AMSC HW6

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1 Solution of Exercise 3

We want to prove that the average size of the non-giant component a vertex v belongs to is:

$$\langle s \rangle = 1 + \frac{zu^2}{[1-S][1-G_1'(u)]}$$

First, the generating function for the total number of vertices reachable from the given vertex v that does not belong to the giant component if there is one is given by

$$H_1(x) = xG_1(H_1(x)).$$

If there is no giant component then the generating function is given by $H_0(x) = xG_0(H_1(x))$. From the definition of H_0 we have $H_0(1) = 1$ and $H_0'(1) = \langle s \rangle$ (when there is no giant component). Here, note that a giant component may exists so we need to renormalize all probabilities for component sizes in the series $H_0(x)$:

$$h_k \to \frac{h_k}{H_0(1)}$$

Since $1 - S = G_0(u)$ and $u \equiv H_1(1)$ is the smallest non-negative solution of $u = G_1(u)$ then $H_0(1) = 1 - S$. Also,

$$\langle s \rangle = \frac{H_0'(1)}{H_0(1)}. \tag{1}$$

Next, we compute the derivative of H_0:

$$\begin{aligned} H_0'(x) &= G_0(H_1(x)) + xG_0'(H_1(x))H_1'(x) \\ \Rightarrow H_0'(1) &= G_0(H_1(1)) + xG_0'(H_1(1))H_1'(1) \\ &= G_0(u) + G_0'(u)H_1'(1) \end{aligned} \tag{2}$$

Similarly, the derivative of H_1 is

$$\begin{aligned} H_1'(x) &= G_1(H_1(x)) + xG_1'(H_1(x))H_1'(x) \\ \Rightarrow H_1'(1) &= G_1(H_1(1)) + xG_1'(H_1(1))H_1'(1) \\ &= G_1(u) + G_1'(u)H_1'(1) \end{aligned}$$

Hence,

$$H_1'(1) = \frac{G_1(u)}{1 - G_1'(u)}. \tag{3}$$

Plugging Eqs.(2),(3) into Eq.(1) we get

$$\begin{aligned}
\langle s \rangle &= \frac{H_0'(1)}{H_0(1)} \\
&= \frac{1}{H_0(1)}\Big(G_0(u) + G_0'(u)H_1'(1)\Big) \\
&= 1 + zu\frac{H_1'(1)}{H_0(1)} \\
&= 1 + \frac{zu^2}{[1-S][1-G_1'(u)]},
\end{aligned}$$

whence I used $G_0'(u) = zu$, where z is the mean degree. This derives from the fact that $G_1(u) = \frac{G_0'(u)}{G_0'(1)}$ and $G_0'(1) = z$ (Section IIA in the paper).

2 Solution of Exercise 1

The figures of both Ex1 and Ex2 are produced from hw6.m at https://github.com/kpantazis/hw6. To reproduce them you will need to download newt0.m file too (Newton's method for finding roots of $f(S)$ numerically.)

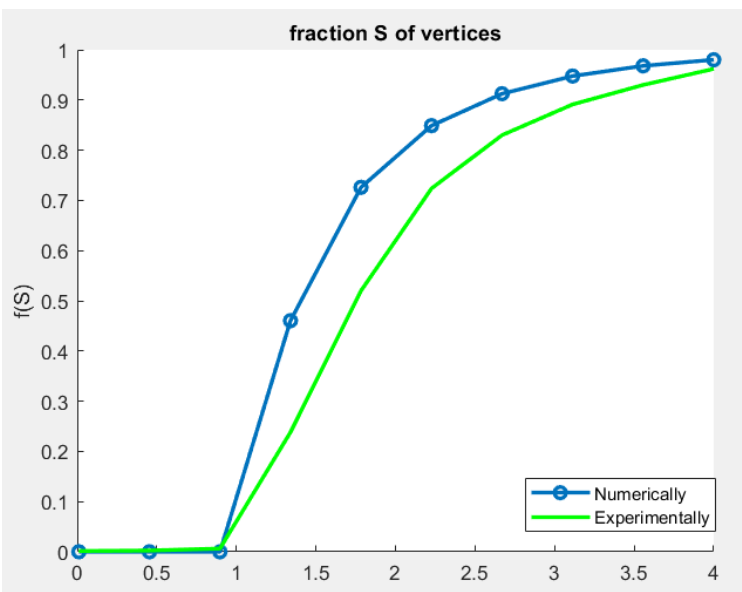

Figure 1: The fraction of vertices S of the giant component over the mean degree z taking values from 0 to 4. Blue line corresponds to finding S numerically using Newton's method. Green line corresponds to finding S using DFS and averaging over 100 Erdos-Renyi random graphs.

