The Effect of Bequests on Wealth Persistence and Inequality across Generations *

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Abstract

Put abstract here.

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1 Introduction

In this paper, we want to test the hypothesis of Wolff (2015, p. 5) that wealth transfers between generations "are equalizing in terms of theoverall distribution of household wealth." We test this hypothesis using a large-scale overlapping generations model similar to DeBacker et al. (2016) with an added distribution function for bequests calibrated from Evans et al. (2016).

2 Model

The dynamic general equilibrium (DGE) model we use is comprised of heterogeneous individuals, perfectly competitive firms, and a government with a balanced budget requirement. Individuals choose labor supply, consumption, and savings to maximize lifetime expected utility. And firms choose how much labor to hire and capital to rent to maximize profits every period. The government levies taxes on individuals and makes lump sum transfers to individuals according to a balanced budget constraint. As in DeBacker et al. (2016), we calibrate the individual tax functions from a microsimulation model that uses U.S. tax data.

2.1 Population dynamics and lifetime earnings profiles

We calibrate population dynamics in this model following the method of DeBacker et al. (2016).[1] We define $\omega_{s,t}$ as the number of individuals of age s alive at time t. A measure $\omega_{1,t}$ of individuals with heterogeneous working ability is born in each period t and live for up to $E+S$ periods, with $S \geq 4$. Individuals are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The individuals enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$. We model the population with individuals age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics.

[1] A more detailed description of the population dynamics can be found in Appendix A-1.

The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\begin{aligned}
\omega_{1,t+1} &= (1-\rho_0)\sum_{s=1}^{E+S} f_s\omega_{s,t} + i_1\omega_{1,t} \quad \forall t \\
\omega_{s+1,t+1} &= (1-\rho_s)\omega_{s,t} + i_{s+1}\omega_{s+1,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1
\end{aligned} \tag{1}$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific net immigration rate, ρ_s is an age specific mortality hazard rate,[2] and ρ_0 is an infant mortality rate. The total population in the economy N_t at any period is simply the sum of individuals in the economy, the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{3}$$

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{4}$$

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{5}$$

At birth, a fraction λ_j of the $\omega_{1,t}$ measure of new agents is randomly assigned to each of the J lifetime income groups, indexed by $j = 1, 2, ...J$, such that $\sum_{j=1}^{J} \lambda_j = 1$. Note that lifetime income is endogenous in the model, therefore we define lifetime income groups by a particular path of earnings abilities. For each lifetime income group, the measure $\lambda_j\omega_{s,t}$ of individuals' effective labor units (which we also call ability) evolve deterministically according to $e_{j,s}$. This gives a different life cycle profile of earnings to each lifetime income group. An individual's working ability evolves over its working-age lifetime $E+1 \leq s \leq E+S$ according to this age-

[2]The parameter ρ_s is the probability that a individual of age s dies before age $s+1$.

dependent deterministic process. The processes for the evolution of the population weights $\omega_{s,t}$ as well as lifetime earnings are exogenous inputs to the model.

Figure 1: Exogenous life cycle income ability paths $\log(e_{j,s})$ with $S = 80$ and $J = 7$

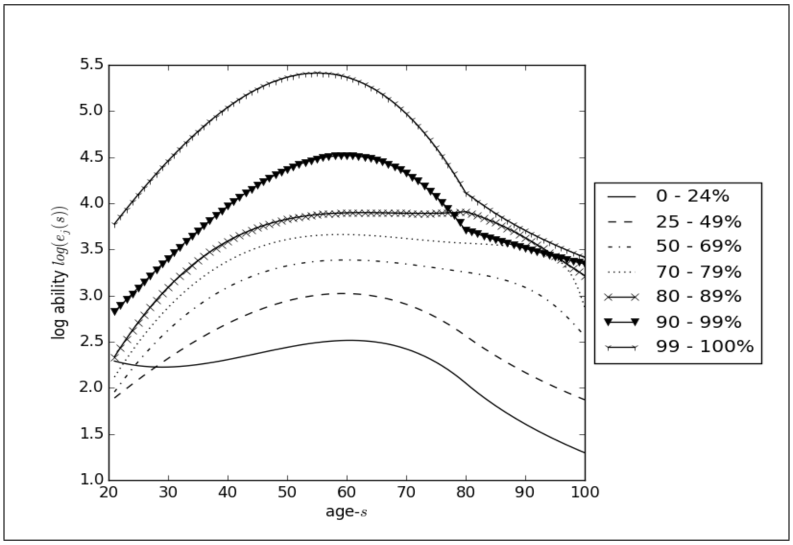


Figure 1 shows the calibrated trajectory of effective labor units (ability) $e_{j,s} \in \mathcal{E} \subset \mathbb{R}_{++}$ by age s for each type j for lifetime income distribution $\{\lambda_j\}_{j=1}^{7} = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$. We show effective labor units in logarithms because the difference in levels between the top one percent and the rest of the distribution is so large. These exogenous earnings processes are taken from DeBacker et al. (2015). All model individuals have the same time endowment and receive the same wage per effective labor unit, but some are endowed with more effective labor units. We utilize a measure of lifetime income, by using potential lifetime earnings, that allows us to define income groups in a way that accounts for the fact that earnings of individuals observed in the data are endogenous. It is in this way that we are able to calibrate the exogenous lifetime earnings profiles form the model with their data counterparts.

2.2 Individual problem

Individuals are endowed with a measure of time $\tilde{l}$ in each period t, and they choose how much of that time to allocate between labor $n_{j,s,t}$ and leisure $l_{j,s,t}$ in each period. That is, an individual's labor and leisure choice is constrained by his total time endowment, which constraint is identical across all individuals.

$$n_{j,s,t} + l_{j,s,t} = \tilde{l} \tag{6}$$

At time t, all age-s individuals with ability $e_{j,s}$ know the real wage rate, w_t, and know the one-period real net interest rate, r_t, on bond holdings, $b_{j,s,t}$, that mature at the beginning of period t. They also receive accidental and intentional bequests. They choose how much to consume $c_{j,s,t}$, how much to save for the next period by loaning capital to firms in the form of a one-period bond $b_{j,s+1,t+1}$, and how much to work $n_{j,s,t}$ in order to maximize expected lifetime utility of the following form,

$$\begin{aligned} U_{j,s,t} &= \sum_{u=0}^{E+S-s} \beta^u \left[\prod_{v=s}^{s+u-1}(1-\rho_v)\right] u\left(c_{j,s+u,t+u}, n_{j,s+u,t+u}, b_{j,s+u+1,t+u+1}\right) \\ \text{and} \quad & u\left(c_{j,s,t}, n_{j,s,t}, b_{j,s+1,t+1}\right) = \frac{\left(c_{j,s,t}\right)^{1-\sigma}-1}{1-\sigma}... \\ & \quad + e^{g_y t(1-\sigma)}\chi^n_s\left(b\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} + k\right) + \rho_s\chi^b_j\frac{\left(b_{j,s+1,t+1}\right)^{1-\sigma}-1}{1-\sigma} \\ & \qquad \forall j,t \quad \text{and } E+1 \leq s \leq E+S \end{aligned} \tag{7}$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption and on intended (precautionary) bequests, $\beta \in (0,1)$ is the agent's discount factor, and the product term in brackets depreciates the individual's discount factor by the cumulative mortality rate. The disutility of labor term in the period utility function looks nonstandard, but is simply the upper right quadrant of an ellipse that closely approximates the standard CRRA utility of leisure functional form.[3] The term χ^n_s is a

[3] Appendix A-2 describes how the elliptical function closely matches the more standard constant Frisch elasticity disutility of labor of the form $-\frac{(n_{j,s,t})^{1+\theta}}{1+\theta}$. This elliptical utility function forces an interior solution that automatically respects both the upper and lower bound of labor supply, which greatly simplifies the computation of equilibrium. In addition, the elliptical disutility of labor has

constant term that varies by age s influencing the disutility of labor relative to the other arguments in the period utility function,[4] and g_y is a constant growth rate of labor augmenting technological progress, which we explain in Section 2.3.[5]

The last term in (7) incorporates a warm-glow bequest motive in which individuals value having savings to bequeath to the next generation in the chance they die before the next period. Including this term is essential to generating the positive wealth levels across the life cycle and across abilities that exist in the data. In addition, the term χ^b_j is a constant term that varies by lifetime income group j influencing the marginal utility of bequests, $b_{j,s+1,t+1}$ relative to the other arguments in the period utility function. Allowing the χ^b_j scale parameter on the warm glow bequest motive vary by lifetime income group is critical for matching the distribution of wealth. As was mentioned in Section 2.1, individuals in the model have no income uncertainty because each lifetime earnings path $e_{j,s}$ deterministic, model agents thus hold no precautionary savings. Calibrating the χ^b_j for each income group j captures in a reduced form way some of the characteristics that individual income risk provides.

The parameter $\sigma \geq 1$ is the coefficient of relative risk aversion on bequests, and the mortality rate ρ_s appropriately discounts the value of this term.[6] Note that, because of this bequest motive, individuals in the last period of their lives ($s = S$) will die with positive savings $b > 0$. Also note that the CRRA utility of bequests term prohibits negative wealth holdings in the model, but is not a strong restriction since none of the wealth data for the lifetime income group j and age s cohorts is negative except for the lowest quartile.

The per-period budget constraints for each agent normalized by the price of con-

a Frisch elasticity that asymptotes to a constant rather than increasing to infinity as it does in the CRRA case. For a more in-depth discussion see Evans and Phillips (2015)

[4]DeBacker et al. (2015) calibrate χ^n_s and χ^b_j to match average labor hours by age and some moments of the distribution of wealth.

