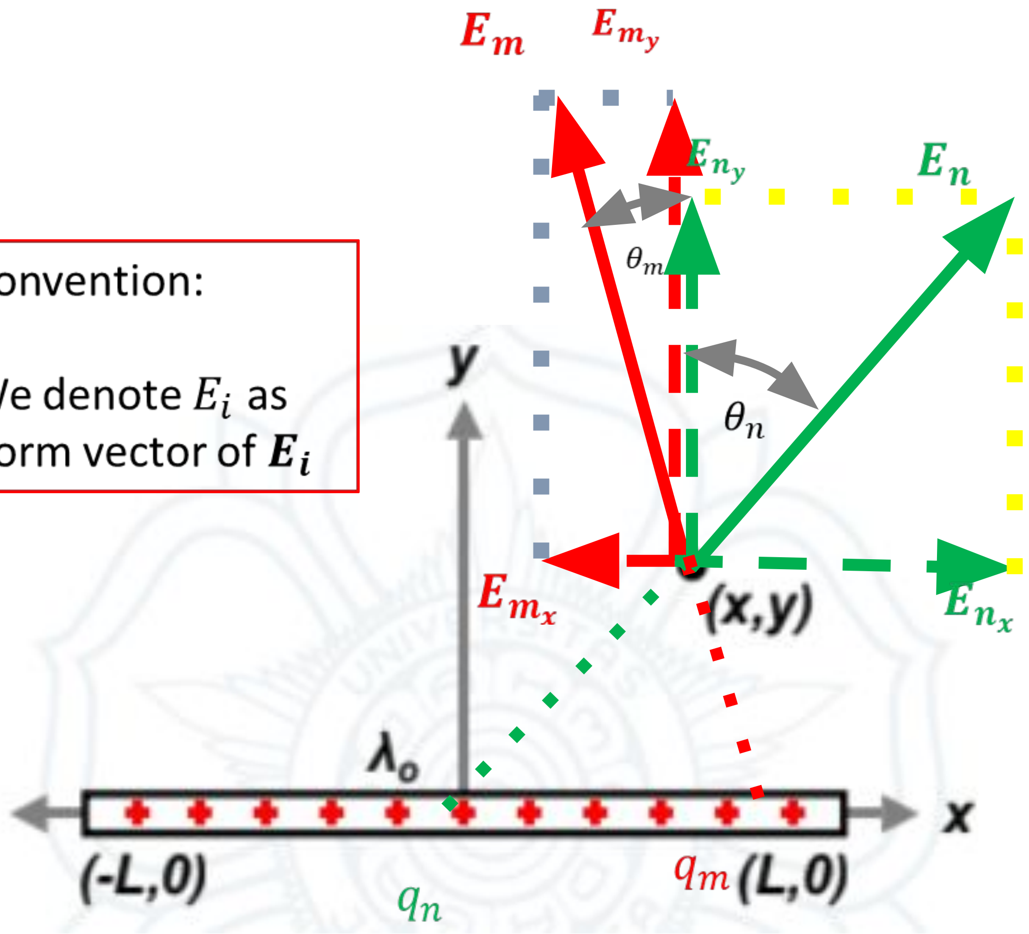
$$\vec{E}(x, y) = E_x(x, y)\hat{i} + E_y(x, y)\hat{j}$$

1. Determine the value of $E_x(x, y)$ and $E_y(x, y)$! Express your answer only in terms of λ_o, x, y, L and ϵ_o!
2. Determine also the value of $E_x(x, y)$ and $E_y(x, y)$ assuming that $L \to \infty$! Express your answer only in terms of λ_o, y and ϵ_o!

Question A

Convention:

We denote E_i as norm vector of $\boldsymbol{E_i}$



- Karena charge density terdistribusi uniform sebesar $\lambda(x) = \lambda_0$
- Jika di soal posisi titik pengamatan adalah di (x, y), kemudain kita definisikan (x_r, y_r) sebagai kordinat muatan segmen pada batang

$$E_n = \frac{1}{4\pi\epsilon_0} \frac{q_n}{(x_r - x)^2 + y^2}$$

$$\boldsymbol{E_n} = E_{nx}\boldsymbol{i} + E_{ny}\boldsymbol{j}$$

$$\boldsymbol{E_{nx}} = E_n \sin(\theta_n)\,\boldsymbol{i} \text{ dan } \boldsymbol{E_{ny}} = E_n \cos(\theta_n)\,\boldsymbol{j}$$

$$\sin(\theta_n) = \frac{x_r - x}{\sqrt{(x_r - x)^2 + y^2}}$$

$$\cos(\theta_n) = \frac{y}{\sqrt{(x_r - x)^2 + y^2}}$$

- Dengan $\lambda(x_r) = \lambda_0$ maka dengan asumsi panjang segment q_n sebesar Δx_r, kita peroleh $q_n = \lambda_0 \Delta x_r$

$$E_{nx} = \frac{1}{4\pi\epsilon_0} \frac{\lambda_0 (x_r - x)\, \Delta x_r}{\left((x_r - x)^2 + y^2\right)^{3/2}}$$

$$E_{nx} = \frac{1}{4\pi\epsilon_0} \frac{\lambda_0 y\, \Delta x_r}{\left((x_r - x)^2 + y^2\right)^{3/2}}$$

- Maka untuk $\Delta x_r \to 0$, kita definisikan panjang segmentnya sebesar dx_r

$$\Sigma E_y = \int_{x_r=-L}^{x_r=L} \frac{1}{4\pi\epsilon_0} \frac{\lambda_0 y\, dx_r}{\left((x_r - x)^2 + y_0^2\right)^{3/2}}$$

$$\Sigma E_x = \int_{x_r=-L}^{x_r=L} \frac{1}{4\pi\epsilon_0} \frac{\lambda_0 (x_r - x) dx_r}{\left((x_r - x)^2 + y\right)^{3/2}}$$

Pengamatan Sumbu Y

Dengan definisikan $(x_r - x) = y \tan(\psi)$

Maka $dx_r = y \sec^2(\psi)\ d\psi$

$$\Sigma E_y = \frac{\lambda_0}{4\pi\epsilon_0} \int_{\psi_1}^{\psi_2} \frac{y^2 \sec^2(\psi)\ d\psi}{y^3 \sec^3\ (\psi)}$$

$$= \frac{\lambda_0}{4\pi\epsilon_0 y} \int_{\psi_1}^{\psi_2} \cos(\psi)\, d\psi$$

$$= \frac{\lambda_0}{4\pi\epsilon_0 y} \left[\sin(\psi)\right]_{\psi_1}^{\psi_2}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\tan\psi = \frac{\sin(\psi)}{\cos(\psi)}$$

$$\frac{d\{\tan(\psi)\}}{d\psi} = \sec^2(\psi)$$

$$1 + \tan^2(\psi) = \sec^2(\psi)$$

$$\Sigma E_y = \frac{\lambda_0}{4\pi\epsilon_0 y}\left[\frac{x_r - x}{\sqrt{(x_r - x)^2 + y^2}}\right]_{x_r=-L}^{x_r=L}$$

$$\Sigma E_y(x,y) = \frac{\lambda_0}{4\pi\epsilon_0 y}\left(\frac{L - x}{\sqrt{(L-x)^2 + y^2}} + \frac{L + x}{\sqrt{(L+x)^2 + y^2}}\right)$$

Karena $(x_r - x) = y\tan(\psi)$

$$\sin(\psi) = \sqrt{1 - \cos^2(\psi)}$$

$$= \sqrt{1 - \frac{1}{\sec^2(\psi)}}$$

$$= \sqrt{1 - \frac{1}{1 + \tan^2(\psi)}}$$

$$= \sqrt{\frac{\tan^2(\psi)}{1 + \tan^2(\psi)}}$$

$$= \frac{(x_r - x)/y}{\sqrt{1 + ((x_r - x)/y)^2}}$$

$$\sin(\psi) = \frac{(x_r - x)}{\sqrt{(x_r - x)^2 + y^2}}$$

Pengamatan Sumbu X

$$\Sigma E_x = \int_{x_r=-L}^{x_r=L} \frac{1}{4\pi\epsilon_0} \frac{\lambda_0 (x_r - x) dx_r}{\left((x_r - x)^2 + y^2\right)^{3/2}}$$

Dengan definisikan $u = (x_r - x)^2$

Maka $du = 2(x_r - x)dx_r$

$$\Sigma E_x = \frac{\lambda_0}{8\pi\epsilon_0} \int_{u1}^{u2} \frac{du}{(u + y^2)^{3/2}}$$

Dengan mendefinisikan $w = u + y^2$

Maka $dw = du$

$$\Sigma E_x = \frac{\lambda_0}{8\pi\epsilon_0} \int_{w1}^{w2} w^{-3/2} dw$$

$$\Sigma E_x = \frac{-2\lambda_0}{8\pi\epsilon_0} \left[\frac{1}{\sqrt{w}}\right]_{w1}^{w2}$$

$$\Sigma E_x = \frac{-\lambda_0}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{u + y^2}}\right]_{u1}^{u2}$$

$$\Sigma E_x = \frac{-\lambda_0}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{(x_r - x)^2 + y^2}}\right]_{x_r=-L}^{x_r=L}$$

$$\Sigma E_x(x_0, y_0) = \frac{\lambda_0}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{(L + x)^2 + y^2}} - \frac{1}{\sqrt{(L - x)^2 + y^2}}\right)$$

Maka kita peroleh:

$$\boldsymbol{E}(x,y) = E_x(x,y)\boldsymbol{i} + E_y(x,y)\boldsymbol{j}$$

Dengan

$$E_x(x,y) = \Sigma E_x(x,y) = \frac{\lambda_0}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(L+x)^2+y^2}} - \frac{1}{\sqrt{(L-x)^2+y^2}}\right)$$

$$E_y(x,y) = \Sigma E_y(x,y) = \frac{\lambda_0}{4\pi\epsilon_0}\frac{1}{y}\left(\frac{L-x}{\sqrt{(L-x)^2+y^2}} + \frac{L+x}{\sqrt{(L+x)^2+y^2}}\right)$$

Intermezzo

Hasil dari soal ini adalah untuk kondisi $x, y \in \mathbb{R}$, untuk batang bermuatan dengan panjang berhingga yang simetris pada sumbu Y

Jika ingin diamati untuk titik di sumbu Y (yaitu $y \in \mathbb{R}$ tapi $x = 0$) akan diperoleh seperti pada PR 2, bahwa

$$\Sigma E_x(0,y) = 0 \text{ dan } \Sigma E_y(0,y) = \frac{\lambda_0}{2\pi\epsilon_0}\frac{1}{y\sqrt{L^2+y^2}}$$

Question B

Untuk $L \to \infty$ maka kita peroleh:

$$\lim_{L\to\infty} E_x(x, y, L) = \frac{\lambda_0}{4\pi\epsilon_0} \lim_{L\to\infty} \left(\frac{1}{\sqrt{(L+x)^2 + y^2}} - \frac{1}{\sqrt{(L-x)^2 + y^2}} \right)$$

$$= \frac{\lambda_0}{4\pi\epsilon_0} \lim_{L\to\infty} \left(\frac{1/L}{\sqrt{\left(1 + \frac{x}{L}\right)^2 + \left(\frac{y}{L}\right)^2}} - \frac{1/L}{\sqrt{\left(1 - \frac{x}{L}\right)^2 + \left(\frac{y}{L}\right)^2}} \right)$$

$$= \frac{\lambda_0}{4\pi\epsilon_0} \left(\frac{0}{\sqrt{(1+0)^2 + (0)^2}} - \frac{0}{\sqrt{(1-0)^2 + (0)^2}} \right)$$

$$\lim_{L\to\infty} E_x(x, y, L) = 0$$

Untuk $L \to \infty$ maka kita peroleh:

$$\lim_{L\to\infty} E_y(x,y,L) = \frac{\lambda_0}{4\pi\epsilon_0 y} \lim_{L\to\infty} \left(\frac{L-x}{\sqrt{(L-x)^2+y^2}} + \frac{L+x}{\sqrt{(L+x)^2+y^2}} \right)$$

$$= \frac{\lambda_0}{4\pi\epsilon_0 y} \lim_{L\to\infty} \left(\frac{1-x/L}{\sqrt{\left(1-\frac{x}{L}\right)^2+\left(\frac{y}{L}\right)^2}} + \frac{1+x/L}{\sqrt{\left(1-\frac{x}{L}\right)^2+\left(\frac{y}{L}\right)^2}} \right)$$

