Question of the Day

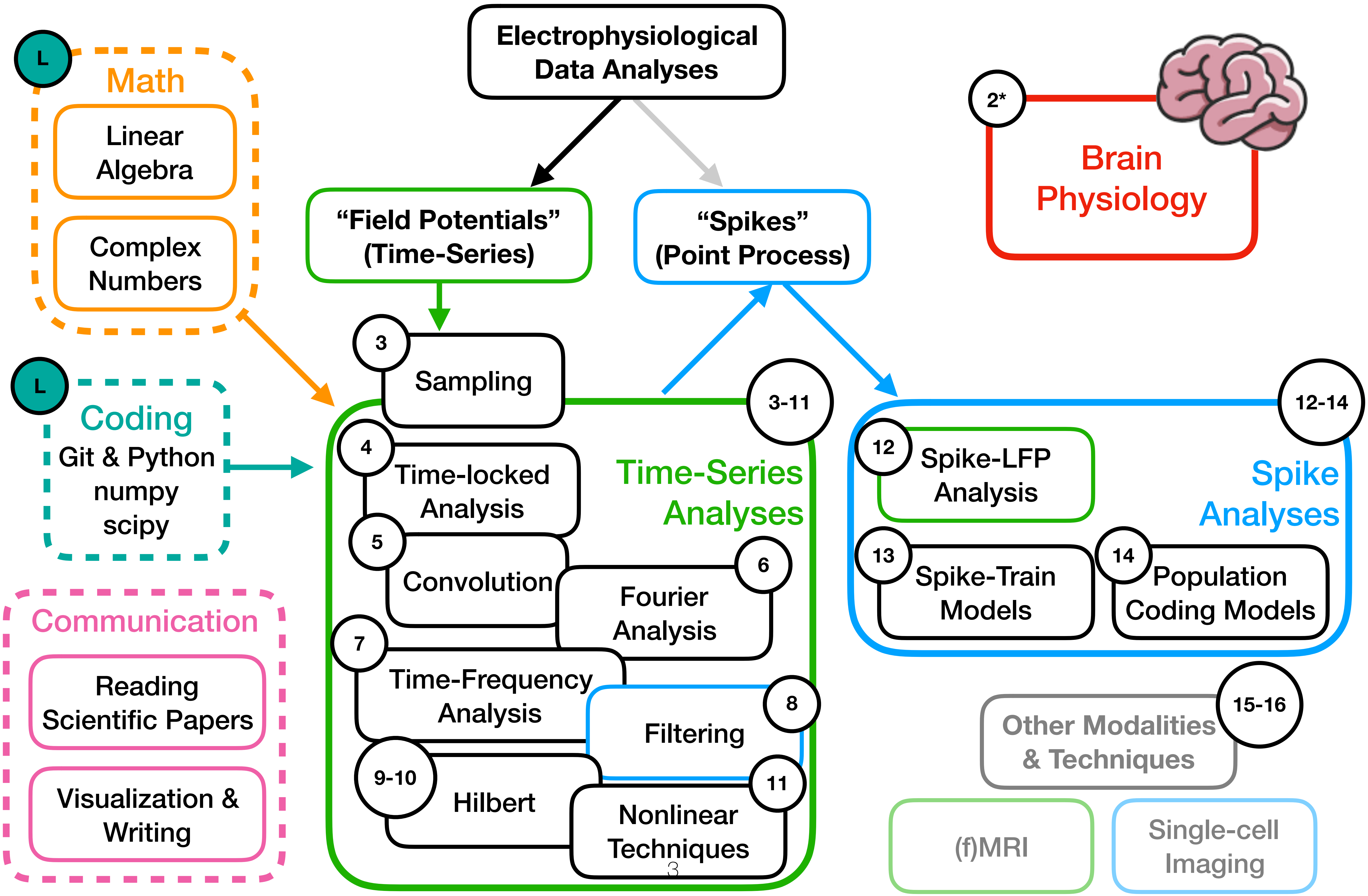
Is eating 10 strawberries and a banana the same as drinking a strawberry banana smoothie?