[5]The term with the growth rate $e^{g_y t(1-\sigma)}$ must be included in the period utility function because consumption and bequests will be growing at rate g_y and this term stationarizes the individual Euler equation by making the marginal disutility of labor grow at the same rate as the marginal benefits of consumption and bequests. This is the same balanced growth technique as that used in Mertens and Ravn (2011).

[6]It is necessary for the coefficient of relative risk aversion σ to be the same on both the utility of consumption and the utility of bequests. If not, the resulting Euler equations are not stationarizable.

sumption are the following,

$$
\begin{aligned}
c_{j,s,t} + b_{j,s+1,t+1} \leq (1 + r_t)\, b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \frac{\varrho_{j,s} BQ_t}{\lambda_j \omega_{s,t}} - T_{s,t} \\
\text{where} \quad b_{j,E+1,t} = 0 \quad \text{for} \quad E+1 \leq s \leq E+S \quad \forall j, t
\end{aligned}
\tag{8}
$$

where $\lambda_j \omega_{s,t}$ is the population of individuals of type j and age s in period t. Note that the price of consumption is normalized to one, so w_t is the real wage and r_t is the net real interest rate. The term BQ_t represents total bequests from individuals any individuals who died in period $t-1$, and $\varrho_{j,s}$ is the exogenous percent of total bequests that get distributed to agents of type j and age s. $T_{s,t}$ is a function representing net taxes paid, which we specify more fully below in equation (11).

We follow Evans et al. (2016) and calibrate the joint distribution of recipients of total bequests $\varrho_{j,s}$. The matrix representation of this joint distribution is the following,

$$
\boldsymbol{\varrho} = \begin{bmatrix} \varrho_{1,1} & \varrho_{1,2} & \cdots & \varrho_{1,S} \\ \varrho_{2,1} & \varrho_{2,2} & \cdots & \varrho_{2,S} \\ \vdots & \vdots & \ddots & \vdots \\ \varrho_{J,1} & \varrho_{J,2} & \cdots & \varrho_{J,S} \end{bmatrix} \tag{9}
$$

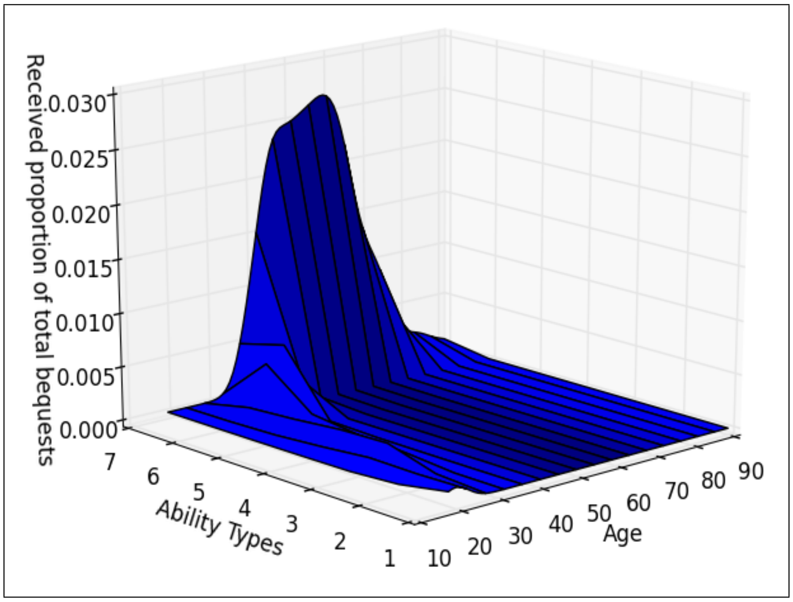
with typical element $\varrho_{j,s}$ and $\sum_{j=1}^{J}\sum_{s=E+1}^{E+S}\varrho_{j,s} = 1$. Figure 2 shows our calibrated joint distribution of bequest recipients.

Because the form of the period utility function in (7) ensures that $b_{j,s,t} > 0$ for all j, s, and t, total bequests will always be positive $BQ_t > 0$ for all t.

$$
BQ_{t+1} = (1 + r_{t+1}) \sum_{s=E+1}^{E+S} \sum_{j=1}^{J} \lambda_j \rho_s \omega_{s,t} b_{j,s+1,t+1} \quad \forall t \tag{10}
$$

In addition to each the budget constraint in each period, the utility function (7) imposes nonnegative consumption through infinite marginal utility, and the elliptical utility of leisure ensures individual labor and leisure must be strictly nonnegative $n_{j,s,t}, l_{j,s,t} > 0$. Because individual savings or wealth is always strictly positive, the

Figure 2: Multivariate kernel density estimate of joint distribution of bequest recipients



aggregate capital stock is always positive.[7] An interior solution to the individual's problem (7) is assured.

In reality, each household is subject to many different taxes, all of which cannot be modeled in a DGE framework. It is the net tax liability function $T_{s,t}$ that we estimate from the microsimulation model output. This output includes information on all federal individual income and payroll taxes in the U.S. tax code. We also assume that every individual also receives an equal lump sum transfer T_t^H which is generated from a balanced budget constraint on the government. Let x represent stationary labor income, and let y represent stationary capital income. We represent the net tax liability function as an effective tax rate times total labor and capital income.

$$\begin{aligned} &T_{s,t}(x,y) = \tau_{s,t}(x,y)\big(x+y\big) - T_t^H \\ &\quad\text{where}\quad x \equiv \frac{w_t e_{j,s} n_{j,s,t}}{e^{g_y t}} \quad\text{and}\quad y \equiv \frac{r_t b_{j,s,t}}{e^{g_y t}} \end{aligned} \tag{11}$$

Note that the both the total tax liability function $T_{s,t}(x,y)$ and the effective tax rate

[7] An alternative would be to allow for individual borrowing as long as an aggregate capital constraint $K_t > 0$ for all t is satisfied.

functions $\tau_{s,t}(x,y)$ are functions of stationarized labor income x and capital income y, separately.[8]

The solution to the lifetime maximization problem (7) of individual with ability j subject to the per-period budget constraint (8) and the specification of taxes in (11) is a system of $2S$ Euler equations. The S static first order conditions for labor supply $n_{j,s,t}$ are the following,

$$\begin{aligned}
(c_{j,s,t})^{-\sigma}\left(w_t e_{j,s} - \frac{\partial T_{s,t}}{\partial n_{j,s,t}}\right) = e^{g_y t(1-\sigma)}\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{v-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^v\right]^{\frac{1-v}{v}} \\
\forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S \qquad (12) \\
\text{where} \quad c_{j,s,t} = (1+r_t)\,b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \frac{\varrho_{j,s} BQ_t}{\lambda_j \omega_{s,t}} - b_{j,s+1,t+1} - T_{s,t} \\
\text{and} \quad b_{j,E+1,t} = 0 \quad \forall j,t
\end{aligned}$$

where the marginal tax rate with respect to labor supply $\frac{\partial T_{s,t}}{\partial n_{j,s,t}}$ is described in Appendix A-3.

An individual also has $S-1$ dynamic Euler equations that govern his saving decisions, $b_{j,s+1,t+1}$, with the included precautionary bequest saving in case of unexpected death. These are given by:

$$\begin{aligned}
(c_{j,s,t})^{-\sigma} = \rho_s\chi^b_j\big(b_{j,s+1,t+1}\big)^{-\sigma} + \beta(1-\rho_s)(c_{j,s+1,t+1})^{-\sigma}\left[(1+r_{t+1}) - \frac{\partial T_{s+1,t+1}}{\partial b_{j,s+1,t+1}}\right] \\
\forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S-1
\end{aligned} \qquad (13)$$

where the marginal tax rate with respect to savings $\frac{\partial T_{s,t}}{\partial b_{j,s,t}}$ is also described in Appendix A-3. Lastly, Each individual also has one static first order condition for the last period of life $s=E+S$, which governs how much to bequeath to the following generation given that the individual will die with certainty. This condition is simply equation

[8]The estimation of these tax functions $\tau_{s,t}(x,y)$ and $T_{s,t}(x,y)$ are described in detail in DeBacker et al. (2016).

(13) with $\rho_s = 1$.

$$(c_{j,E+S,t})^{-\sigma} = \chi^b_j(b_{j,E+S+1,t+1})^{-\sigma} \quad \forall j,t \tag{14}$$

Define $\hat{\mathbf{\Gamma}}_t$ as the distribution of stationary individual savings across individuals at time t, including the intentional bequests of the oldest cohort.

$$\hat{\mathbf{\Gamma}}_t \equiv \Big\{\big\{\hat{b}_{j,s,t}\big\}_{j=1}^{J}\Big\}_{s=E+2}^{E+S+1} \quad \forall t \tag{15}$$

As will be shown in Section 2.5, the state in every period t for the entire equilibrium system described in the stationary, non-steady-state equilibrium characterized in Definition 2 is the stationary distribution of individual savings $\hat{\mathbf{\Gamma}}_t$ from (15). Because individuals must forecast wages, interest rates, and aggregate bequests received in every period in order to solve their optimal decisions and because each of those future variables depends on the entire distribution of savings in the future, we must assume some individual beliefs about how the entire distribution will evolve over time. Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\hat{\mathbf{\Gamma}}^{\boldsymbol{e}}_{\boldsymbol{t+u}} = \Omega^u\left(\hat{\mathbf{\Gamma}}_{\boldsymbol{t}}\right) \quad \forall t, \quad u \geq 1 \tag{16}$$

where the e superscript signifies that $\hat{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[9]

2.3 Firm problem

A unit measure of identical, perfectly competitive firms exist in the economy. The representative firm is characterized by the following Cobb-Douglas production technology,

$$Y_t = ZK_t^\alpha\left(e^{g_y t}L_t\right)^{1-\alpha} \quad \forall t \tag{17}$$

[9] In Section 2.5 we will assume that beliefs are correct (rational expectations) for the stationary non-steady-state equilibrium in Definition 2.

where Z is the measure of total factor productivity, $\alpha \in (0,1)$ is the capital share of income, g_y is the constant growth rate of labor augmenting technological change, and L_t is aggregate labor measured in efficiency units. The firm uses this technology to produce a homogeneous output which is consumed by individuals and used in firm investment. The interest rate r_t paid to the owners of capital is the real interest rate net of depreciation. The real wage is w_t. The real profit function of the firm is the following.

$$\text{Real Profits} = ZK_t^{\alpha}\left(e^{g_y t}L_t\right)^{1-\alpha} - (r_t+\delta)K_t - w_t L_t \tag{18}$$

As in the individual budget constraint (8), note that the price output has been normalized to one.