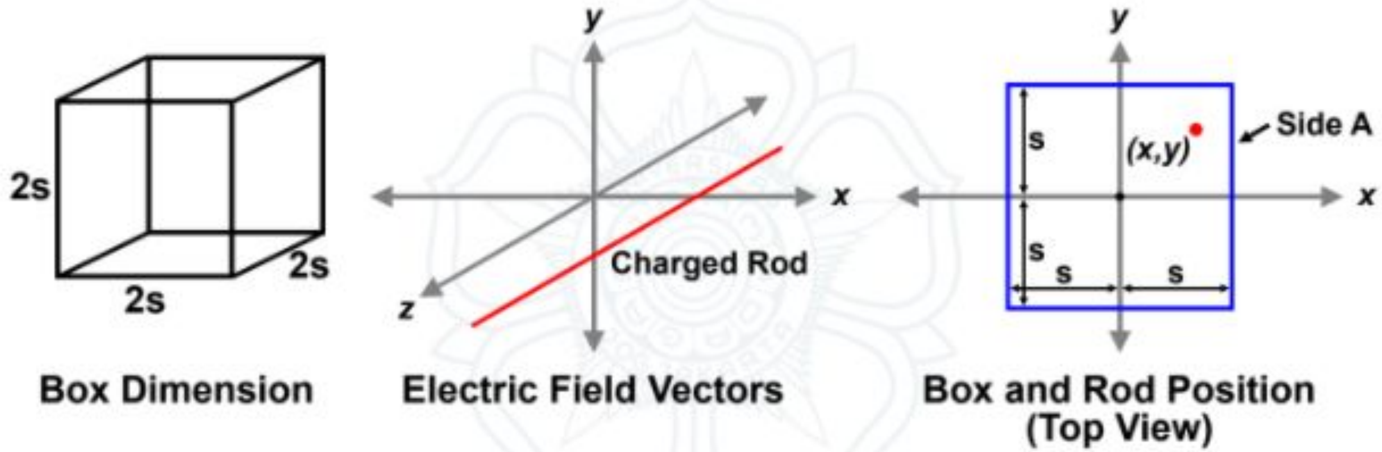
$$= \frac{\lambda_0}{4\pi\epsilon_0 y} \left(\frac{1-0}{\sqrt{(1+0)^2+(0)^2}} + \frac{1-0}{\sqrt{(1-0)^2+(0)^2}} \right)$$

$$\lim_{L\to\infty} E_y(x,y,L) = \frac{\lambda_0}{2\pi\epsilon_0 y}$$

Maka untuk $L \to \infty$ kita peroleh:

$$\lim_{L\to\infty} E(x,y,L) = \lim_{L\to\infty} E_x(x,y,L)\,\boldsymbol{i} + \lim_{L\to\infty} E_y(x,y,L)\,\boldsymbol{j} = \frac{\lambda_0}{2\pi\epsilon_0 y}\boldsymbol{j}$$

Problem #2 : Electric Flux due to a Long Charged Rod



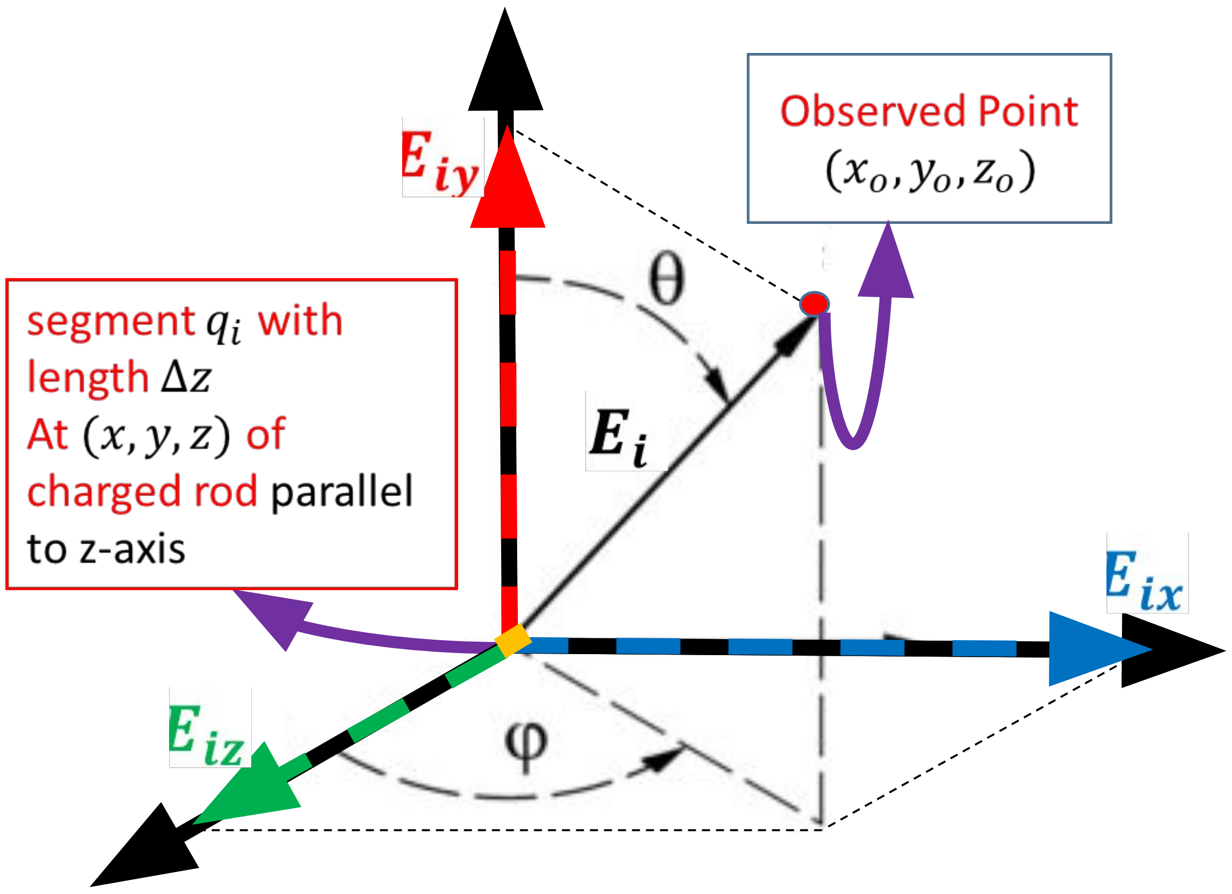
Consider a long rod ($L \to \infty$) whose charge is uniformly distributed so that the charge density per unit length is given by :

$$\lambda(x) = \lambda_o$$

This rod is placed at coordinate (x, y) and enclosed by a cube with dimension $2s \times 2s \times 2s$ as shown in the figure above. The center of the cube is placed at the center of coordinate $(0, 0, 0)$.

A. Determine the electric flux that passes through side A of this cube! Express your answer only in terms of λ_o, x, y, s and ϵ_o!

Pre Processing: Electric Field in $\mathbb{R}^3$ by infinite-long rod



$$q_i = \lambda_o \, \Delta z$$

For $\Delta z \to 0$

$$q_i = \lambda_o dz$$

Sumbu X

$$E_{ix} = E_i \sin(\theta) \sin(\psi) = \frac{1}{4\pi\epsilon_0} \frac{q_i \, (x - x_o)}{\left((x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right)^{3/2}}$$

$$E_{ix} = \frac{1}{4\pi\epsilon_0} \frac{\lambda_o \, (x - x_o) dz}{\left((x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right)^{3/2}}$$

Sumbu Z

$$E_{iz} = E_i \sin(\theta) \cos(\psi) = \frac{1}{4\pi\epsilon_0} \frac{\lambda_o (z - z_o) dz}{\left((x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right)^{3/2}}$$

Sumbu Y

$$E_{iy} = E_i \cos(\theta) = \frac{1}{4\pi\epsilon_0} \frac{\lambda_o (y - y_o) dz}{\left((x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right)^{3/2}}$$

Karena objek pengamatan kita pada soal ini pada batang bermuatan dengan panjang tak hingga, maka batang ini simetris dengan panjang di sumbu $z(+)$ dan $z(-)$ sama panjang $L = \infty$

Untuk proses analisis nya, kita wakilkan panjang batang dengan variable L, dan kita amati batang yang memiliki panjang $2L$ yang membentang dari $z = -L$ hingga $z = L$

Hal ini tetap valid untuk kemudian dijadikan patokan untuk $L = \infty$, karena $2L = 2(\infty) = \infty$

Dari nilai medan listrik pada segment q_i, dapat dihitung total medan listrik oleh batang bermuatan tersebut, sebagai:

$$\Sigma E_x = \frac{1}{4\pi\epsilon_0} \int_{-L}^{L} \frac{\lambda_o\,(x - x_o)dz}{\left((x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right)^{3/2}}$$

$$\Sigma E_y = \frac{1}{4\pi\epsilon_0} \int_{-L}^{L} \frac{\lambda_o\,(y - y_o)dz}{\left((x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right)^{3/2}}$$

$$\Sigma E_y = \frac{1}{4\pi\epsilon_0} \int_{-L}^{L} \frac{\lambda_o\,(z - z_o)dz}{\left((x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right)^{3/2}}$$

Sumbu X & Y

Asumsikan $(z - z_0) = \sqrt{(x - x_o)^2 + (y - y_o)^2}\,\tan(\zeta)$

Sumbu Z

Asumsikan $u = (z - z_o)^2$

Kemudian asumsikan $w = u + (x - x_o)^2 + (y - y_o)^2$

Maka kita akan menemukan bentuk seperti pada **Problem #1**

Sehingga diperoleh hasil:

NEXT SLIDE ☺

$$\boldsymbol{\Sigma E_x} = \frac{\lambda_0}{4\pi\epsilon_0}\frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}\left(\frac{(L - z_0)}{\sqrt{(L - z_0)^2 + (x - x_0)^2 + (y - y_0)^2}} + \frac{(L + z_0)}{\sqrt{(L + z_0)^2 + (x - x_0)^2 + (y - y_0)^2}}\right)\boldsymbol{i}$$

$$\boldsymbol{\Sigma E_y} = \frac{\lambda_0}{4\pi\epsilon_0}\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}\left(\frac{(L - z_0)}{\sqrt{(L - z_0)^2 + (x - x_0)^2 + (y - y_0)^2}} + \frac{(L + z_0)}{\sqrt{(L + z_0)^2 + (x - x_0)^2 + (y - y_0)^2}}\right)\boldsymbol{j}$$

$$\boldsymbol{\Sigma E_z} = \frac{\lambda_0}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(L + z_0)^2 + (x - x_o)^2 + (y - y_o)^2}} - \frac{1}{\sqrt{(L - z_0)^2 + (x - x_o)^2 + (y - y_o)^2}}\right)\boldsymbol{k}$$