LTI Systems and Convolution

Lecture 5

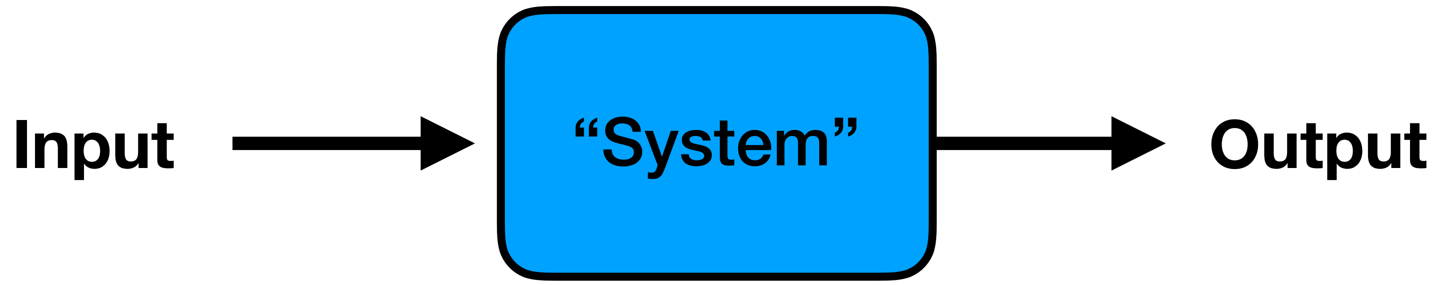
July 9, 2019

Course Outline: Road Map

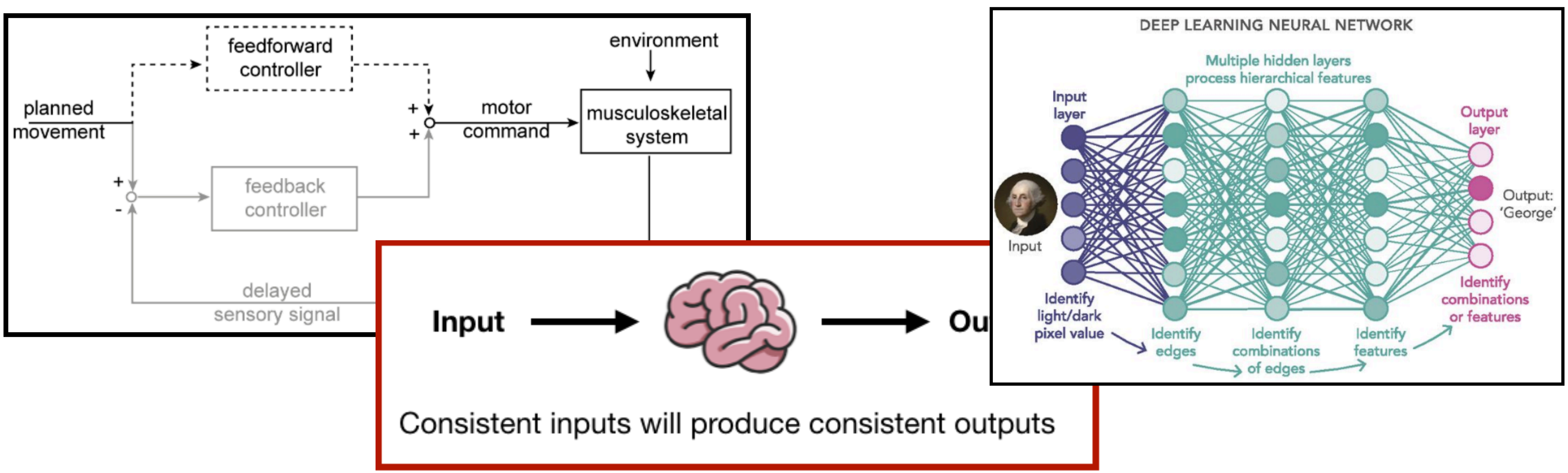


1. Formally define LTI systems
2. Convolution & impulse response
3. Introduce the frequency domain

Systems Perspective



The “system” performs a set of transformations on the input, to produce the output.



List 5 examples of systems: their input, output, and transformation.
Be imaginative!