Profit maximization results in the real wage, w_t, and the real rental rate of capital r_t being determined by the marginal products of labor and capital, respectively:

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \quad \forall t \tag{19}$$

$$r_t = \alpha\frac{Y_t}{K_t} - \delta \qquad \forall t \tag{20}$$

2.4 Government fiscal policy

The government is represented by a balanced budget constraint. The government collects taxes $\tau_{s,t}(x,y)(x+y)$ from all individuals and divides total revenues equally among all economically active individuals in the economy to determine the lump-sum transfer.

$$T_t^H = \frac{1}{\tilde{N}_t}\sum_s\sum_j \omega_{s,t}\lambda_j\tau_{s,t}(w_t e_{j,s}n_{j,s,t}, r_t b_{j,s,t})\big(w_t e_{j,s}n_{j,s,t} + r_t b_{j,s,t}\big) \tag{21}$$

Lump sum transfers have an impact on the distribution of income and wealth. However, if one constrains policy experiments to have the same steady-state revenue impact, the changes in inequality in economic outcomes due to changes in government transfers is equivalent in each policy experiment in the steady-state.

2.5 Market clearing and stationary equilibrium

Labor market clearing requires that aggregate labor demand L_t measured in efficiency units equal the sum of individual efficiency labor supplied $e_{j,s}n_{j,s,t}$. Capital market clearing requires that aggregate capital demand K_t equal the sum of capital investment by individuals $b_{j,s,t}$. Aggregate consumption C_t is defined as the sum of all individual consumptions, and aggregate investment is defined by the resource constraint $Y_t = C_t + I_t$ as shown in (24). That is, the following conditions must hold:

$$L_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{22}$$

$$K_t = \sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\omega_{s-1,t-1}\lambda_j b_{j,s,t} \quad \forall t \tag{23}$$

$$\begin{split}Y_t = C_t + K_{t+1} - (1-\delta)K_t \quad \forall t \\ \text{where} \quad C_t \equiv \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j c_{j,s,t}\end{split} \tag{24}$$

The usual definition of equilibrium would be allocations and prices such that individuals optimize (12), (13), and (14), firms optimize (19) and (20), and markets clear (22) and (23). However, the variables in the equations characterizing the equilibrium are potentially non-stationary due to the growth rate in the total population $g_{n,t}$ each period coming from the cohort growth rates in (1) and from the deterministic growth rate of labor augmenting technological change g_y in (17).

Table 1 characterizes the stationary versions of the variables of the model in terms of the variables that grow because of labor augmenting technological change, population growth, both, or none. With the definitions in Table 1, it can be shown that the equations characterizing the equilibrium can be written in stationary form in the following way. The static and intertemporal first-order conditions from the individual's optimization problem corresponding to (12), (13), and (14) are the following:

Table 1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{j,s,t} \equiv \frac{c_{j,s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	$n_{j,s,t}$
$\hat{b}_{j,s,t} \equiv \frac{b_{j,s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{BQ}_t \equiv \frac{BQ_t}{e^{g_y t}\tilde{N}_t}$	
$\hat{y}_{j,s,t} \equiv \frac{y_{j,s,t}}{e^{g_y t}}$		$\hat{C}_t \equiv \frac{C_t}{e^{g_y t}\tilde{N}_t}$	
$\hat{T}_{s,t} \equiv \frac{T_{j,s,t}}{e^{g_y t}}$			

[a] The interest rate r_t in (20) is already stationary because Y_t and K_t grow at the same rate. Individual labor supply $n_{j,s,t}$ is stationary.

$$(\hat{c}_{j,s,t})^{-\sigma}\left(\hat{w}_t e_{j,s} - \frac{\partial \hat{T}_{s,t}}{\partial n_{j,s,t}}\right) = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}$$
$$\forall j,t, \quad \text{and} \quad E+1 \leq s \leq E+S \tag{25}$$
$$\text{where} \quad \hat{c}_{j,s,t} = (1+r_t)\,\hat{b}_{j,s,t} + \hat{w}_t e_{j,s} n_{j,s,t} + \frac{\varrho_{j,s}\hat{BQ}_t}{\lambda_j \hat{\omega}_{s,t}} - e^{g_y}\hat{b}_{j,s+1,t+1} - \hat{T}_{s,t}$$
$$\text{and} \quad \hat{b}_{j,E+1,t} = 0 \quad \forall j,t$$

$$(\hat{c}_{j,s,t})^{-\sigma} = \ldots$$
$$e^{-g_y\sigma}\left(\rho_s\chi^b_j\left(\hat{b}_{j,s+1,t+1}\right)^{-\sigma} + \beta(1-\rho_s)(\hat{c}_{j,s+1,t+1})^{-\sigma}\left[1 + r_{t+1} - \frac{\partial \hat{T}_{s+1,t+1}}{\partial \hat{b}_{j,s+1,t+1}}\right]\right) \tag{26}$$
$$\forall j,t, \quad \text{and} \quad E+1 \leq s \leq E+S-1$$

$$(\hat{c}_{j,E+S,t})^{-\sigma} = \chi^b_j e^{-g_y\sigma}\left(\hat{b}_{j,E+S+1,t+1}\right)^{-\sigma} \quad \forall j,t \tag{27}$$

The stationary firm first order conditions for optimal labor and capital demand corresponding to (19) and (20) are the following.

$$\hat{w}_t = (1-\alpha)\frac{\hat{Y}_t}{\hat{L}_t} \quad \forall t \tag{28}$$

$$r_t = \alpha\frac{\hat{Y}_t}{\hat{K}_t} - \delta = \alpha\frac{Y_t}{K_t} - \delta \quad \forall t \tag{20}$$

And the two stationary market clearing conditions corresponding to (22) and (23)—with the goods market clearing by Walras' Law—are the following.

$$\hat{L}_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j e_{j,s} n_{j,s,t} \quad \forall t \tag{29}$$

$$\hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}}\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\hat{\omega}_{s-1,t-1}\lambda_j \hat{b}_{j,s,t}\right) \quad \forall t \tag{30}$$

where $\tilde{g}_{n,t}$ is the growth rate in the working age population between periods $t-1$ and t described in (5). The stationary version of the goods market clearing condition (aggregate resource constraint) is the following.

$$\hat{Y}_t = \hat{C}_t + e^{g_y}(1+\tilde{g}_{n,t+1})\hat{K}_{t+1} - (1-\delta)\hat{K}_t \quad \forall t \tag{31}$$

It is also important to note the stationary version of the characterization of total bequests $BQ_{j,t+1}$ from (10) and for the government budget constraint in (21).

$$\hat{BQ}_{t+1} = \left(\frac{1+r_{t+1}}{1+\tilde{g}_{n,t+1}}\right)\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\lambda_j\rho_s\hat{\omega}_{s,t}\hat{b}_{j,s+1,t+1} \quad \forall t \tag{32}$$

$$\hat{T}^H_t = \sum_s\sum_j\hat{\omega}_{s,t}\lambda_j\hat{T}_{s,t} \tag{33}$$

We can now define the stationary steady-state equilibrium for this economy in the following way.

Definition 1 (**Stationary steady-state equilibrium**). A non-autarkic stationary steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as constant allocations $n_{j,s,t} = \bar{n}_{j,s}$ and $\hat{b}_{j,s+1,t+1} = \bar{b}_{j,s+1}$ and constant prices $\hat{w}_t = \bar{w}$ and $r_t = \bar{r}$ for all j, s, and t such that the following conditions hold:

i. individuals optimize according to (25), (26), and (27),

ii. Firms optimize according to (28) and (20),

iii. Markets clear according to (29) and (30), and

iv. The population has reached its stationary steady state distribution $\bar{\omega}_s$ for all ages s, characterized in Appendix A-1.

The steady-state equilibrium is characterized by the system of $2JS$ equations and $2JS$ unknowns $\bar{n}_{j,s}$ and $\bar{b}_{j,s+1}$. Appendix A-4 details how to solve for the steady-state equilibrium.

The non-steady state equilibrium is characterized by $2JST$ equations and $2JST$ unknowns, where T is the number of periods along the transition path from the current state to the steady state. The definition of the stationary non-steady-state equilibrium is similar to Definition 1, with the stationary steady-state equilibrium definition being a special case of the stationary non-steady-state equilibrium.