Sedangkan untuk $L \to \infty$

$$\lim_{L\to\infty} \boldsymbol{\Sigma E_x(x, y, z, L)} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}\boldsymbol{i}$$

$$\lim_{L\to\infty} \boldsymbol{\Sigma E_y(x, y, z, L)} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}\boldsymbol{j}$$

$$\lim_{L\to\infty} \boldsymbol{\Sigma E_z(x, y, z, L)} = 0\boldsymbol{k}$$

Sehinga medan listrik yang dihasilkan batang tak berhingga di soal adalah sebesar:

$$\boldsymbol{E} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{1}{(x-x_0)^2+(y-y_0)^2}\left((x-x_0)\boldsymbol{i}+(y-y_0)\boldsymbol{j}\right)$$

Question A

$$\Phi = \oint \boldsymbol{E}\cdot\boldsymbol{n}\,dA$$

Untuk Side A, $x_0^A = s$ dan $\boldsymbol{n_A} = \boldsymbol{i}$**,** maka

$$\boldsymbol{E}\cdot\boldsymbol{n_A} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{(x-s)}{(x-s)^2+(y-y_0)^2}$$

Sedangkan $dA_A = dz_0\,dy_0$, sehingga

$$\Phi_A = \int_{-s}^{s}\int_{-s}^{s}\frac{\lambda_0}{2\pi\epsilon_0}\frac{(x-s)}{(x-s)^2+(y-y_0)^2}\,dy_0\,dz_0$$

Dengan

$$(y-y_0) = (x-s)\tan(\alpha) \rightarrow \alpha = \tan^{-1}\left(\frac{y-y_0}{x-s}\right)$$

$$dy_0 = -(x-s)\sec^2(\alpha)\,d\alpha$$

$$\Phi_A = \frac{-\lambda_0}{2\pi\epsilon_0}\int_{-s}^{s}\int_{\alpha_1}^{\alpha_2}d\alpha\,dz_0$$

$$\Phi_{\mathrm{A}} = \frac{-\lambda_0}{2\pi\epsilon_0}\int_{-s}^{s}\int_{\alpha_1}^{\alpha_2} d\alpha\, dz_0 = \frac{-\lambda_0}{2\pi\epsilon_0}\int_{-s}^{s}[\alpha]_{\alpha_1}^{\alpha_2}\, dz_0$$

$$= \frac{-\lambda_0}{2\pi\epsilon_0}\int_{-s}^{s}\left[\tan^{-1}\left(\frac{y-y_0}{x-s}\right)\right]_{y_0^{down}=-s}^{y_0^{up}=s} dz_0$$

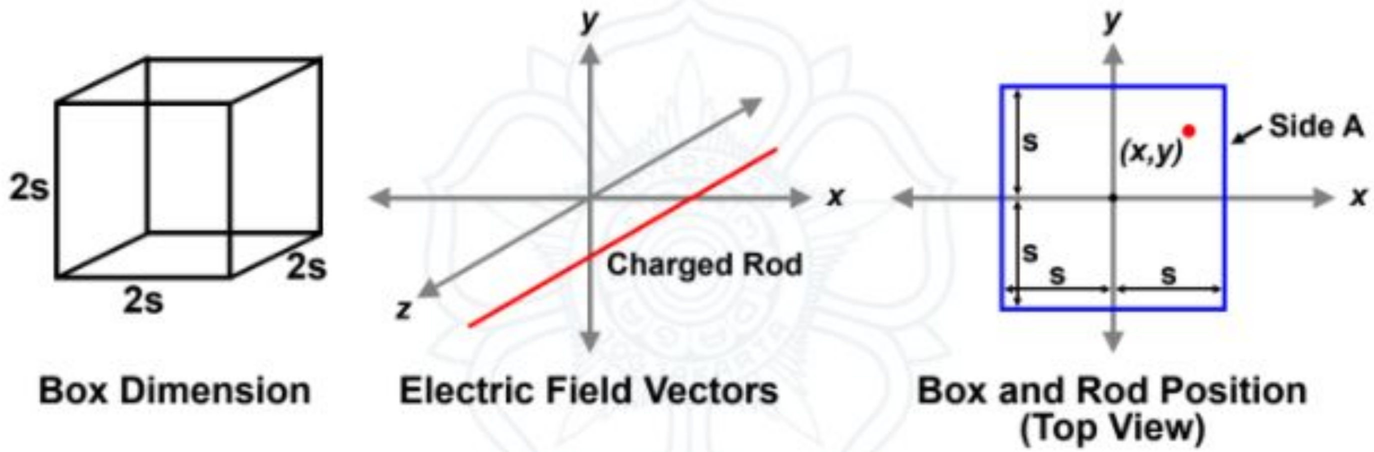
$$\Phi_{\mathrm{A}} = \frac{-\lambda_0}{2\pi\epsilon_0}\int_{-s}^{s}\left(\tan^{-1}\left(\frac{y-s}{x-s}\right) - \tan^{-1}\left(\frac{y+s}{x-s}\right)\right) dz_0$$

$$\Phi_{\mathrm{A}} = \frac{\lambda_0}{2\pi\epsilon_0}\int_{-s}^{s}\left(-\tan^{-1}\left(\frac{y-s}{x-s}\right) + \tan^{-1}\left(\frac{y+s}{x-s}\right)\right) dz_0$$

$$\Phi_{\mathrm{A}} = \frac{\lambda_0}{2\pi\epsilon_0}\left[\left(-\tan^{-1}\left(\frac{y-s}{x-s}\right) + \tan^{-1}\left(\frac{y+s}{x-s}\right)\right) z_0\right]_{z_0^{down}=-s}^{z_0^{up}=s}$$

$$\Phi_{\mathrm{A}} = \frac{\lambda_0 s}{\pi\epsilon_0}\left(-\tan^{-1}\left(\frac{y-s}{x-s}\right) + \tan^{-1}\left(\frac{y+s}{x-s}\right)\right)$$

Problem #2 : Electric Flux due to a Long Charged Rod



- Determine the electric flux that passes through the whole cube! Express your answer only in terms of λ_o, s and ϵ_o!

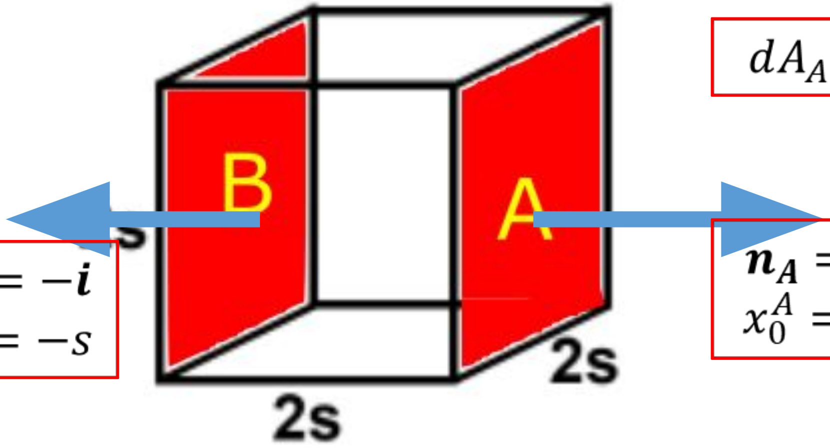
Hint :

$$\tan^{-1}(x) + \tan^{-1}(1/x) = \pi/2$$

$$\cos^{-1}(x) + \cos^{-1}\left(\sqrt{1-x^2}\right) = \pi/2$$

$$\sin^{-1}(x) + \sin^{-1}\left(\sqrt{1-x^2}\right) = \pi/2$$

$$dA_A = dA_B = dy_0 dz_0$$



$$\boldsymbol{n_B} = -\boldsymbol{i}$$
$$x_0^A = -s$$

$$\boldsymbol{n_A} = \boldsymbol{i}$$
$$x_0^A = s$$

Untuk

$$\boldsymbol{E} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{1}{(x-x_0)^2+(y-y_0)^2}\left((x-x_0)\boldsymbol{i} + (y-y_0)\boldsymbol{j}\right)$$

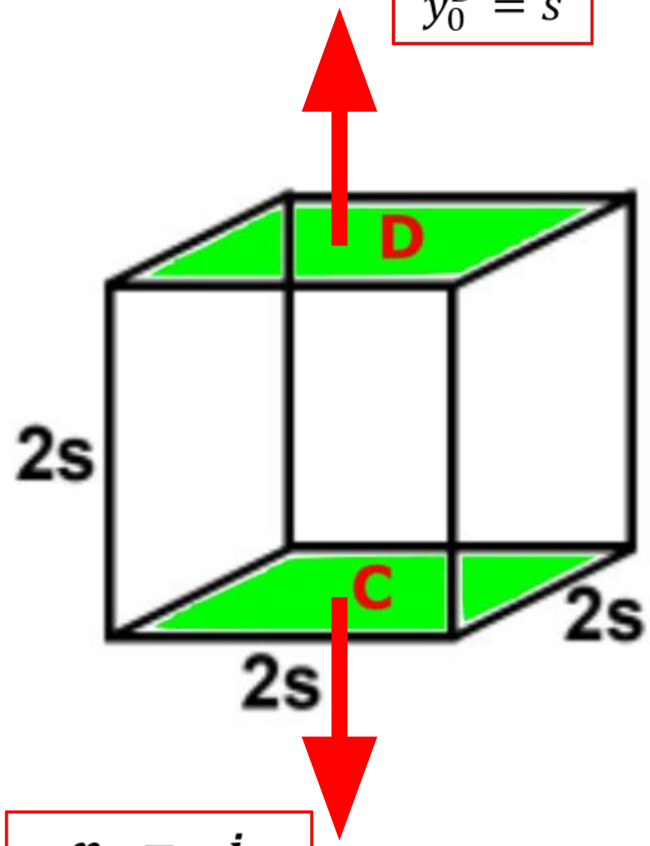
diperoleh

$$\boldsymbol{E}\cdot\boldsymbol{n_B} = \frac{-\lambda_0}{2\pi\epsilon_0}\frac{(x+s)}{(x+s)^2+(y-y_0)^2}$$

Dengan cara yang sama pada **Question A,** diperoleh:

$$\Phi_B = \frac{\lambda_0 s}{\pi\epsilon_0}\left(\tan^{-1}\left(\frac{y-s}{x+s}\right) - \tan^{-1}\left(\frac{y+s}{x+s}\right)\right)$$

$$\boldsymbol{n_D} = \boldsymbol{j}$$
$$y_0^D = s$$



$$\boldsymbol{n_C} = -\boldsymbol{j}$$
$$y_0^C = -s$$

$$dA_C = dA_D = dx_0 dz_0$$

Untuk

$$\boldsymbol{E} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{1}{(x-x_0)^2+(y-y_0)^2}\left((x-x_0)\boldsymbol{i} + (y-y_0)\boldsymbol{j}\right)$$

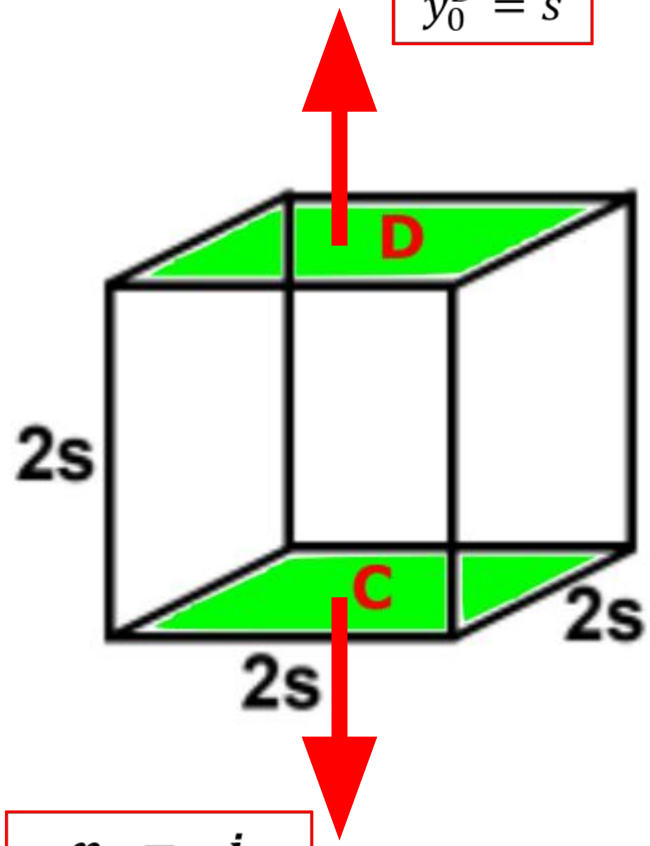
diperoleh

$$\boldsymbol{E}\cdot\boldsymbol{n_D} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{(x-x_0)}{(x-x_0)^2+(y-s)^2}$$