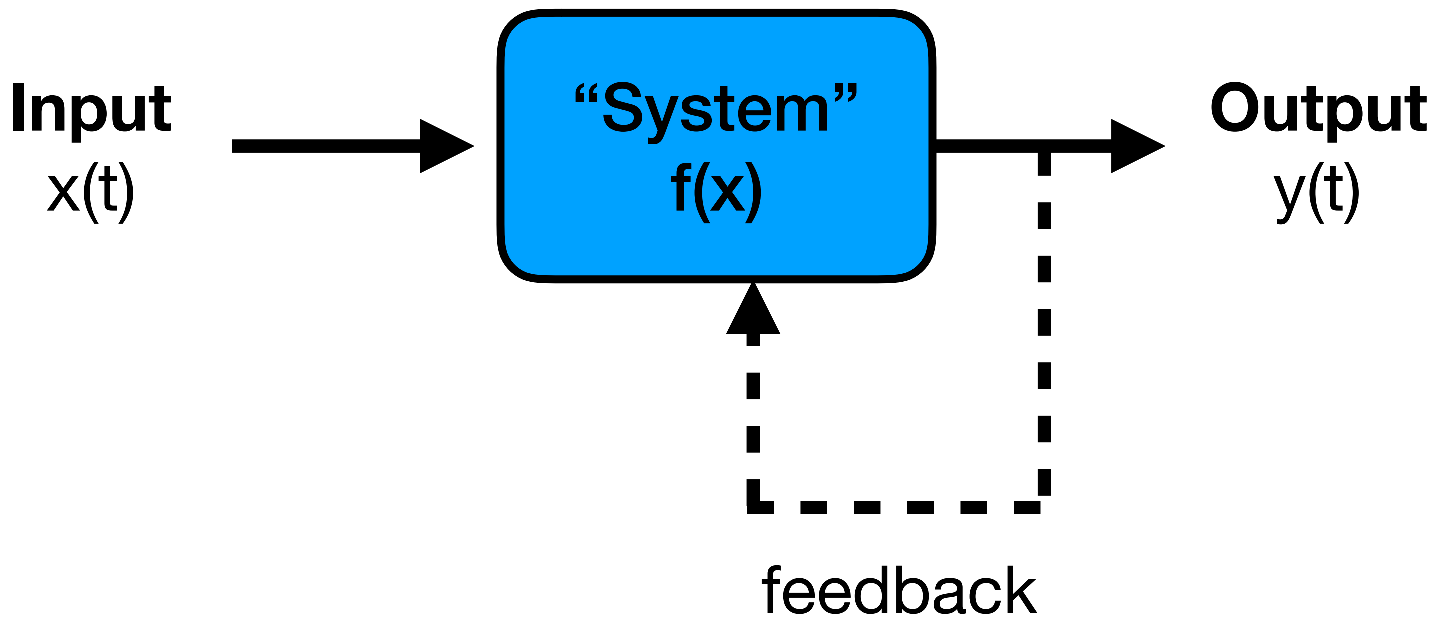
Systems Perspective

System	Input	Transformation	Output	Linear?	Time Invariant?
school	students teachers resources/ funding	education/classes	degrees/skills		
computer	money people	communication/ money processing	Coachella tickets		
lactose intolerant digestive system	milk	digest everything else	lack of output		

Linear Time Invariant (LTI) Systems

Real-life systems are very complex (non-linear).

Engineering approach: assume **“linearity”** and/or find linear range, and usually “**time invariance**”