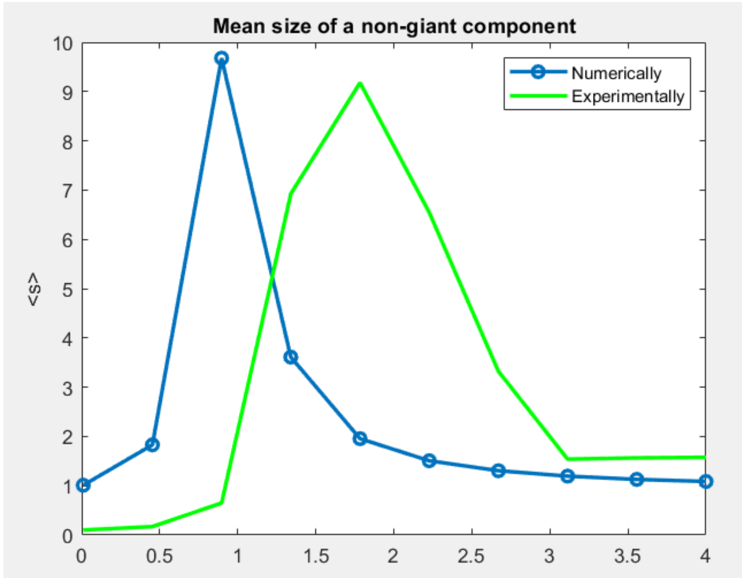

Figure 2: The mean size of a non-giant component $\langle s \rangle$ over the (constant) mean degree z ranging from 0 to 4. Blue line corresponds to $< s >$ using the optimal solution S from the previous part (Newton's method). Green line corresponds to $< s >$ using DFS and averaging over 100 Erdos-Renyi random graphs.

3 Solution of Exercise 2

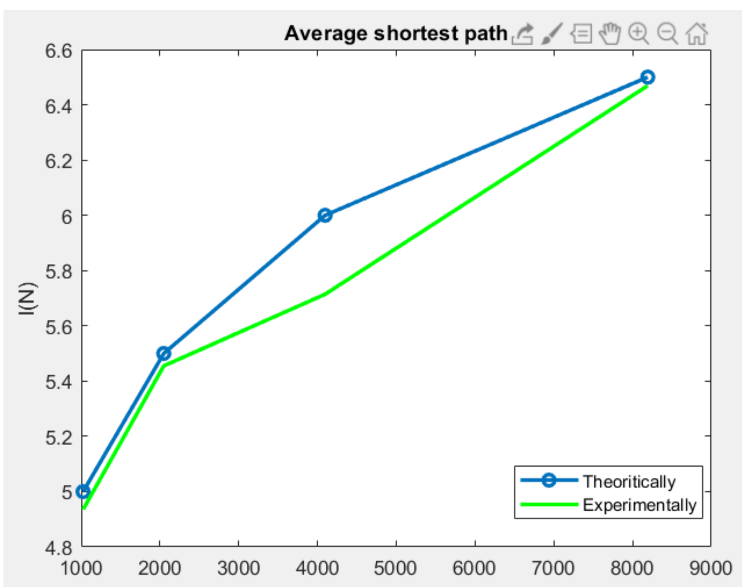

Figure 3: The average shortest path $\ell(N)$ over different values of the number of vertices N. The mean degree is set $z = 4$. Blue line corresponds to the theoritical value $\ell(N) = \frac{\log N}{\log z}$, while the green line corresponds to $\ell(n)$ by using BFS and averaging the shortest paths of $r = 100$ randomly chosen vertices.