Dengan cara yang sama pada **<u>Question A,</u>** diperoleh:

$$\Phi_{\mathrm{D}} = \frac{\lambda_0 s}{\pi\epsilon_0}\left(-\tan^{-1}\left(\frac{x-s}{y-s}\right) + \tan^{-1}\left(\frac{x+s}{y-s}\right)\right)$$

$$\boldsymbol{n_D} = \boldsymbol{j}$$
$$y_0^D = s$$



$$\boldsymbol{n_C} = -\boldsymbol{j}$$
$$y_0^C = -s$$

$$dA_C = dA_D = dx_0 dz_0$$

Untuk

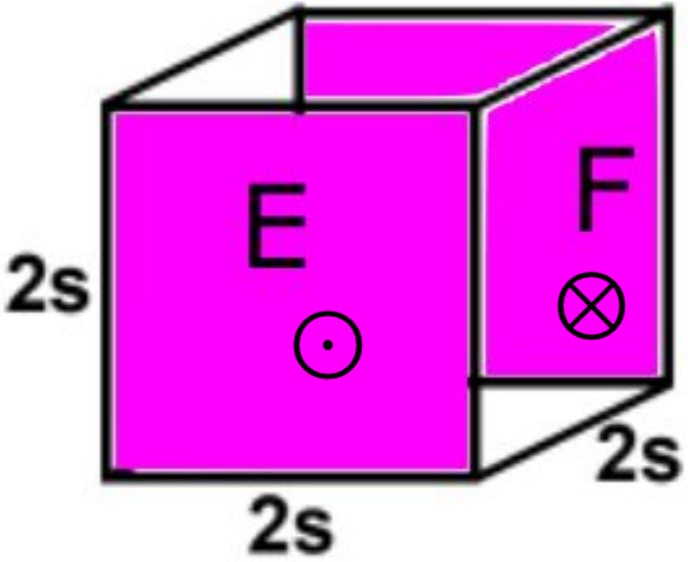
$$\boldsymbol{E} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{1}{(x-x_0)^2+(y-y_0)^2}\left((x-x_0)\boldsymbol{i} + (y-y_0)\boldsymbol{j}\right)$$

diperoleh

$$\boldsymbol{E}\cdot\boldsymbol{n_C} = \frac{-\lambda_0}{2\pi\epsilon_0}\frac{(x-x_0)}{(x-x_0)^2+(y+s)^2}$$

Dengan cara yang sama pada **Question A,** diperoleh:

$$\Phi_C = \frac{\lambda_0 s}{\pi\epsilon_0}\left(\tan^{-1}\left(\frac{x-s}{y+s}\right) - \tan^{-1}\left(\frac{x+s}{y+s}\right)\right)$$



Karena $\boldsymbol{n_E} = \boldsymbol{k}$ dan $\boldsymbol{n_F} = -\boldsymbol{k}$, sedangkan

$$\boldsymbol{E} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{1}{(x-x_0)^2+(y-y_0)^2}\left((x-x_0)\boldsymbol{i} + (y-y_0)\boldsymbol{j}\right)$$

Maka

$$\boldsymbol{E}\cdot\boldsymbol{n_E} = 0$$

$$\boldsymbol{E}\cdot\boldsymbol{n_F} = 0$$

Sehingga

$$\Phi_E = \Phi_F = 0$$

Then the total flux passes through whole cube is:

$$\Phi_{tot} = \Phi_A + \Phi_B + \Phi_C + \Phi_D + \Phi_E + \Phi_F$$

Dengan

$$\Phi_A = \frac{\lambda_0 s}{\pi\epsilon_0}\left(-\tan^{-1}\left(\frac{y-s}{x-s}\right) + \tan^{-1}\left(\frac{y+s}{x-s}\right)\right)$$

$$\Phi_B = \frac{\lambda_0 s}{\pi\epsilon_0}\left(\tan^{-1}\left(\frac{y-s}{x+s}\right) - \tan^{-1}\left(\frac{y+s}{x+s}\right)\right)$$

$$\Phi_C = \frac{\lambda_0 s}{\pi\epsilon_0}\left(\tan^{-1}\left(\frac{x-s}{y+s}\right) - \tan^{-1}\left(\frac{x+s}{y+s}\right)\right)$$

$$\Phi_D = \frac{\lambda_0 s}{\pi\epsilon_0}\left(-\tan^{-1}\left(\frac{x-s}{y-s}\right) + \tan^{-1}\left(\frac{x+s}{y-s}\right)\right)$$

$$\Phi_E = \Phi_F = 0$$

$$\Phi_{tot} = \frac{\lambda_0 s}{\pi\epsilon_0}\left(\tan^{-1}\left(\frac{s-y}{x-s}\right) + \tan^{-1}\left(\frac{1}{\left(\frac{s-y}{x-s}\right)}\right)\right)$$

$$+\frac{\lambda_0 s}{\pi\epsilon_0}\left(\tan^{-1}\left(\frac{y+s}{x+s}\right) + \tan^{-1}\left(\frac{1}{\left(\frac{y+s}{x+s}\right)}\right)\right)$$

$$+\frac{\lambda_0 s}{\pi\epsilon_0}\left(\tan^{-1}\left(\frac{y-s}{x+s}\right) + \tan^{-1}\left(\frac{1}{\left(\frac{y-s}{x+s}\right)}\right)\right)$$

$$+\frac{\lambda_0 s}{\pi\epsilon_0}\left(\tan^{-1}\left(\frac{-y-s}{x+s}\right) + \tan^{-1}\left(\frac{1}{\left(\frac{-y-s}{x-s}\right)}\right)\right)$$

Dari Hint:

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

Maka

$$\Phi_{tot} = \frac{2s\lambda_0}{\epsilon_0}$$

Sedikit Selingan

Dari hasil bahwa:

$$\Phi_{tot} = \frac{2s\lambda_0}{\epsilon_0}$$

Mengingat λ_0 adalah rapat muatan per satuan panjang, sedangkan $2s$ adalah panjang batang bermuatan yang ada di dalam kubus, maka diperoleh bahwa $2s\lambda_0$ merupakan muatan total yang berada di dalam kubus

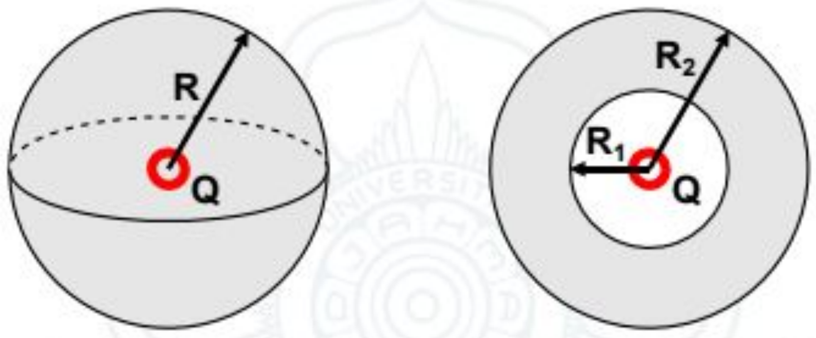
$$Q_{enc} = 2s\lambda_0$$

Sehingga dapat ditulis sebagai

$$\Phi_{tot} = \frac{Q_{enc}}{\epsilon_0}$$

Sehingga hasil perhitungan ini juga telah membuktikan validnya Hukum Gauss pada Gaussian Surface berupa Kubus

Problem #3 : Electric Field around a Metal Conductor



Consider a solid metal conductor with total charge Q as shown in the figure above.

1. Determine the electric field due to this sphere for $r \leq R$ and $r > R$! Express your answer only in terms of Q, r and ϵ_o!

Question A

I. *Conductors* contain *free charges* that move easily

II. Coulomb's Law: same charge will repeal, different charge will attract

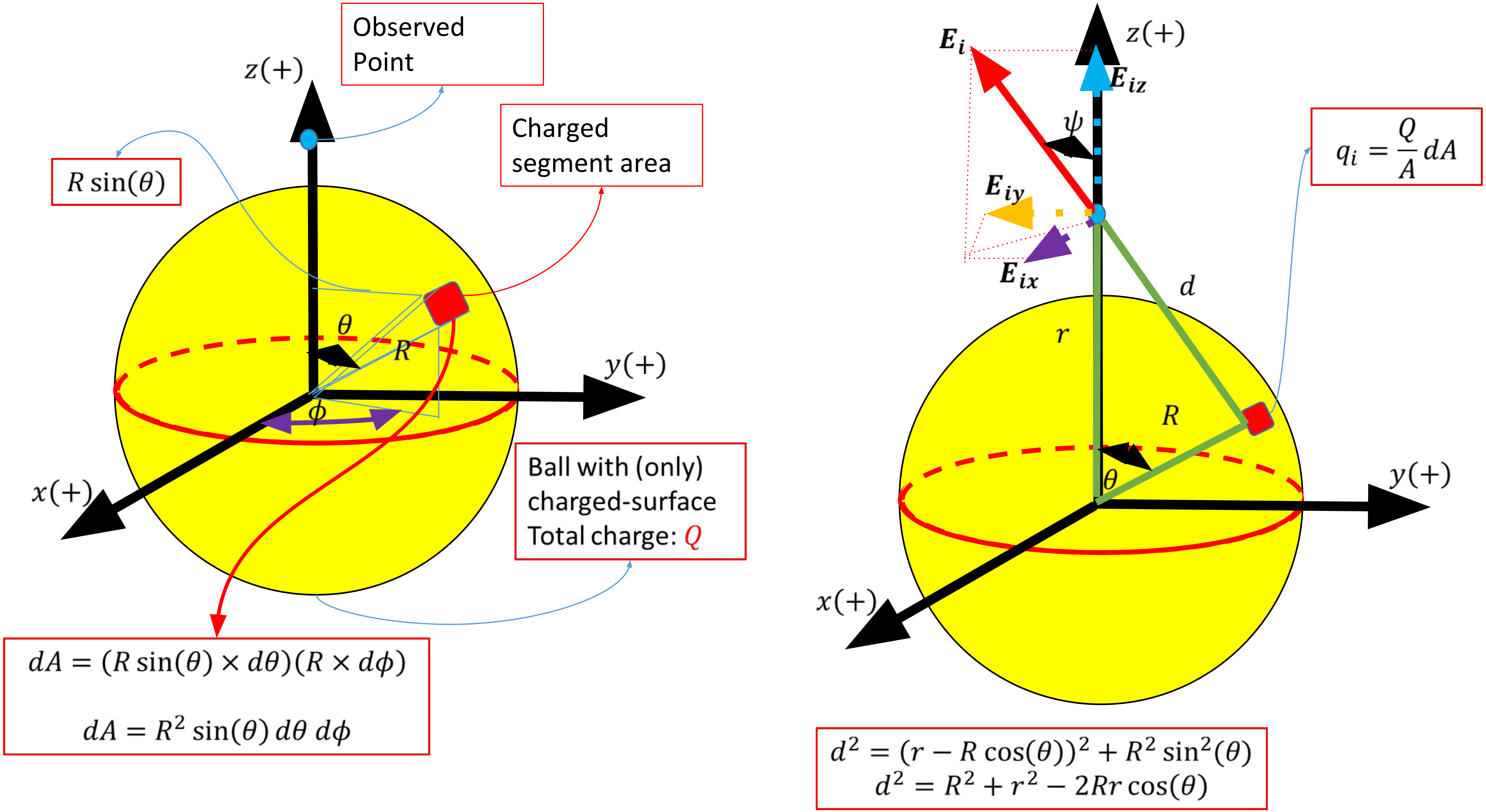
III. If we assume that the ball contains positive or negative charges only, then each charge will repeal another, and finally all charges are in it's surface only, so there is no charge inside the ball