Definition 2 (**Stationary non-steady-state equilibrium**). A non-autarkic stationary non-steady-state equilibrium in the overlapping generations model with S-period lived agents and heterogeneous ability $e_{j,s}$ is defined as allocations $n_{j,s,t}$ and $\hat{b}_{j,s+1,t+1}$ and prices $\hat{w}_t$ and r_t for all j, s, and t such that the following conditions hold:

i. individuals have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings, and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\hat{\mathbf{\Gamma}}_{t+u} = \hat{\mathbf{\Gamma}}^e_{t+u} = \Omega^u\left(\hat{\mathbf{\Gamma}}_t\right) \quad \forall t, \quad u \geq 1$$

ii. individuals optimize according to (25), (26), and (27)

iii. Firms optimize according to (28) and (20), and

iv. Markets clear according to (29) and (30).

We describe the methodology to compute the solution to the non-steady-state equilibrium to Appendix A-5. We use the equilibrium transition path solution to find effects of tax policies on macroeconomic variables over the budget window.

Table 2 shows the calibrated values for the exogenous variables and parameters. Note that the scale parameter χ^n_s takes on 80 values (one for each model age) that increase with age, representing an increasing disutility of labor that is not modeled anywhere else in the utility function. An hour of labor for an older person becomes more costly due to biological reasons related to aging. Such a parametrization helps to fit fact that hours worked decline much more sharply later in life than do hourly earnings.

Heterogeneity in the scale parameter multiplying useful in having the model generate a distribution of wealth similar to that observed in the data. Note that without such heterogeneity in this parameter, individuals at the high end of the earnings distribution in our model would not save as much as their real world counterparts given the deterministic earnings process in our model. They have no precautionary savings motive, only the warm-glow bequest motive for savings. One can view the assumption of heterogeneous utility weights as not just variation in preference across households, but also as reflecting differences in family size, expectations of income growth, or other variations that are not explicitly modeled here. We thus allow $\{\chi^b_j\}_{j=1}^7$ to take on seven values, one for each lifetime income group.

3 Experiments

Put experiments here. Look at inequality across the lifecycle dimension as well as across the income group dimension.

i. baseline model, Gini time path, other measures of inequality.

ii. Wealth tax paper parameterization (most persistent)

iii. Equal distribution of bequests

Table 2: List of exogenous variables and baseline calibration values

Symbol	Description	Value
$\hat{\boldsymbol{\Gamma}}_1$	Initial distribution of savings	$\bar{\boldsymbol{\Gamma}}$
N_0	Initial population	1
$\{\omega_{s,0}\}_{s=1}^{S}$	Initial population by age	(see App. A-1)
$\{f_s\}_{s=1}^{S}$	Fertility rates by age	(see App. A-1)
$\{i_s\}_{s=1}^{S}$	Immigration rates by age	(see App. A-1)
$\{\rho_s\}_{s=1}^{S}$	Mortality rates by age	(see App. A-1)
$\{e_{j,s}\}_{j,s=1}^{J,S}$	Deterministic ability process	(see DeBacker et al., 2015)
$\{\lambda_j\}_{j=1}^{J}$	Lifetime income group percentages	$[0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$
$\{\varrho_{j,s}\}_{j,s=1}^{J,S}$	Distribution of bequest recipients	(see Evans et al., 2016)
J	Number of lifetime income groups	7
S	Maximum periods in economically active individual life	80
E	Number of periods of youth economically outside the model	20
R	Retirement age (period)	65
$\tilde{l}$	Maximum hours of labor supply	1
β	Discount factor	$(0.96)^{\frac{80}{S}}$
σ	Coefficient of constant relative risk aversion	1.5
b	Scale parameter in utility of leisure	0.573
υ	Shape parameter in utility of leisure	2.856
k	constant parameter in utility of leisure	0.000
χ^n_s	Disutility of labor level parameters	[19.041, 76.623]
χ^b_j	Utility of bequests level parameters	$[9.264 \times 10^{-5}, 118, 648]$
Z	Level parameter in production function	1.0
α	Capital share of income	0.35
δ	Capital depreciation rate	$1 - (1 - 0.05)^{\frac{80}{S}} = 0.05$
g_y	Growth rate of labor augmenting technological progress	$(1 + 0.03)^{\frac{80}{S}} - 1 = 0.03$
T	Number of periods to steady state	160
ν	Dampening parameter for TPI	0.4

iv. Turn off bequest motive so that all bequests are accidental and only based on precautionary savings.

4 Conclusion

Put conclusion here.

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APPENDIX

A-1 Characteristics of exogenous population dynamics

In this appendix, we detail how we generate the exogenous population dynamics that are inputs to the model described in Section 2.1. We follow the approach of DeBacker et al. (2016).

Figure 3: Correspondence of model timing to data timing for model periods of one year

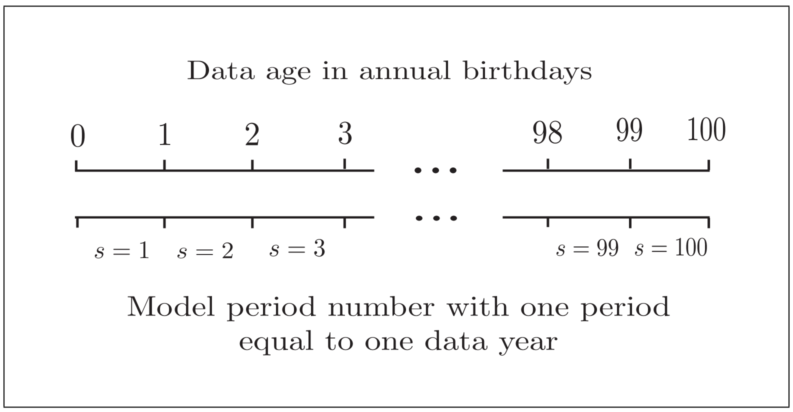


Figure 3 shows the correspondence between model periods and data periods. Period $s = 1$ corresponds to the first year of life between birth and when an individual turns one year old. We use this convention to match our model periods to those in the data.

A-1.1 Nonstationary and stationary population dynamics

We define $\omega_{s,t}$ as the number of individuals of age s alive at time t. A measure $\omega_{1,t}$ of individuals with heterogeneous working ability is born in each period t and live for up to $E+S$ periods, with $S \geq 4$.[10] Individuals are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The individuals enter the workforce and economy in period $E + 1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$. We model the population with individuals age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics.

[10]Theoretically, the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), we need S to be at least 4.

The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\begin{aligned} \omega_{1,t+1} &= (1-\rho_0)\sum_{s=1}^{E+S} f_s\omega_{s,t} + i_1\omega_{1,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1-\rho_s)\omega_{s,t} + i_{s+1}\omega_{s+1,t} \quad \forall t \quad \text{and} \quad 1\leq s\leq E+S-1 \end{aligned} \tag{1}$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific net immigration rate, ρ_s is an age specific mortality hazard rate,[11] and ρ_0 is an infant mortality rate. The total population in the economy N_t at any period is simply the sum of individuals in the economy, the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.

$$N_t \equiv \sum_{s=1}^{E+S}\omega_{s,t} \quad \forall t \tag{2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{3}$$

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S}\omega_{s,t} \quad \forall t \tag{4}$$

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{5}$$

We can transform the nonstationary equations in (1) into stationary laws of motion by dividing both sides by the total economically relevant population in the current period $\tilde{N}_t$ and then multiplying the left-hand-side of the equation by $\tilde{N}_{t+1}/\tilde{N}_{t+1}$,

$$\begin{aligned} \hat{\omega}_{1,t+1} &= \frac{(1-\rho_0)\sum_{s=1}^{E+S} f_s\hat{\omega}_{s,t} + i_1\hat{\omega}_{1,t}}{1+\tilde{g}_{n,t+1}} \quad \forall t \\ \hat{\omega}_{s+1,t+1} &= \frac{(1-\rho_s)\hat{\omega}_{s,t} + i_{s+1}\hat{\omega}_{s+1,t}}{1+\tilde{g}_{n,t+1}} \quad \forall t \quad \text{and} \quad 1\leq s\leq E+S-1 \end{aligned} \tag{A.1.1}$$

where $\hat{\omega}_{s,t}$ is the percent of the total economically relevant population $\tilde{N}_t$ in age cohort s in period t, and $\tilde{g}_{n,t+1}$ is the population growth rate between periods t and $t+1$ defined in (5).[12]

---

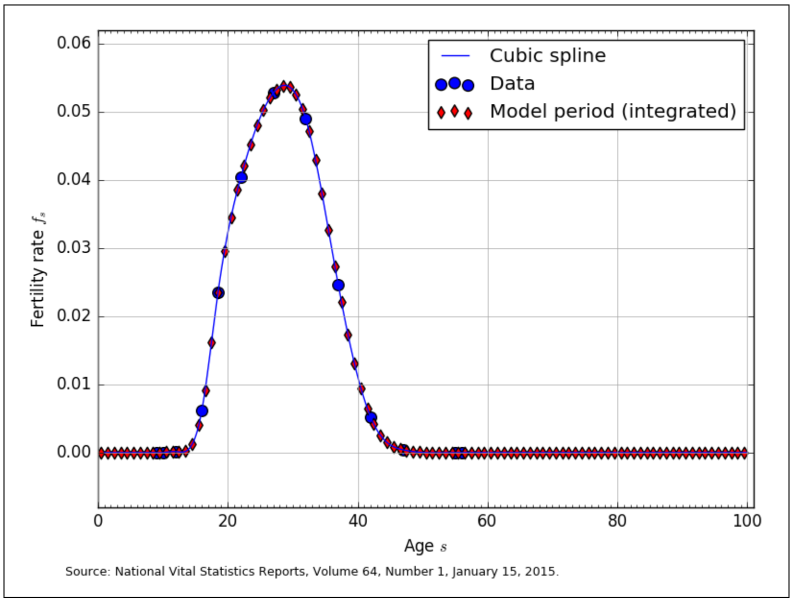
[11] The parameter ρ_s is the probability that a individual of age s dies before age $s+1$.