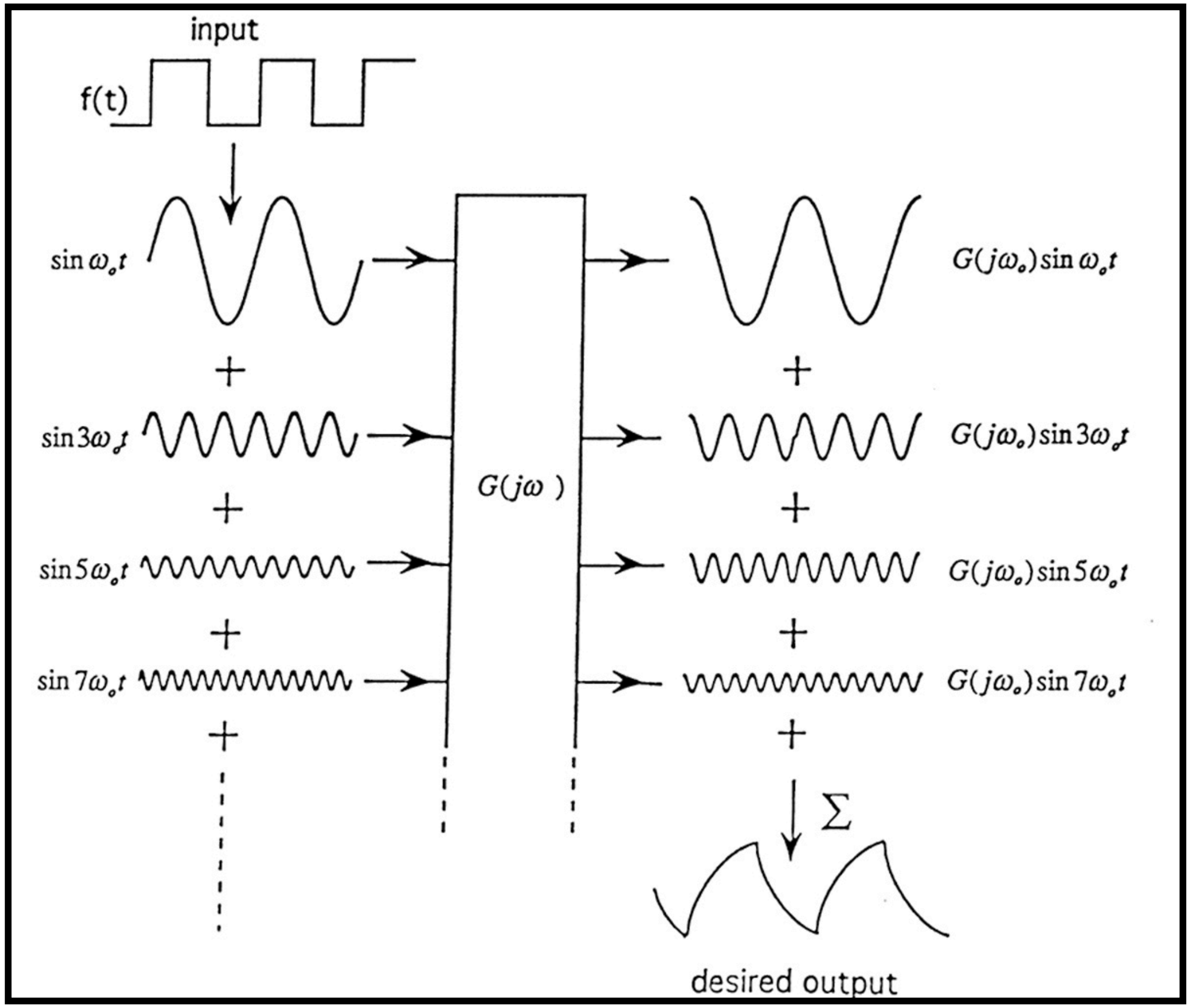
Linearity

$$y(t) = f(x_1(t) + x_2(t)) = f(x_1(t)) + f(x_2(t))$$

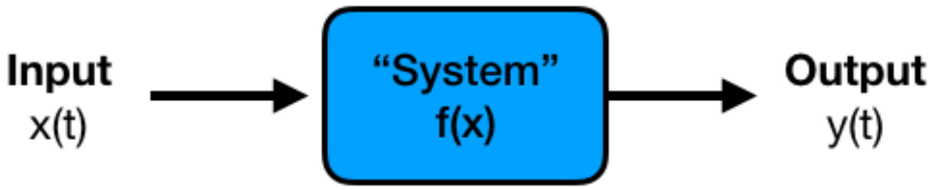
$$y(t) = f(Cx_1(t)) = Cf(x_1(t)), C = constant$$

In English: adding together the result of the operation on the components separately **is equivalent** to acting on the sum of the components.

The whole is NOT greater than the sum of the parts.



Linear or Not?



$$y(t) = f(x_1(t) + x_2(t)) = f(x_1(t)) + f(x_2(t))$$

$$y(t) = f(Cx_1(t)) = Cf(x_1(t)), C = constant$$

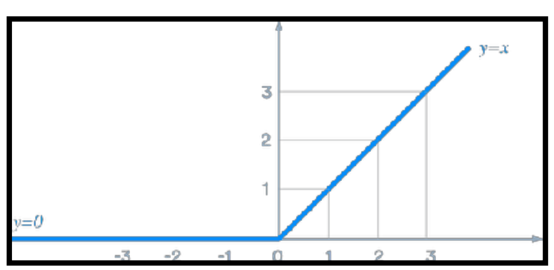
$$f(x) = x$$

$$f(x) = x^2$$

$$f(x) = x + C, C = constant$$

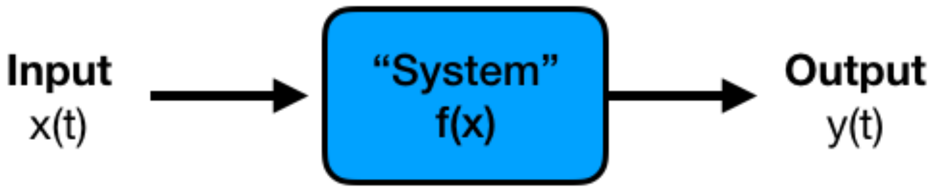
$$f(x) = dot(\overrightarrow{w}, \overrightarrow{x})$$

$$f(x) = ReLu$$



Selling fruits ($1/apples and $0.5/oranges)

Linear or Not?