IV. if we assume that the ball contains both positive and negative charges with different amount, then each charge will attract and be neutral, whereas residual charges just do **point III)**

V. From **point III)** to calculate electric field inside the ball, just place Gaussian surface inside it, then we will get $Q_{enclosed} = 0$, Using Gauss's Law: $E(4\pi r^2) = \frac{Q_{enclosed}}{\epsilon_0} \rightarrow \boldsymbol{E = 0}$ **for** $r \leq R$

VI. For $r > R$, we get $Q_{enclosed} = Q$, then with same formula, we get $\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$ **for** $r > R$

VII. Curious using Coulomb's Law only? Just move to next slide ☺



Observed Point
Charged segment area
$R\sin(\theta)$
$z(+)$
$y(+)$
$x(+)$
θ
R
ϕ
Ball with (only) charged-surface Total charge: Q
$dA = (R\sin(\theta)\times d\theta)(R\times d\phi)$
$dA = R^2\sin(\theta)\,d\theta\,d\phi$
$\boldsymbol{E_i}$
$\boldsymbol{E_{iz}}$
ψ
$\boldsymbol{E_{iy}}$
$\boldsymbol{E_{ix}}$
d
r
$q_i = \frac{Q}{A}dA$
$d^2 = (r - R\cos(\theta))^2 + R^2\sin^2(\theta)$
$d^2 = R^2 + r^2 - 2Rr\cos(\theta)$

Surface area of Ball:

$$A = 8 \int_0^{\pi/2} \int_0^{\pi/2} R^2 \sin(\theta)\ d\theta\, d\phi$$

$$A = 8R^2 \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin(\theta)\, d\theta$$

$$A = 4\pi R^2$$

Medan listrik pada Observed Point:

$$E_i = \frac{1}{4\pi\epsilon_0} \frac{q_i}{d^2}$$

Dengan

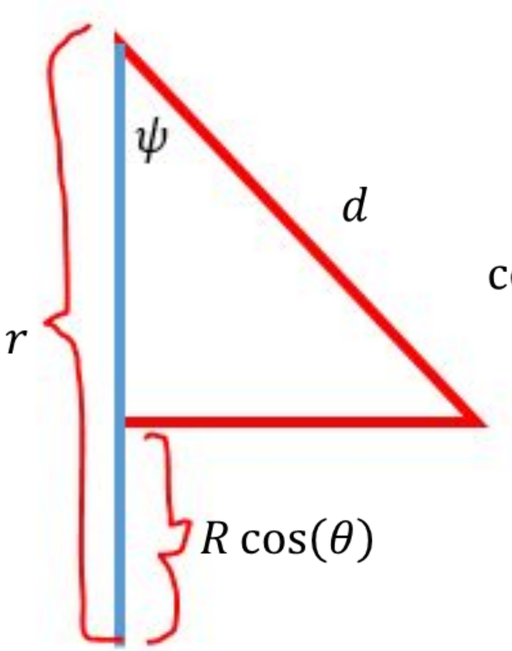
$$q_i = \frac{Q}{A} dA = \frac{Q}{4\pi R^2} R^2 \sin(\theta)\, d\theta\, d\phi$$

$$q_i = \frac{Q}{4\pi} \sin(\theta)\, d\theta\, d\phi$$

Maka:

$$E_i = \frac{1}{4\pi\epsilon_0} \frac{Q/4\pi}{R^2 + r^2 - 2Rr\cos(\theta)} \sin(\theta)\, d\theta\, d\phi$$

Untk mendekomposisikan E_i menjadi komponen pada masing – masing sumbu, maka



$$\cos(\psi) = \frac{r - R\cos(\theta)}{d}$$

$$\cos(\psi) = \frac{r - R\cos(\theta)}{\sqrt{R^2 + r^2 - 2Rr\cos(\theta)}}$$

$$E_{iz} = E_i \cos(\psi)$$

$$E_{iz} = \frac{1}{4\pi\epsilon_0} \frac{\left(\frac{Q}{4\pi}\right)(r - R\cos(\theta))}{(R^2 + r^2 - 2Rr\cos(\theta))^{3/2}} \sin(\theta)\, d\theta\, d\phi$$

Asumsikan $v = \cos(\theta)$ maka $dv = \sin(\theta)\, d\theta$

$$E_{iz} = \frac{Q}{(4\pi)^2\epsilon_0} \frac{(r - Rv)dv}{(R^2 + r^2 - 2Rrv)^{3/2}} d\phi$$

$$E_{iz} = \frac{Q}{(4\pi)^2\epsilon_0}\frac{(r - Rv)dv}{(R^2 + r^2 - 2Rrv)^{3/2}}d\phi$$

Dengan $u = R^2 + r^2 - 2Rrv$ maka $dv = -\frac{du}{2Rr}$, sehingga

$$u = R^2 + 2r\left(\frac{r}{2} - Rv\right) \rightarrow \left(\frac{r}{2} - Rv\right) = \frac{1}{2r}(u - R^2) \rightarrow (r - Rv) = \frac{1}{2r}(u + r^2 - R^2)$$

Sehingga

$$\Sigma E_z = \frac{Q}{(4\pi)^2\epsilon_0}\int_{u1}^{u2}\left(\frac{1}{2r}\frac{(u + r^2 - R^2)}{u^{3/2}}\right)\left(-\frac{du}{2Rr}\right)\int_0^{2\pi} d\phi$$

$$\Sigma E_z = \frac{-Q(2\pi)}{(4\pi)^2\epsilon_0}\frac{1}{4Rr^2}\int_{u1}^{u2}\left((r^2 - R^2)u^{-3/2} + u^{-1/2}\right)du$$

$$\Sigma E_z = \frac{-Q(2\pi)}{(4\pi)^2\epsilon_0}\frac{2}{4Rr^2}\left[\left(\frac{(r^2 - R^2)}{\sqrt{u}} - \sqrt{u}\right)\right]_{u1}^{u2}$$

$$\Sigma E_z = \frac{-Q}{8\pi\epsilon_0}\frac{1}{2Rr^2}\left[\left(\frac{(r^2 - R^2)}{\sqrt{R^2 + r^2 - 2Rrv}} - \sqrt{R^2 + r^2 - 2Rrv}\right)\right]_{v1}^{v2}$$

$$\Sigma E_z = \frac{-Q}{8\pi\epsilon_0}\frac{1}{2Rr^2}\left[\left(\frac{(r^2-R^2)}{\sqrt{R^2+r^2-2Rr\cos(\theta)}} - \sqrt{R^2+r^2-2Rr\cos(\theta)}\right)\right]_{\theta=0}^{\theta=\pi}$$

$$\Sigma E_z = \frac{-Q}{8\pi\epsilon_0}\frac{1}{2Rr^2}\left\{\left(\frac{(r^2-R^2)}{\sqrt{R^2+r^2+2Rr}} - \sqrt{R^2+r^2+2Rr}\right) - \left(\frac{(r^2-R^2)}{\sqrt{R^2+r^2-2Rr}} - \sqrt{R^2+r^2-2Rr}\right)\right\}$$

Karena dalam memperoleh suku $\sqrt{R^2+r^2-2Rr}$ bukan berasal dari $\sqrt{(r-R)^2}$ namun dari subtitusi nilai $\cos(0)=1$, maka tidak tepat jika ditulis $\cancel{\sqrt{R^2+r^2-2Rr} = \sqrt{(r-R)^2}}$

Contradiction Proof:

- Hal ini karena jika $(r-R)<0$ maka walaupun $(r-R)^2>0$, nilai dari $\sqrt{(r-R)^2}<0$
- Karena $\sqrt{(-1)^2} \neq 1$ tapi $\sqrt{(-1)^2} = -1$
- Karena jika $\sqrt{(-1)^2}=1$, maka kita akan peroleh bahwa $(1-1) = \big(1+(-1)\big) = \left(1+\sqrt{(-1)^2}\right) = (1+1) = 2$
- Tentu ini salah karena $(1-1)=0\neq 2$

Sehingga suku $\sqrt{R^2+r^2-2Rr}$ akan **selalu** bernilai positif walaupun $(r-R)<0$ oleh karena itu, kita tuliskan sebagai:

$$\sqrt{R^2+r^2-2Rr} = \sqrt{|r-R|^2}$$

Sedangkan untuk suku $\sqrt{R^2+r^2+2Rr}$ tidak ada masalah baik ditulis sebagai $\sqrt{(r+R)^2}$ maupun $\sqrt{|r+R|^2}$, hal ini karena $r>0$ dan $R>0$ selalu.

$$\Sigma E_z = \frac{-Q}{8\pi\epsilon_0}\frac{1}{2Rr^2}\left\{\left(\frac{(r-R)(r+R)}{\sqrt{(r+R)^2}} - \sqrt{(r+R)^2}\right) - \left(\frac{(r-R)(r+R)}{\sqrt{|r-R|^2}} - \sqrt{|r-R|^2}\right)\right\}$$

$$\Sigma E_z = \frac{-Q}{8\pi\epsilon_0}\frac{1}{2Rr^2}\left\{\left((r-R) - (r+R)\right) - \left(\frac{(r-R)(r+R)}{|r-R|} - |r-R|\right)\right\}$$

$$\Sigma E_z = \frac{Q}{8\pi\epsilon_0}\frac{1}{2Rr^2}\left\{2R + \left(\frac{(r-R)(r+R)}{|r-R|} - |r-R|\right)\right\}$$

Dengan definisi nilai mutlak sebagai:

$$|r-R| = \begin{cases} (r-R), & \text{for } r > R \\ -(r-R), & \text{for } r \le R \end{cases}$$

- Maka untuk $r < R$, kita peroleh sebagai:

$$\Sigma E_z = \frac{Q}{8\pi\epsilon_0}\frac{1}{2Rr^2}\left\{2R + \left(\frac{(r-R)(r+R)}{-(r-R)} + (r-R)\right)\right\}$$

$$\Sigma E_z = \frac{Q}{8\pi\epsilon_0}\frac{1}{2Rr^2}\{2R - 2R\}$$

$$\Sigma E_z = 0, \qquad \text{for } r \le R \text{ (inside ball)}$$

- Maka untuk $r \geq R$, kita peroleh sebagai:

$$\Sigma E_z = \frac{Q}{8\pi\epsilon_0}\frac{1}{2Rr^2}\left\{2R + \left(\frac{(r-R)(r+R)}{(r-R)} - (r-R)\right)\right\}$$

$$\Sigma E_z = \frac{Q}{8\pi\epsilon_0}\frac{1}{2Rr^2}\{2R + 2R\}$$

$$\Sigma E_z = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, \qquad \text{for } r > R \text{ (at surface and outer ball)}$$

Has been shown that electric field at outer/surface (at the $z - axis$) of charged ball conductor is like caused by point charged

- With same step, we will get that $\Sigma E_x = \Sigma E_y = 0$ caused by symmetricity (do it yourself ☺)

- Above we have proven validity of observed point along $z - axis$, but caused of symmetricity, to prove another location of observed point, we just need to rotate the ball such as it's $z - axis$ coincide with your observed point, and you will see same face ☺