[12] Note in the specification of the stationary laws of motion (A.1.1) that $\sum_{s=1}^{E+S}\hat{\omega}_{s,t} > 1$ while $\sum_{s=E+1}^{E+S}\hat{\omega}_{s,t} = 1$. This is because in the model we only look at the economically relevant population $\hat{\omega}_{s,t}$ for $E+1\leq s\leq E+S$.

A-1.2 Fertility Rates

In this model, we assume that the fertility rates for each age cohort f_s are constant across time. However, this assumption is conceptually straightforward to relax. Our data for U.S. fertility rates by age come from Martin et al. (2015, Table 3, p. 18) National Vital Statistics Report, which is final fertility rate data for 2013. Figure 4 shows the fertility-rate data and the estimated average fertility rates for $E+S=100$.

Figure 4: Fertility rates by age (f_s) for $E+S=100$



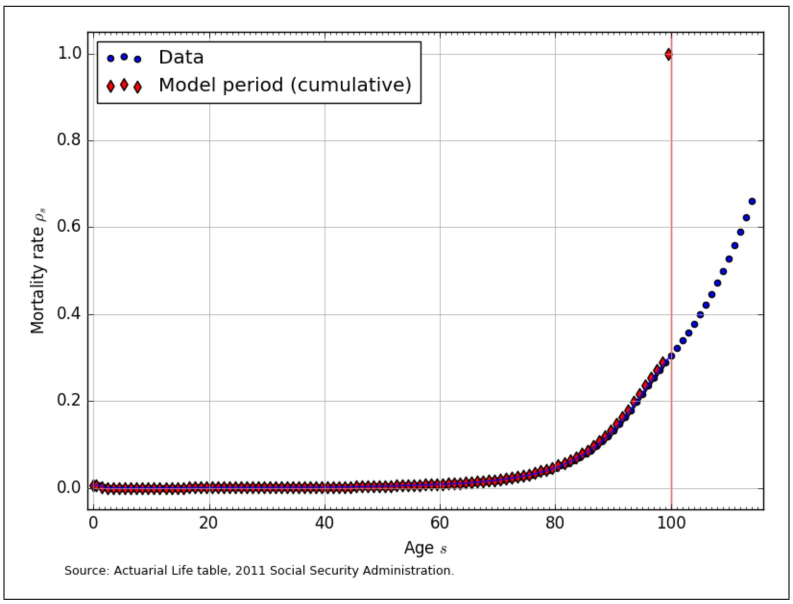
The large blue circles are the 2013 U.S. fertility rate data from Martin et al. (2015). These are 9 fertility rates $[0.3, 12.3, 47.1, 80.7, 105.5, 98.0, 49.3, 10.4, 0.8]$ that correspond to the midpoint ages of the following age (in years) bins $[10-14, 15-17, 18-19, 20-24, 25-29, 30-34, 35-39, 40-44, 45-49]$. In order to get our cubic spline interpolating function to fit better at the endpoints we added to fertility rates of zero to ages 9 and 10, and we added two fertility rates of zero to ages 55 and 56. The blue line in Figure 4 shows the cubic spline interpolated function of the data.

The red diamonds in Figure 4 are the average fertility rate in age bins spanning individuals born at the beginning of period 1 (time $=0$) and dying at the end of their 100th year. Let the total number of model years that an individual lives be `totpers`, which is just $E+S\leq 100$. Then the span from 0 to 100 is divided up into `totpers` bins of equal length. We calculate the average fertility rate in each of the `totpers` model-period bins as the average population-weighted fertility rate in that span. The red diamonds in Figure 4 are the average fertility rates displayed at the midpoint in each of the `totpers` model-period bins.

A-1.3 Mortality Rates

The mortality rates in our model ρ_s are a one-period hazard rate and represent the probability of dying within one year, given that an individual is alive at the beginning of period s. We assume that the mortality rates for each age cohort ρ_s are constant across time. The infant mortality rate of $\rho_0 = 0.00587$ comes from the 2015 U.S. CIA World Factbook. Our data for U.S. mortality rates by age come from the Actuarial Life Tables of the U.S. Social Security Administration (see Bell and Miller, 2015), from which the most recent mortality rate data is for 2011. Figure 5 shows the mortality rate data and the corresponding model-period mortality rates for $E + S = 100$.

Figure 5: Mortality rates by age (ρ_s) for $E + S = 100$



The mortality rates in Figure 5 are a population-weighted average of the male and female mortality rates reported in Bell and Miller (2015). Figure 5 also shows that the data provide mortality rates for ages up to 111-years-old. We truncate the maximum age in years in our model to 100-years old. In addition, we constrain the mortality rate to be 1.0 or 100 percent at the maximum age of 100.

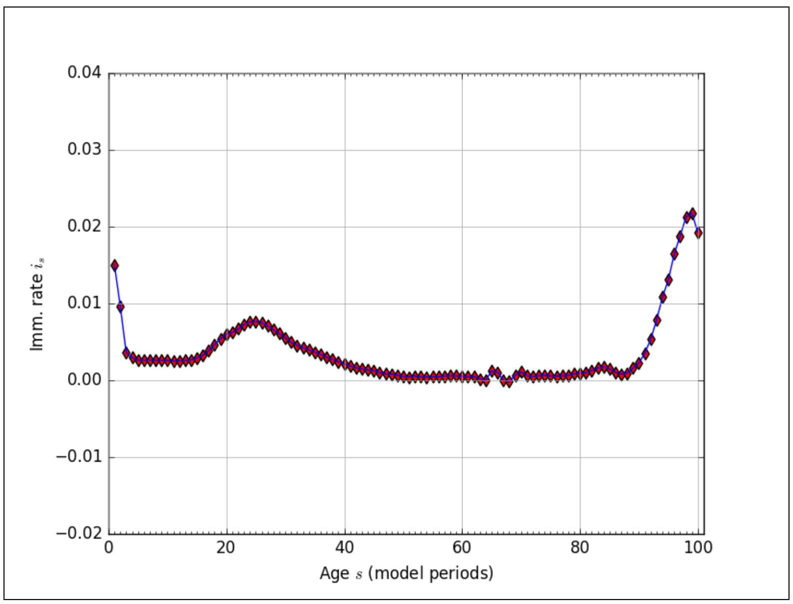
The red diamonds in Figure 5 are the interpolated mortality rates for individuals that live for $E + S = 100$ periods that range between ages 1 and 100. Our mortality rate interpolation function `get_mort()` takes as inputs the total number of periods and the range of data year ages that those periods cover.

A-1.4 Immigration Rates

Because of the difficulty in getting accurate immigration rate data by age, we estimate the immigration rates by age in our model i_s as the average residual that reconciles the current-period population distribution with next period's population distribution given fertility rates f_s and mortality rates ρ_s. Solving equations (1) for the immigration rate i_s gives the following characterization of the immigration rates in given population levels in any two consecutive periods $\omega_{s,t}$ and $\omega_{s,t+1}$ and the fertility rates f_s and mortality rates ρ_s.

$$\begin{aligned} i_1 &= \frac{\omega_{1,t+1} - (1-\rho_0)\sum_{s=1}^{E+S} f_s \omega_{s,t}}{\omega_{1,t}} \quad && \forall t \\ i_{s+1} &= \frac{\omega_{s+1,t+1} - (1-\rho_s)\omega_{s,t}}{\omega_{s+1,t}} \quad && \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{A.1.2}$$

Figure 6: Immigration rates by age (i_s), residual, $E + S = 100$



We calculate our immigration rates for three different consecutive-year-periods of population distribution data (2010 through 2013). Our four years of population distribution by age data come from Census Bureau (2015). The immigration rates i_s that we use in our model are the the residuals described in (A.1.2) averaged across the three periods. Figure 6 shows the estimated immigration rates generated from our `get_imm_resid()` function for $E + S = 100$ and given the fertility rates from Section A-1.2 and the mortality rates from Section A-1.3.

A-1.5 Population Steady State and Transition

This model requires information about mortality rates ρ_s in order to solve for the household's problem each period. It also requires the steady-state stationary population distribution $\bar{\omega}_s$ as well as the full transition path of the stationary population distribution $\hat{\omega}_{s,t}$ from the current state to the steady-state. To solve for the steady-state and the transition path of the stationary population distribution, we write the stationary population dynamic equations from (A.1.1) in matrix form.

$$\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{E+S-1,t+1} \\ \hat{\omega}_{E+S,t+1} \end{bmatrix} = \frac{1}{1+g_{n,t+1}} \times ...$$

$$\begin{bmatrix} (1-\rho_0)f_1 + i_1 & (1-\rho_0)f_2 & (1-\rho_0)f_3 & \dots & (1-\rho_0)f_{E+S-1} & (1-\rho_0)f_{E+S} \\ 1-\rho_1 & i_2 & 0 & \dots & 0 & 0 \\ 0 & 1-\rho_2 & i_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & i_{E+S-1} & 0 \\ 0 & 0 & 0 & \dots & 1-\rho_{E+S-1} & i_{E+S} \end{bmatrix} \begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{E+S-1,t} \\ \hat{\omega}_{E+S,t} \end{bmatrix} \tag{A.1.3}$$

We can write system (A.1.3) more simply in the following way.

$$\boldsymbol{\hat{\omega}}_{t+1} = \frac{1}{1+g_{n,t+1}}\boldsymbol{\Omega}\boldsymbol{\hat{\omega}}_t \quad \forall t \tag{A.1.4}$$

The stationary steady-state population distribution $\boldsymbol{\bar{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1+\bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (A.1.4).

$$(1+\bar{g}_n)\boldsymbol{\bar{\omega}} = \boldsymbol{\Omega}\boldsymbol{\bar{\omega}} \tag{A.1.5}$$

Proposition 1. If the age $s = 1$ immigration rate is $i_1 > -(1-\rho_0)f_1$ and the other immigration rates are strictly positive $i_s > 0$ for all $s \geq 2$ such that all elements of $\boldsymbol{\Omega}$ are nonnegative, then there exists a unique positive real eigenvector $\boldsymbol{\bar{\omega}}$ of the matrix $\boldsymbol{\Omega}$, and it is a stable equilibrium.