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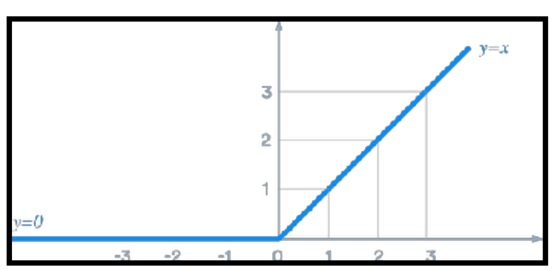
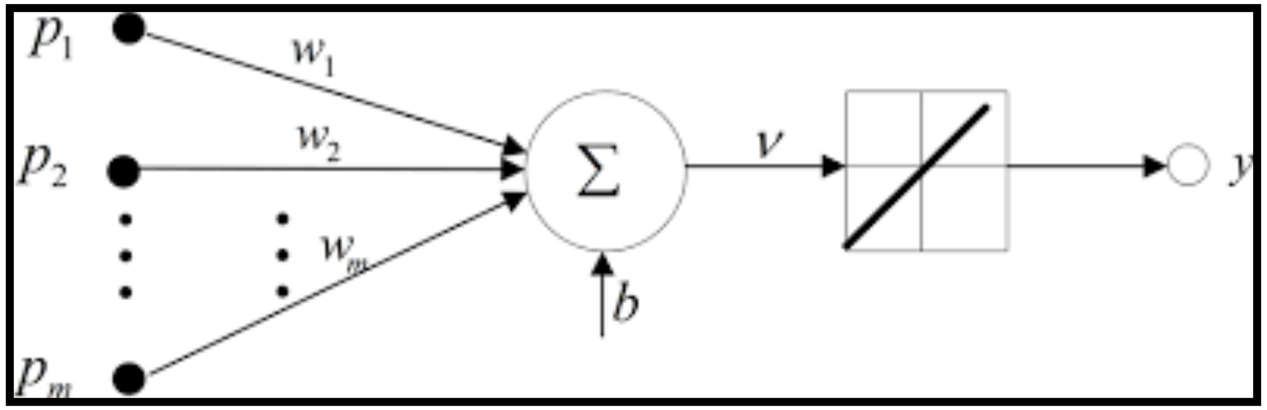


In general, some combination of adding **signals** and multiplying by **constants** will produce a linear operation.