- So, by proofing at $z - axis$, it is sufficient to prove at another (arbitrary) location of observed point

GOOD NIGHT ☺

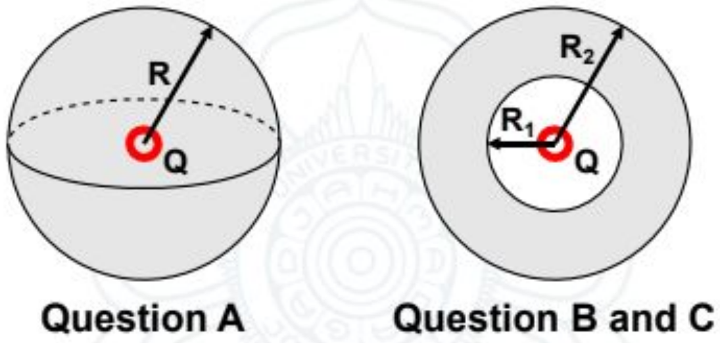
Sedikit Selingan

Dari hasil derivasi pada **Solution HW3 No.3(a)**, nilai medan listrik yang diakibatkan oleh suatu bola dengan (hanya) permukaan bermuatan:

$$E = \begin{cases} 0, & for\, r \leq R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}, & for\, r > R \end{cases}$$

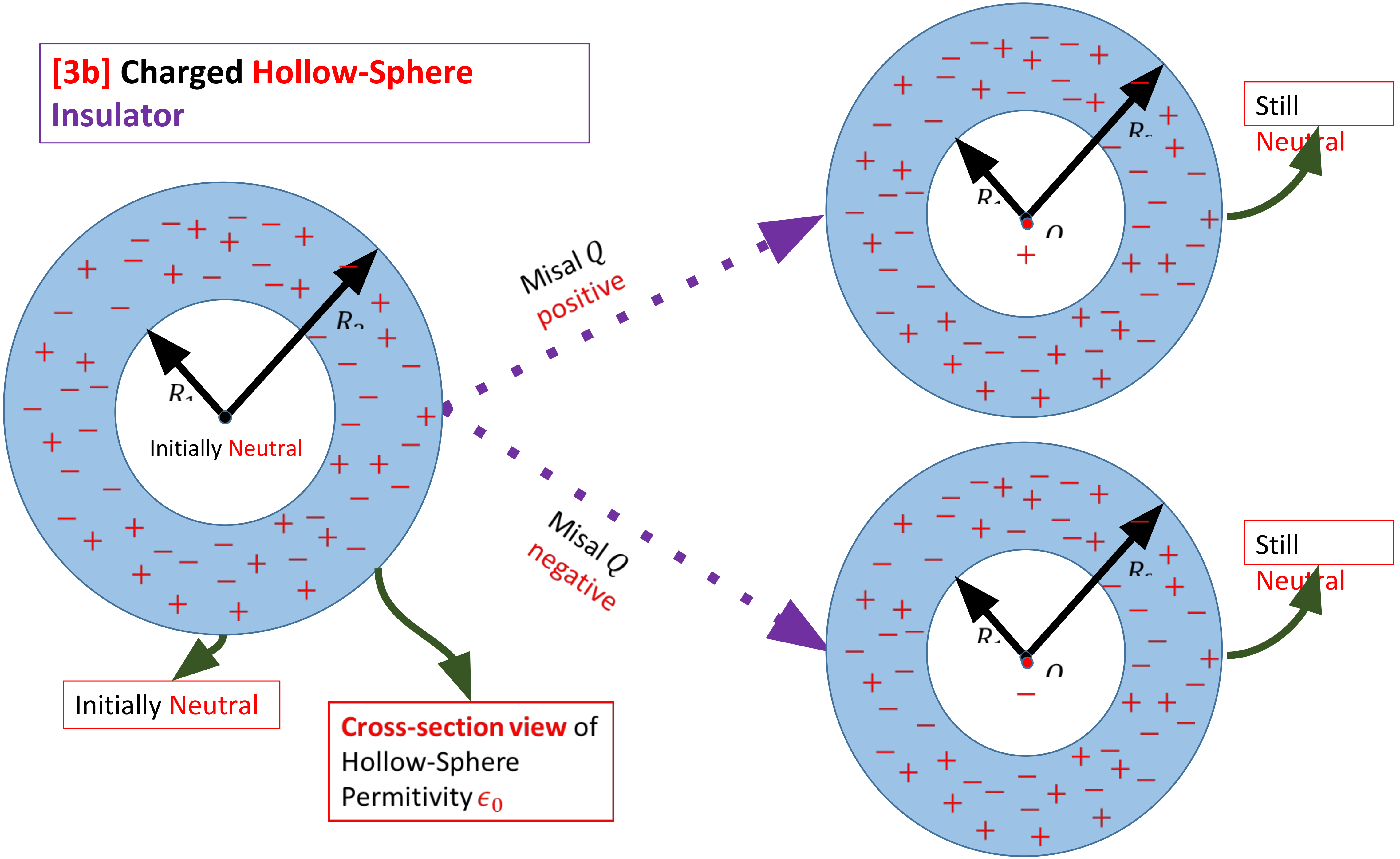
Hasil ini menunjukkan **berapapun jari – jari bola $\boldsymbol{R}$**, tidak mempengaruhi besar medan listrik di luar bola, karena hanya bergantung pada jarak r (jarak titik pengamatan dari pusat bola), dan jumlah muatan total Q

Problem #3 : Electric Field around a Metal Conductor

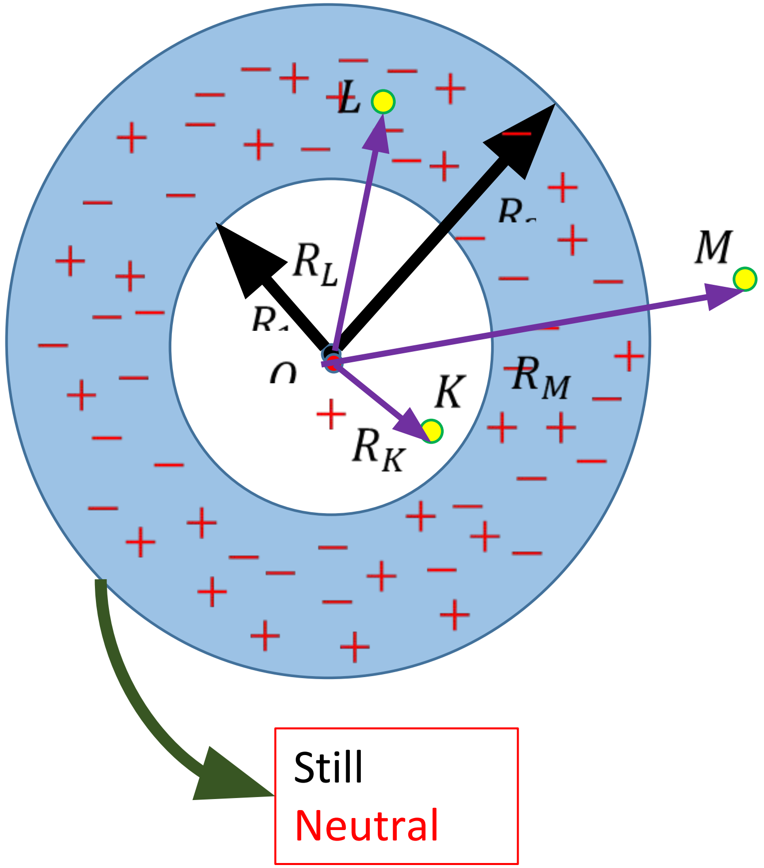


Consider a case when a neutral solid sphere (whose inner and outer radius are R_1 and R_2, respectively), contains a point charge Q located at its center as shown in the figure above.

- **B** Let us assume that the sphere is made from an insulator material whose permittivity is ϵ_o. Determine the electric field as a function of radius r from the center! Express your answer only in terms of Q, r and ϵ_o!
- **C** Let us assume that the sphere is made from a metal conductor. Determine the electric field as a function of radius r from the center! Express your answer only in terms of Q, r and ϵ_o!



[3b] Charged Hollow-Sphere
Insulator
Initially Neutral
Initially Neutral
Cross-section view of
Hollow-Sphere
Permitivity ϵ_0
Misal Q
positive
Misal Q
negative
Still
Neutral
Still
Neutral



Karena electron pada bahan insulator tidak dapat bergerak bebas, maka walaupun di pusat bola diberi muatan titik sebesar Q, electron pada bahan insulator tersebut posisi nya tetap, sehingga satu – satu nya muatan yang berkontribusi pada muncul nya medan listrik hanya muatan titik Q, sehingga:

- $r = R_K \rightarrow r < R_1$ dan $r = R_M \rightarrow r > R_2$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

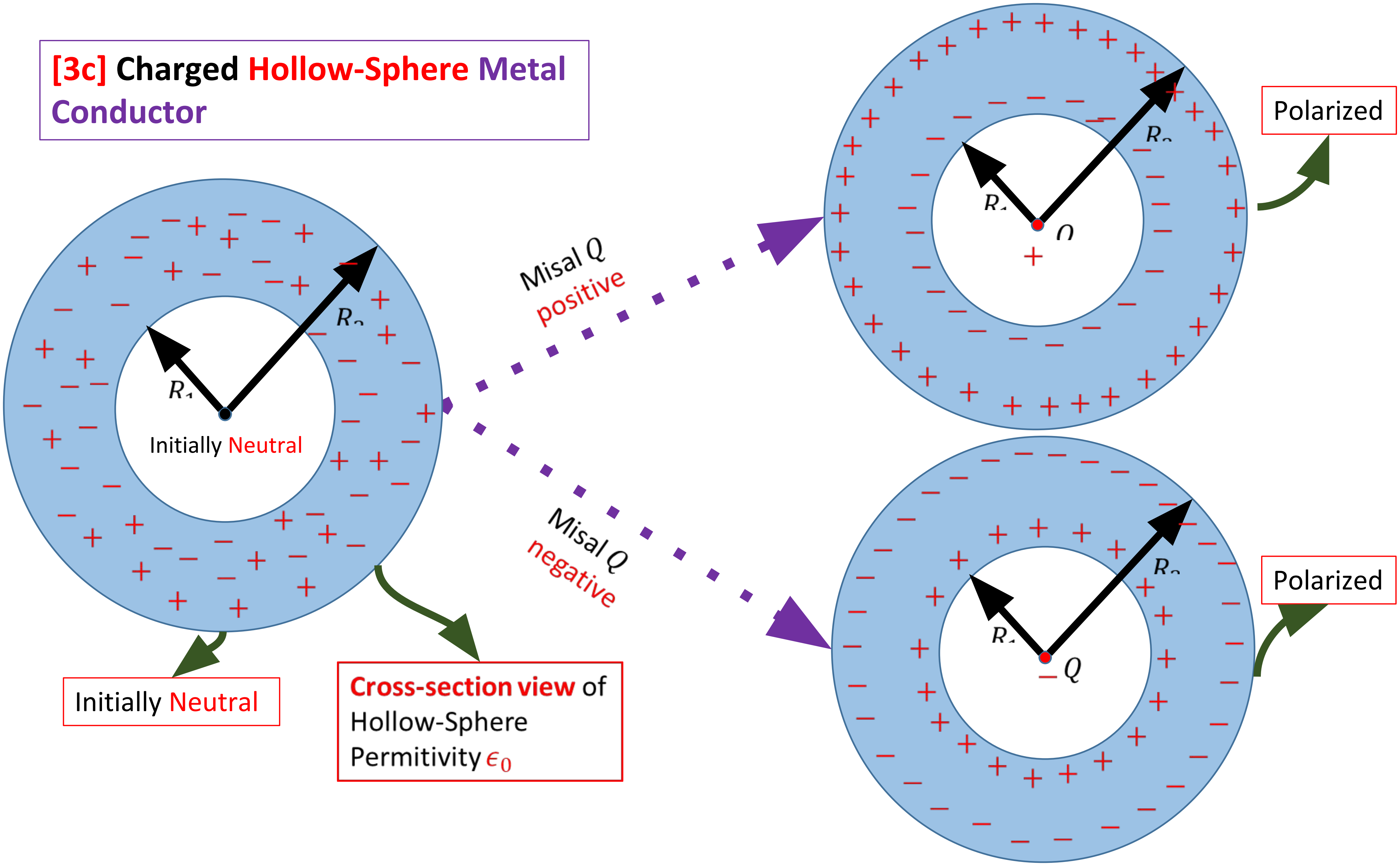
- $r = R_L \rightarrow R_1 \leq r \leq R_2$