Proof. First, note that the matrix $\boldsymbol{\Omega}$ is square and non-negative. This is enough for a general version of the Perron-Frobenius Theorem to state that a positive real eigenvector exists with a positive real eigenvalue. This is not yet enough for uniqueness. For it to be unique by a version of the Perron-Fobenius Theorem, we need to know that the matrix is irreducible. This can be easily shown. The matrix is of the form

$$\boldsymbol{\Omega} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & 0 & \dots & 0 & 0 & 0 \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

Where each * is strictly positive. It is clear to see that taking powers of the matrix causes the sub-diagonal positive elements to be moved down a row and another row of positive entries is added at the top. None of these go to zero since the elements were all non-negative to begin with.

$$\boldsymbol{\Omega}^2 = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix} ; \quad \boldsymbol{\Omega}^{S+E-1} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

$$\boldsymbol{\Omega}^{S+E} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

Existence of an $m \in \mathbb{N}$ such that $\left(\boldsymbol{\Omega}^{\mathbf{m}}\right)_{ij} \neq 0 \quad (> 0)$ is one of the definitions of an irreducible (primitive) matrix. It is equivalent to saying that the directed graph associated with the matrix is strongly connected. Now the Perron-Frobenius Theorem for irreducible matrices gives us that the equilibrium vector is unique.

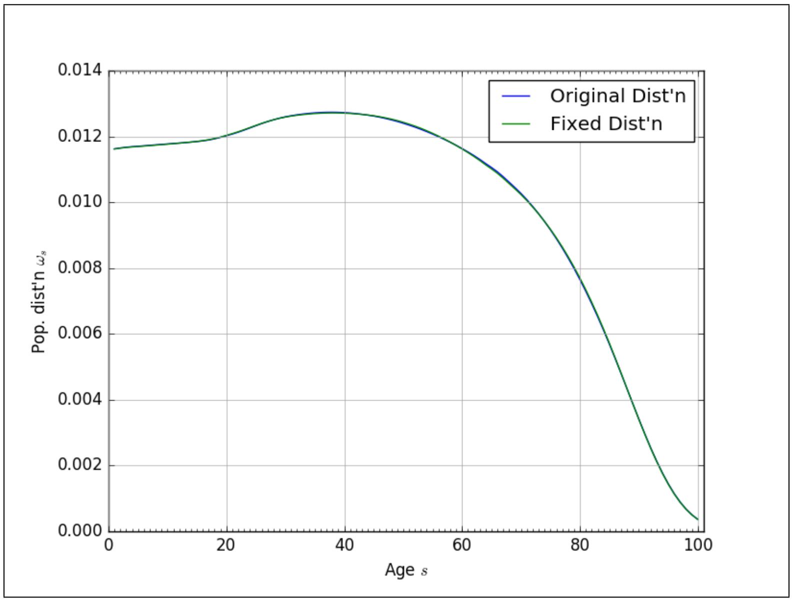
We also know from that theorem that the eigenvalue associated with the positive real eigenvector will be real and positive. This eigenvalue, p, is the Perron eigenvalue and it is the steady state population growth rate of the model. By the PF Theorem for irreducible matrices, $|\lambda_i| \leq p$ for all eigenvalues λ_i and there will be exactly h eigenvalues that are equal, where h is the period of the matrix. Since our matrix $\boldsymbol{\Omega}$ is aperiodic, the steady state growth rate is the unique largest eigenvalue in magnitude. This implies that almost all initial vectors will converge to this eigenvector under iteration. $\square$

For a full treatment and proof of the Perron-Frobenius Theorem, see Suzumura (1983). Because the population growth process is exogenous to the model, we calibrate it to annual age data for age years $s = 1$ to $s = 100$.

Figure 7 shows the steady-state population distribution $\bar{\boldsymbol{\omega}}$ and the population distribution after 120 periods $\hat{\boldsymbol{\omega}}_{120}$. Although the two distributions look very close to each other, they are not exactly the same.

Further, we found that the maximum absolute difference between the population levels $\hat{\omega}_{s,t}$ and $\hat{\omega}_{s,t+1}$ was 1.3852×10^{-5} after 160 periods. Although this sounds small, the population is still changing after many periods. For equilibrium convergence in our solution method over the transition path of the economy, we need things to be settled down to a steady state after T periods. To do this, we artificially impose that the population distribution in period $t = 120$ is the steady-state. As can be seen from Figure 7, this assumption is not very restrictive. Figure 8 shows the change in

Figure 7: Theoretical steady-state population distribution vs. population distribution at period $t = 120$



immigration rates that would make the period $t = 120$ population distribution equal be the steady-state. This change is not very big. The maximum absolute difference between any two corresponding immigration rates in Figure 8 is 0.0028.

The most recent year of population data comes from Census Bureau (2015) population estimates for both sexes for 2013. We those data and use the population transition matrix (A.1.4) to age it to the current model year of 2015. We then use (A.1.4) to generate the transition path of the population distribution over the time period of the model. Figure 9 shows the progression from the 2013 population data to the fixed steady-state at period $t = 120$. The time path of the growth rate of the economically active population $\tilde{g}_{n,t}$ is shown in Figure 10.

Figure 8: Original immigration rates vs. adjusted immigration rates to make fixed steady-state population distribution

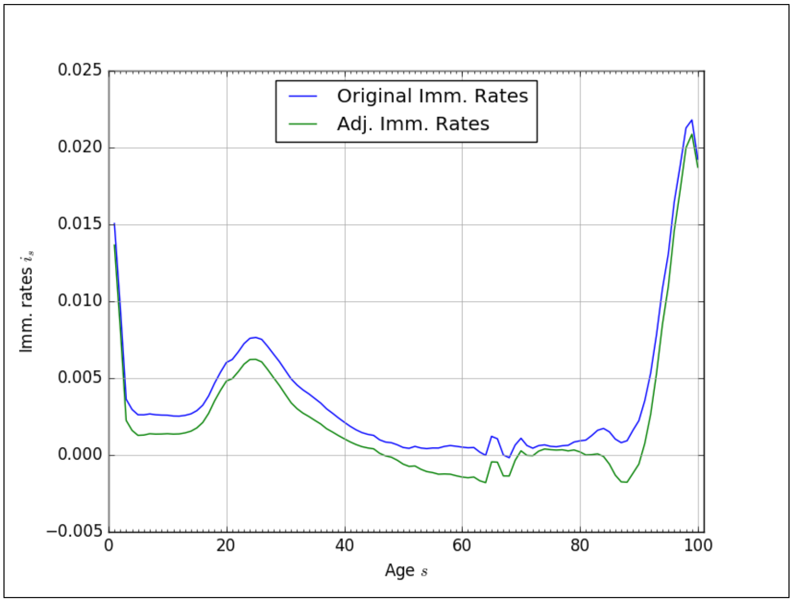


Figure 9: Stationary population distribution at periods along transition path

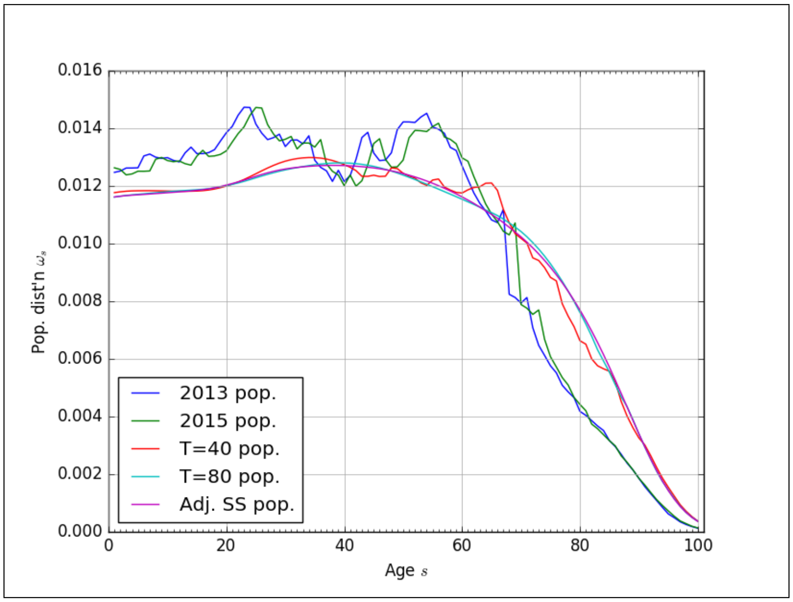
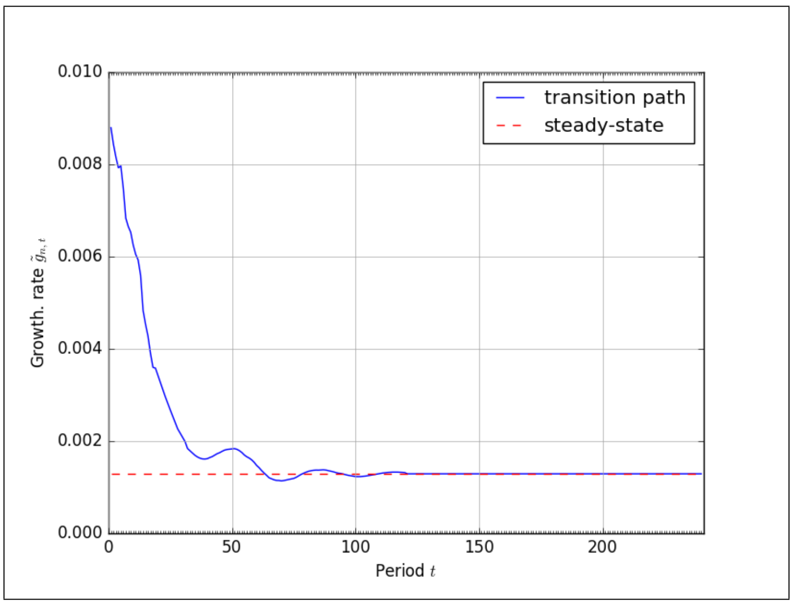


Figure 10: Time path of the population growth rate $\tilde{g}_{n,t}$



A-2 Derivation of elliptical disutility of labor supply

Evans and Phillips (2015) provide an exposition of the value of using elliptical disutility of labor specification as well as its relative properties to such standard disutility of labor functions such as constant relative risk aversion (CRRA) and constant Frisch elasticity (CFE). A standard specification of additively separable period utility in consumption and labor supply similar to one used in King et al. (1988) is the following,

$$u(c,n) = \frac{c^{1-\sigma}-1}{1-\sigma} - \chi^n\frac{(n)^{1+\theta}}{1+\theta} \tag{A.2.1}$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption and $\theta \geq 0$ is proportional to the inverse of the Frisch elasticity of labor supply. The constant χ^n is a scale parameter influencing the relative disutility of labor to the utility of consumption.

Although labor supply is only defined for $n \in [0,\tilde{l}]$—where $\tilde{l}$ is the time endowment or the maximum labor supply possible—the disutility of labor function in (A.2.1) is defined for values of n greater than $\tilde{l}$ and less than 0. Further, for $n < 0$, the marginal utility of labor is positive. To avoid the well known and significant computational difficulty of computing the solution to the complementary slackness conditions in the Karush, Kuhn, Tucker constrained optimization problem, we impose an approximating utility function that has properties bounding the solution for n away from both $n = \tilde{l}$ and $n = 0$. The upper right quadrant of an ellipse has exactly this property and also has many of the properties of the original utility function. Figure 11 shows how our estimated elliptical utility function compares to the utility of labor from (A.2.1) over the allowed support of n.

The general equation for an ellipse in x and y space with centroid at coordinates (h,k), horizontal radius of a, vertical radius of b, and curvature υ is the following.

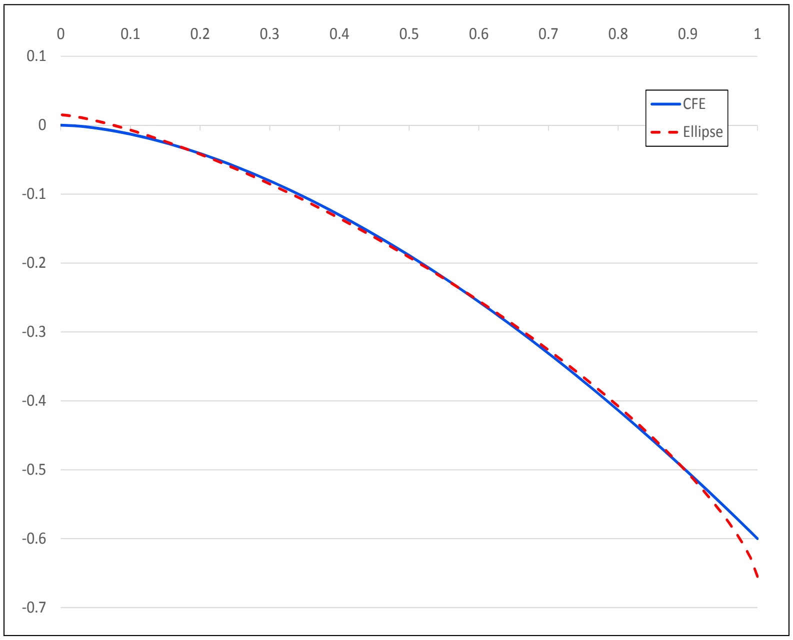
$$\left(\frac{x-h}{a}\right)^\upsilon + \left(\frac{y-k}{b}\right)^\upsilon = 1 \tag{A.2.2}$$

Figure 12 shows an ellipse with the parameterization $[h,k,a,b,\upsilon] = [1,-1,1,2,2]$.

The graph of the ellipse in the upper-right quadrant of Figure 12 ($x \in [1,2]$ and $y \in [-1,1]$) has similar properties to the utility of labor term in (A.2.1). If we let the x variable be labor supply n, the utility of labor supply be $g(n)$, the x-coordinate of the centroid be zero $h = 0$, and the horizontal radius of the ellipse be $a = \tilde{l}$, then the equation for the ellipse corresponding to the standard utility specification is the following.

$$\left(\frac{n}{\tilde{l}}\right)^\upsilon + \left(\frac{g-k}{b}\right)^\upsilon = 1 \tag{A.2.3}$$

Figure 11: Comparison of standard utility of labor n to elliptical utility



Solving the equation for g as a function of n, we get the following.

$$g(n) = b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} + k \tag{A.2.4}$$

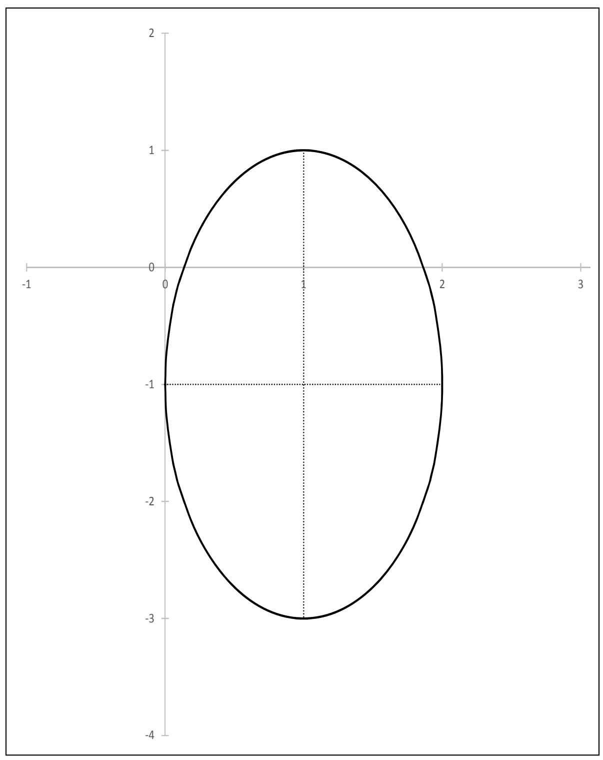
The υ parameter acts like a constant elasticity of substitution, and the parameter b is a shape parameter similar to χ^n in (A.2.1).

We use the upper-right quadrant of the elliptical utility function because the utility of n is strictly decreasing on $n \in (0, \tilde{l})$, because the slope of the utility function goes to negative infinity as n approaches its maximum of $\tilde{l}$ and because the slope of the utility function goes to zero as n approaches its minimum of 0. This creates interior solutions for all optimal labor supply choices $n^* \in (0, \tilde{l})$. Although it is more realistic to allow optimal labor supply to sometimes be zero, the complexity and dimensionality of our model requires this approximating assumption to render the solution method tractable.

Figure 11 shows how closely the estimated elliptical utility function matches the original utility of labor function in (A.2.1) with a Frisch elasticity of 0.4.[13] We choose the ellipse parameters b, k, and υ to best match the points on the original utility of labor function for $n \in [0, 0.9]$. We minimize the sum of absolute errors for 101 evenly spaced points on this domain. The estimated values of the parameters for the elliptical utility shown in Figure 11 and represented in equation (A.2.4) are $[b, k, \upsilon] = [0.573, 0.000, 2.856]$.

[13]See Chetty et al. (2011), Keane and Rogerson (2012) and Peterman (2014) for discussion of this choice.

Figure 12: Ellipse with $[h, k, a, b, v] = [1, -1, 1, 2, 2]$



A-3 Estimation of individual tax functions and marginal tax rate functions

Put tax function appendix here.

A-4 Solving for stationary steady-state equilibrium

This section describes the solution method for the stationary steady-state equilibrium described in Definition 1. The steady-state is characterized by $2JS$ equations and $2JS$ unknowns. However, because some of the other equations cannot be solved for analytically and substituted into the Euler equations, we must take a two-stage approach to the equilibrium solution. We first make a guess at steady-state wage $\bar{w}$, interest rate $\bar{r}$, lump-sum transfer $\bar{T}^H$, and income multiplier $factor$. Then, given those four aggregate variables, we can solve for the second-stage household decisions of steady-state savings $\bar{b}_{j,s}$ and labor supply $\bar{n}_{j,s}$.