$$E = \frac{1}{4\pi\epsilon_{insulator}}\frac{Q}{r^2}$$

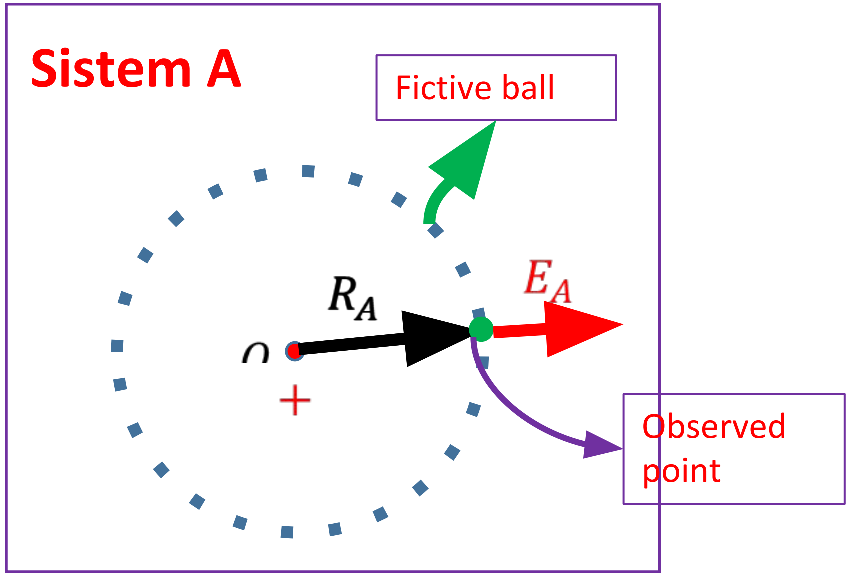
Dengan $\epsilon_{insulator} = \epsilon_0$ maka diperoleh:

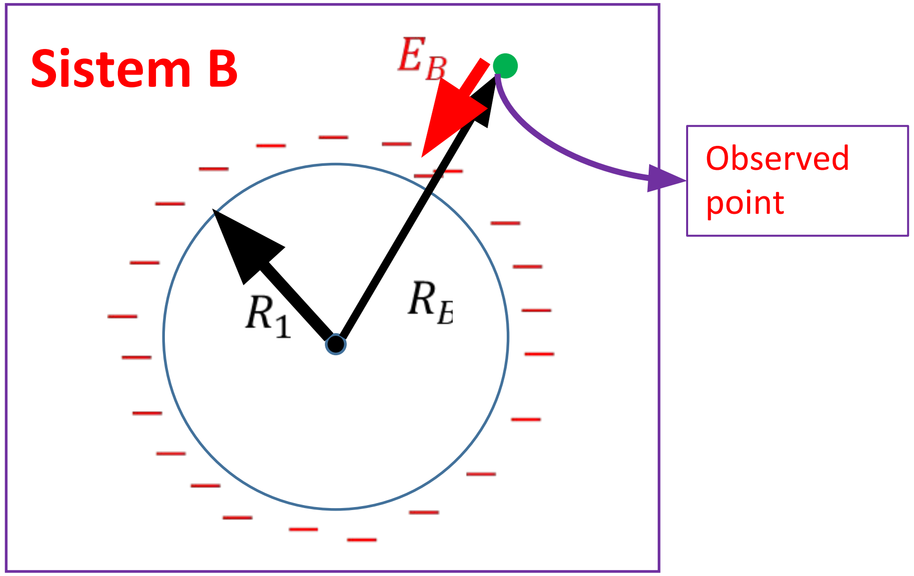
$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, \qquad \forall r \in \mathbb{R}^+$$

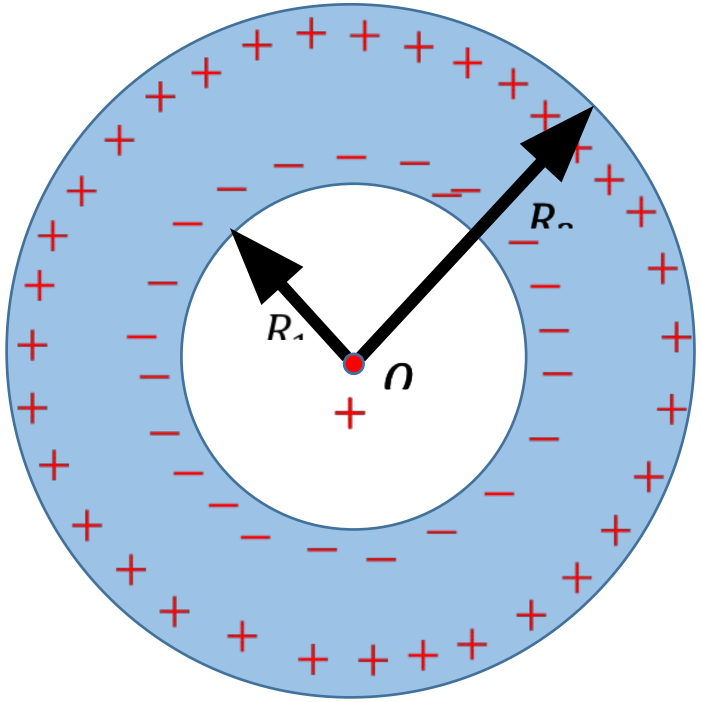
[3c] Charged Hollow-Sphere Metal Conductor



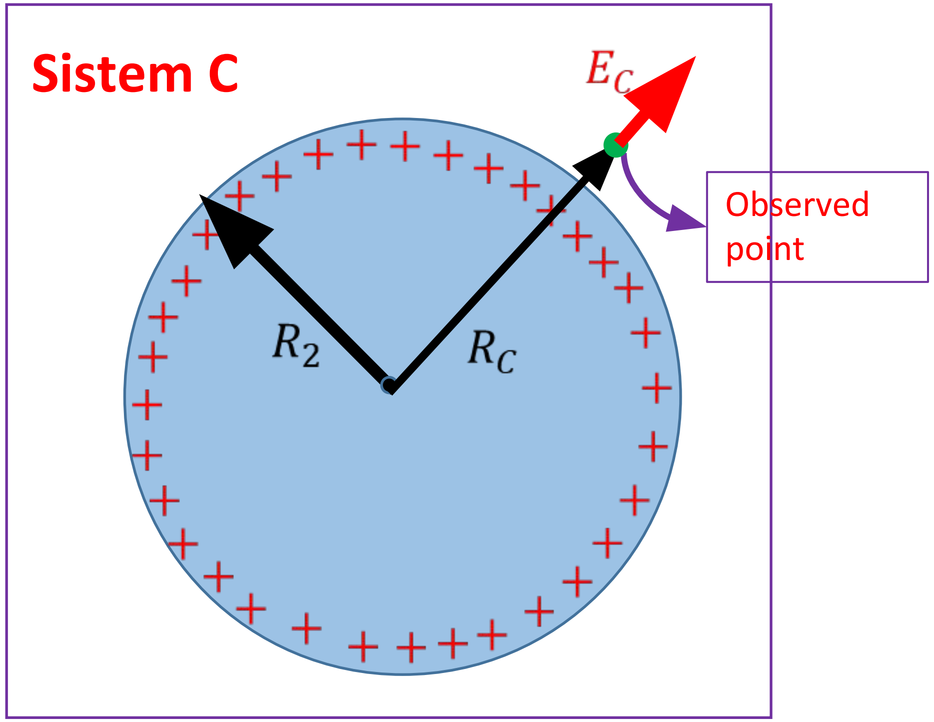
Analisa pada positive Q

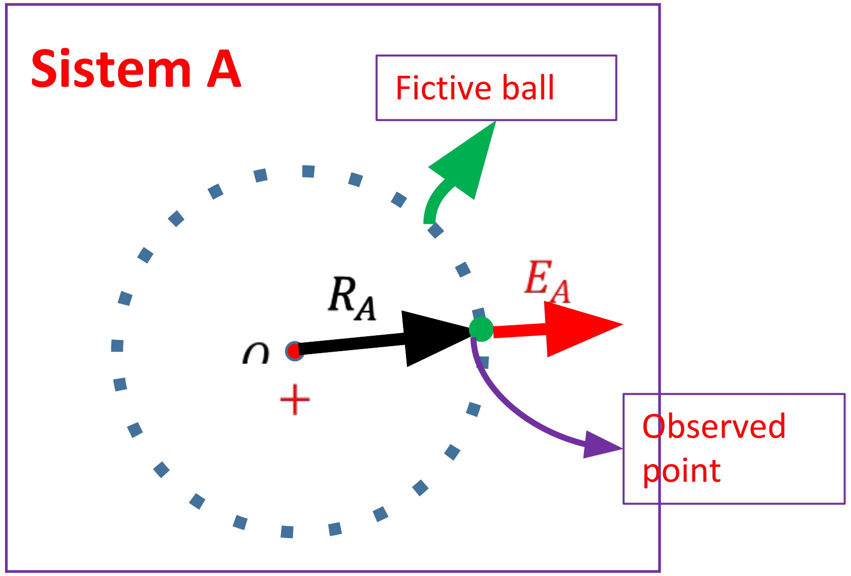






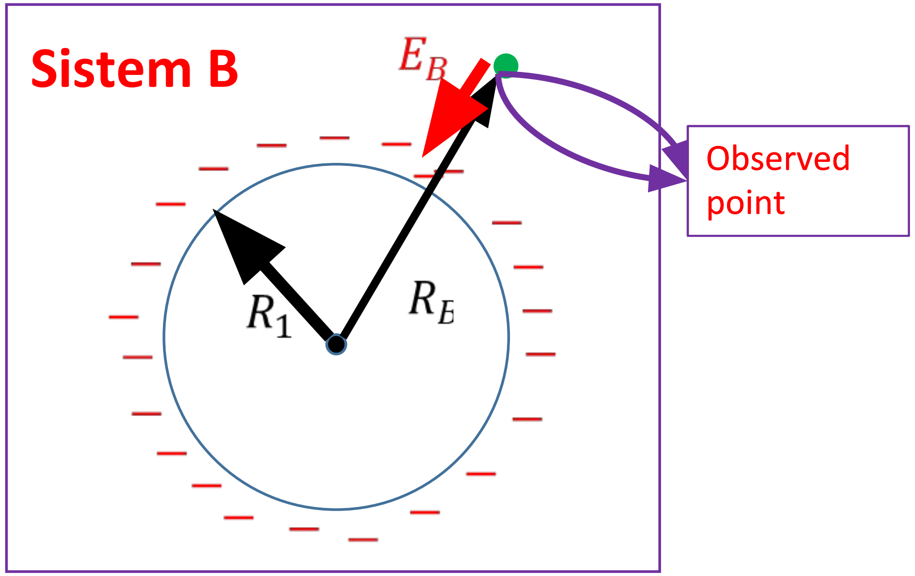
NOTE:
Pemisahan ketiga system ini **hanya untuk analisa**, karena system B dan C terjadi **karena adanya system A** pada konduktor, adapun jika ketiga system tersebut dipisah secara kenyataan, maka system B dan C **tidak akan terjadi**, karena muatan (+) dan (-) akan kembali saling berikatan