1. Use the techniques in Appendix A-1 to solve for the steady-state population distribution vector $\bar{\boldsymbol{\omega}}$ of the exogenous population process.

2. Choose an initial guess for the values of the steady-state wage $\bar{w}$, interest rate $\bar{r}$, lump-sum transfer $\bar{T}^H$, and income multiplier $factor$.

3. Given guesses for $\bar{w}$, $\bar{r}$, $\bar{T}^H$, and $factor$, solve for the steady-state household savings $\bar{b}_{j,s}$ and labor supply $\bar{n}_{j,s}$ decisions using $2JS$ equations (25), (26).

    - A good first guess for $\bar{b}_{j,s}$ and $\bar{n}_{j,s}$ is a number close to but less than $\tilde{l}$ for all the $\bar{n}_{j,s}$ and to choose some small positive number for $\bar{b}_{j,s}$ that is small enough to be less than the minimum income that an individual might have $\bar{w}e_{j,s}\bar{n}_{j,s}$.
    - Make sure that all of the $2JS$ Euler errors is sufficiently close to zero to constitute a solution.

4. Given the solutions $\bar{b}_{j,s}$ and $\bar{n}_{j,s}$ from step (3), make sure that the four characterizing equations for $\bar{w}$, $\bar{r}$, $\bar{T}^H$, and $factor$ are solved. These characterizing equations are the zero equations corresponding to the steady-state versions of (28), (20), (33), and (**??**).

$$\bar{w} - (1-\alpha)\frac{\bar{Y}}{\bar{L}} = 0 \tag{A.4.1}$$

$$\bar{r} - \alpha\frac{\bar{Y}}{\bar{K}} + \delta = 0 \tag{A.4.2}$$

$$\bar{T}^H - \sum_s\sum_j \bar{\omega}_s\lambda_j\bar{T}_s = 0 \tag{A.4.3}$$

$$factor\sum_s\sum_j \bar{\omega}_s\lambda_j\left(\bar{w}e_{j,s}\bar{n}_{j,s} + \bar{r}\bar{b}_{j,s}\right) - (\text{data avg. income}) = 0 \tag{A.4.4}$$

5. Iterate on guesses for outer loop values of $\bar{w}$, $\bar{r}$, $\bar{T}^H$, and $factor$ until the Euler equations from step (3) and the characterizing equations from step (4) are all solved.

A-5 Solving for stationary non-steady-state equilibrium by time path iteration

This section describes the solution to the non-steady-state transition path equilibrium of the model described in Definition 2 and outlines the time path iteration (TPI) method of Auerbach and Kotlikoff (1987) for solving for this equilibrium. The following are the steps for computing a stationary non-steady-state equilibrium time path for the economy.

1. Input all initial parameters. See Table 2.

    (a) The value for T at which the non-steady-state transition path should have converged to the steady state should be at least as large as the number of periods it takes the population to reach its steady state $\bar{\boldsymbol{\omega}}$ as described in Appendix A-1.

2. Choose an initial distribution of savings and intended bequests $\hat{\boldsymbol{\Gamma}}_1$ and then calculate the initial state of the stationarized aggregate capital stock $\hat{K}_1$ and total bequests received $\hat{BQ}_{j,1}$ consistent with $\hat{\boldsymbol{\Gamma}}_1$ according to (30) and (32).

    (a) Note that you must have the population weights from the previous period $\hat{\omega}_{s,0}$ and the growth rate between period 0 and period 1 $\tilde{g}_{n,1}$ to calculate $\hat{BQ}_{j,1}$.

3. Conjecture transition paths for the stationarized wage $\hat{\boldsymbol{w}}^1 = \{\hat{w}_t^1\}_{t=1}^{\infty}$, stationarized interest rate $\boldsymbol{r}^1 = \{r_t^1\}_{t=1}^{\infty}$, total bequests received $\hat{\boldsymbol{BQ}}_j^1 = \{\hat{BQ}_{j,t}^1\}_{t=1}^{\infty}$ for each household type j, and the lump-sum transfer from the government $\hat{\boldsymbol{T}}^{\boldsymbol{H,1}} = \{\hat{T}_t^{H,1}\}_{t=1}^{\infty}$. The only requirements are that $\hat{K}_1^i$ and $\hat{BQ}_{j,1}^i$ are functions of the initial distribution of savings $\hat{\boldsymbol{\Gamma}}_1$ for all iterations i in your initial state and that the time paths of $\hat{\boldsymbol{w}}^i$, $\boldsymbol{r}^i$, $\hat{\boldsymbol{BQ}}_j^i$, and $\hat{\boldsymbol{T}}^{\boldsymbol{H,i}}$ equal their respective steady-state values for all $t \geq T$.

    (a) Initial guesses for $\hat{w}_1$ and r_1 can be disciplined a little bit by whether $\hat{K}_1$ is greater than or less than $\bar{K}$. If $\hat{K}_1 > \bar{K}$, then choose $\hat{w}_1 > \bar{w}$ and $r_1 < \bar{r}$. If $\hat{K}_1 < \bar{K}$, then choose $\hat{w}_1 < \bar{w}$ and $r_1 > \bar{r}$.

4. With the conjectured transition paths $\hat{\boldsymbol{w}}^i$, $\boldsymbol{r}^i$, $\hat{\boldsymbol{BQ}}_j^i$, and $\hat{\boldsymbol{T}}^{\boldsymbol{H,i}}$, one can solve for the lifetime labor and savings decisions for each individual in the model who will be alive between periods $t = 1$ and T. Each individual's lifetime decisions can be solved independently using the systems of $2S$ Euler equations of the form (25), (26), and (27).

    (a) Make sure all the Euler errors for both the savings and labor supply decisions are sufficiently close to zero in order to ensure that the household equilibrium is being solved.

5. Use the implied distribution of savings and labor supply in each period to compute the new implied time paths for the wage $\hat{\boldsymbol{w}}^{i'} = \{\hat{w}_1^i, \hat{w}_2^{i'}, ...\hat{w}_T^{i'}\}$, interest rate $\boldsymbol{r}^{i'} = \{r_1^i, r_2^{i'}, ...r_T^{i'}\}$, total bequests received $\hat{\boldsymbol{BQ}}_j^{i'} = \{\hat{BQ}_{j,1}^i, \hat{BQ}_{j,2}^{i'}, ...\hat{BQ}_{j,T}^{i'}\}$ for each ability group j, and lump-sum transfer from the government $\hat{\boldsymbol{T}}^{\boldsymbol{H,i'}} = \{\hat{T}_1^{H,i'}, \hat{T}_2^{H,i'}, ...\hat{T}_T^{H,i'}\}$.

6. Check the distance between the two sets time paths.

$$\left\| \left[\hat{\boldsymbol{w}}^{i'}, \boldsymbol{r}^{i'}, \{\hat{\boldsymbol{BQ}}_j^{i'}\}_{j=1}^J, \hat{\boldsymbol{T}}^{\boldsymbol{H,i'}} \right] - \left[\hat{\boldsymbol{w}}^{i}, \boldsymbol{r}^{i}, \{\hat{\boldsymbol{BQ}}_j^{i}\}_{j=1}^J, \hat{\boldsymbol{T}}^{\boldsymbol{H,i}} \right] \right\|$$

   (a) If the distance between the initial time paths and the implied time paths is less-than-or-equal-to some convergence criterion $\varepsilon > 0$, then the fixed point has been achieved and the equilibrium time path has been found.

   (b) If the distance between the initial time paths and the implied time paths is greater than some convergence criterion $\|\cdot\| > \varepsilon$, then update the guess for the time paths and repeat steps (4) through (6) until a fixed point is reached.

Figures 13 and 14 show the equilibrium time paths of the aggregate capital stock K_t and aggregate labor supply L_t for the calibration of the model in this paper.

Figure 13: Equilibrium time path of K_t for $S = 80$ and $J = 7$ in baseline model

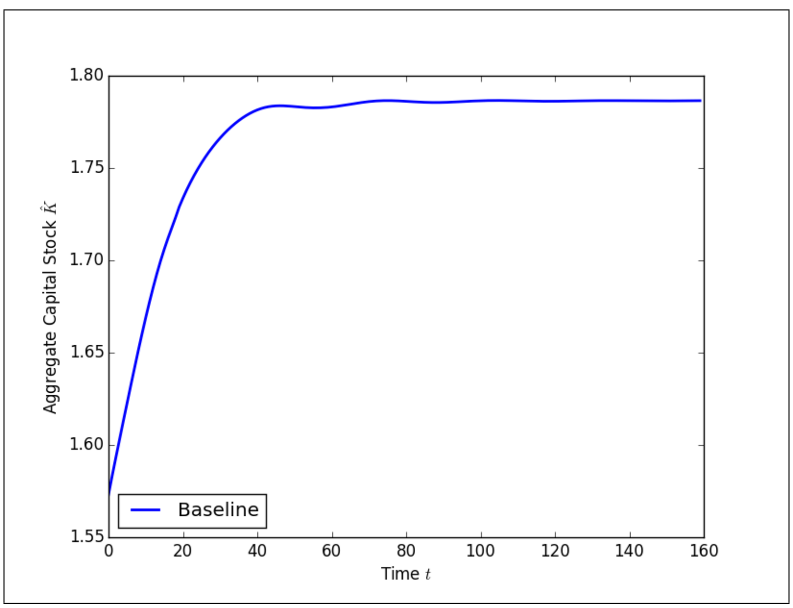


Figure 14: Equilibrium time path of L_t for $S = 80$ and $J = 7$ in baseline model