Medan listrik oleh partikel bermuatan:

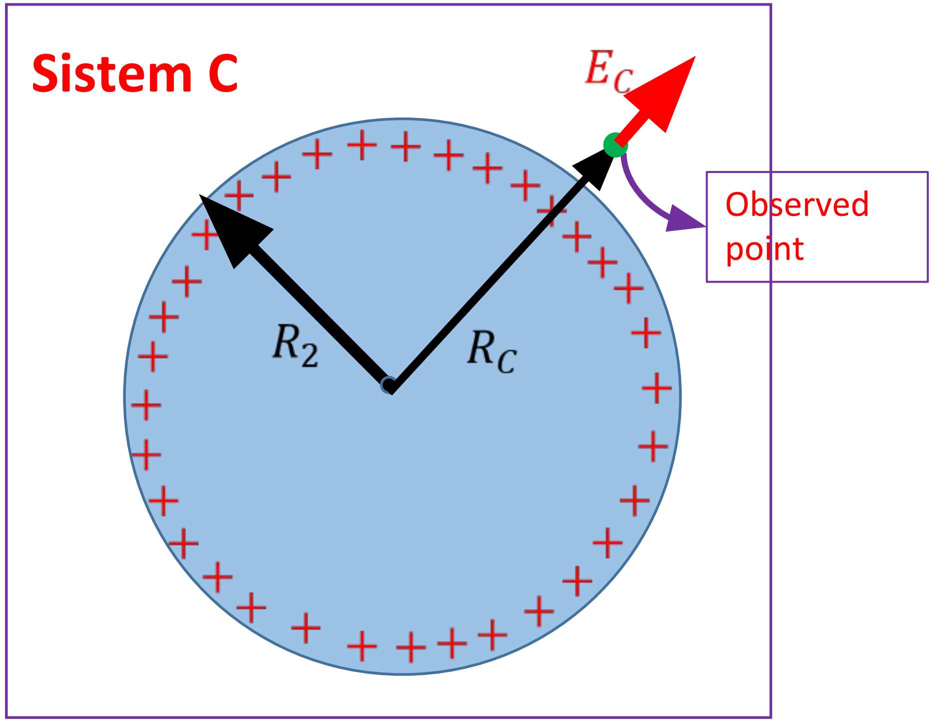
$$E_A = \frac{1}{4\pi\epsilon_0}\frac{Q}{R_A^2}\boldsymbol{r}, \qquad for\ R_A \neq 0$$



Medan listrik oleh **charged (only)-surface ball**:

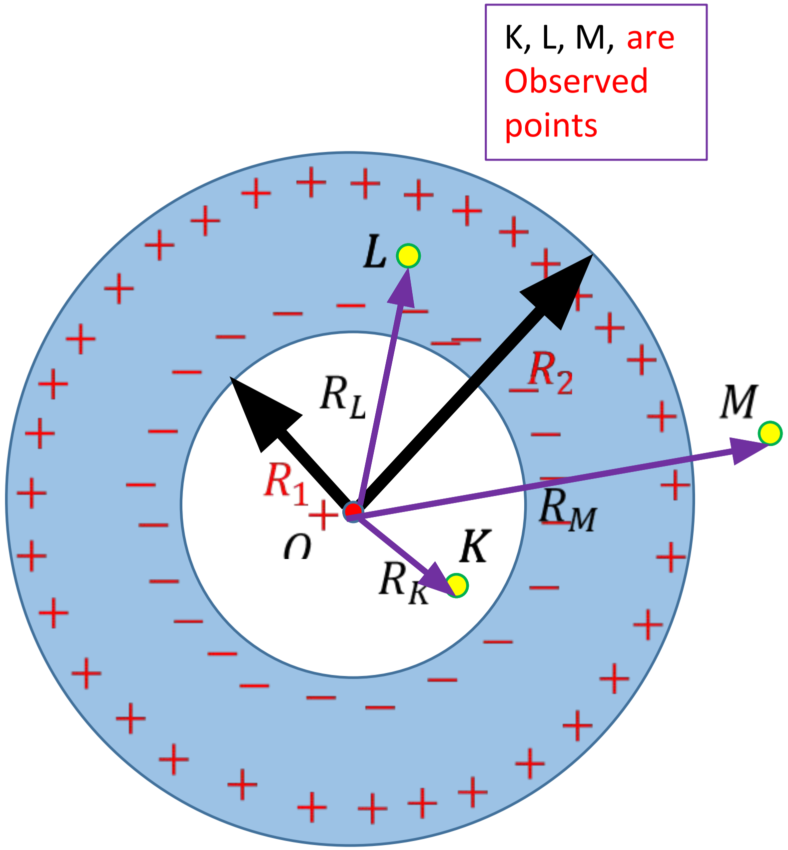
$$E_B = \begin{cases} 0, & for\ R_B \leq R_1 \\ -\frac{1}{4\pi\epsilon_0}\frac{Q}{R_B^2}\boldsymbol{r}, & for\ R_B > R_1 \end{cases}$$

$\boldsymbol{r}$ adalah vector satuan ke arah radial dalam system koordinat lingkaran



Medan listrik oleh **charged (only)-surface ball**:

$$E_C = \begin{cases} 0, & for\ R_C \leq R_2 \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{R_C^2}\boldsymbol{r}, & for\ R_C > R_2 \end{cases}$$



Jika kita ingin mengamati nilai medan listrik pada K:

$$R_K \leq R_1 \ \text{ dan } \ R_K \leq R_2$$

$$E_K = E_A(R_K) + E_B(R_K) + E_C(R_K)$$

$$E_K = \frac{1}{4\pi\epsilon_0} \frac{Q}{R_K^2} \boldsymbol{r} + 0 + 0$$

$$E_K = \frac{1}{4\pi\epsilon_0} \frac{Q}{R_K^2} \boldsymbol{r}$$

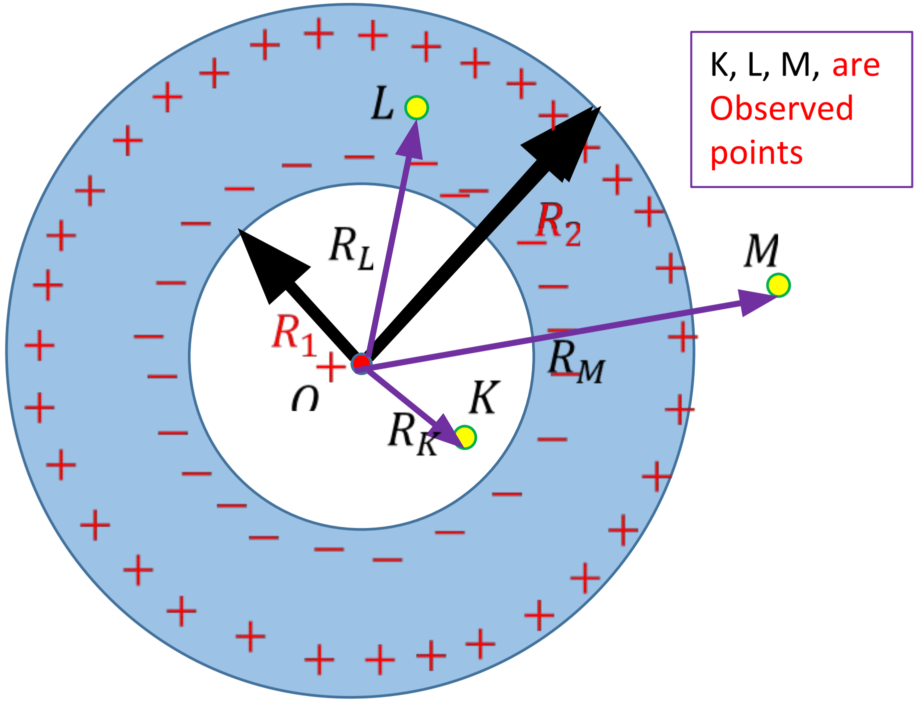
Jika kita ingin mengamati nilai medan listrik pada L:

$$R_L > R_1 \ \text{ dan } R_L \leq R_2$$

$$E_L = E_A(R_L) + E_B(R_L) + E_C(R_L)$$

$$E_L = \frac{1}{4\pi\epsilon_0} \frac{Q}{R_L^2} \boldsymbol{r} - \frac{1}{4\pi\epsilon_0} \frac{Q}{R_L^2} \boldsymbol{r} + 0$$

$$\boldsymbol{E_L} = 0$$



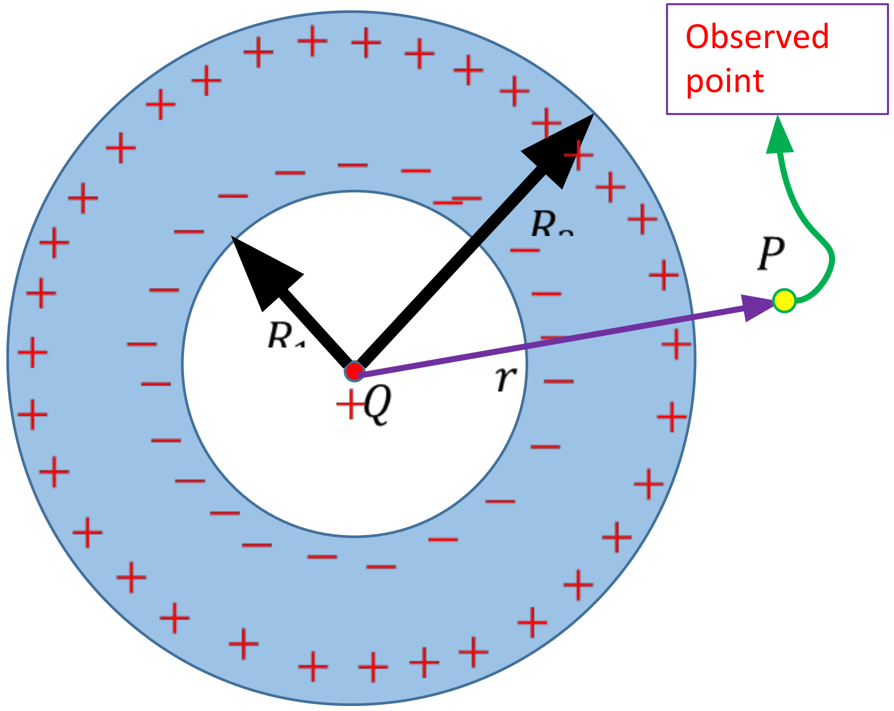
Jika kita ingin mengamati nilai medan listrik pada M:

$$R_M > R_1 \text{ dan } R_M > R_2$$

$$E_M = E_A(R_L) + E_B(R_L) + E_C(R_L)$$

$$E_M = \frac{1}{4\pi\epsilon_0}\frac{Q}{R_M^2}\boldsymbol{r} - \frac{1}{4\pi\epsilon_0}\frac{Q}{R_M^2}\boldsymbol{r} + \frac{1}{4\pi\epsilon_0}\frac{Q}{R_M^2}\boldsymbol{r}$$

$$E_M = \frac{1}{4\pi\epsilon_0}\frac{Q}{R_M^2}\boldsymbol{r}$$



Sehingga secara umum untuk hollow-sphere yang awalnya **netral**, kemudian diberikan muatan titik di pusat nya, akan diperoleh:

$$E_P = \begin{cases} 0, & \text{for } R_1 \le r \le R_2 \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\boldsymbol{r}, & \text{for } r < R_1 \text{ atau } r > R_2 \end{cases}$$