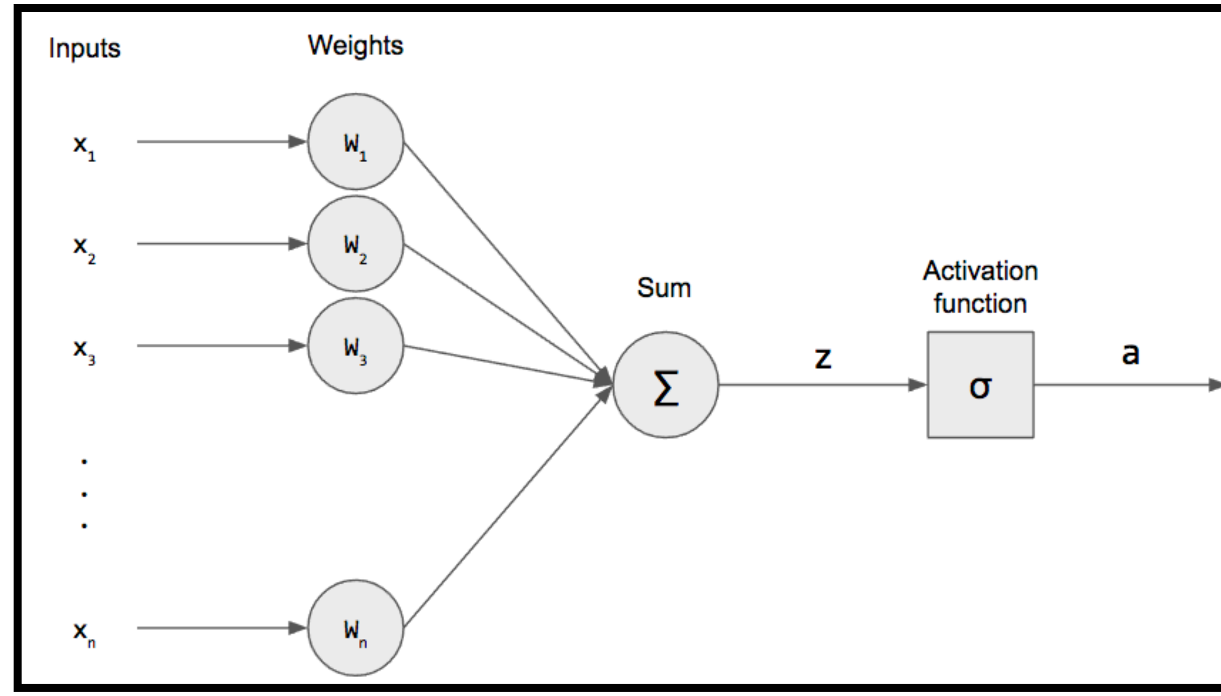
Selling fruits ($1/apples and $0.5/oranges)

Neural Networks

Linear Network



Non-Linear Network



Nonlinearity introduces **a lot** of computational complexity.

Dendritic nonlinearities are tuned for efficient spike-based computations in cortical circuits

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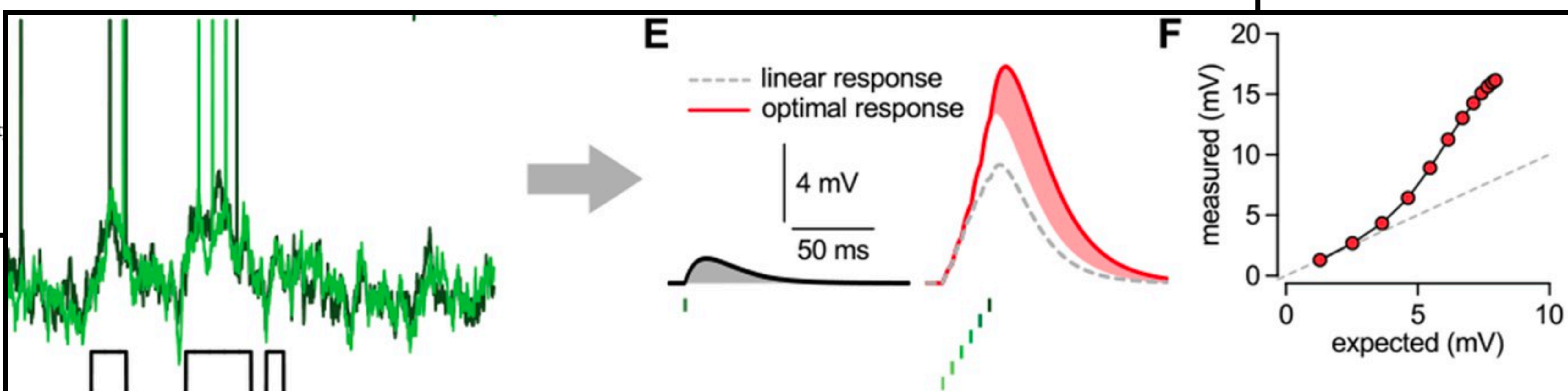

Time-Invariance

$$\boxed{y(t) = f(x_1(t)) \rightarrow y(t+\tau) = f(x_1(t+\tau)), \tau = constant}$$

In English: operating on the delayed signal will produce a delayed output.

Or, acting on the same signal later will produce the same result later.

Time-Invariant
e.g., selling fruits (mostly)

Not Time-Invariant
e.g., time dilation

$$f(x(t)) = x(2t)$$

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Systems Perspective

System	Input	Transformation	Output	Linear?	Time Invariant?
school	students teachers resources/ funding	education/classes	degrees/skills	no	no
computer	money people	communication/ money processing	Coachella tickets	yes*	no
lactose intolerant digestive system	milk	digest everything else	lack of output	yes*	yes*

Real-world & natural systems are almost never completely LTI.
Always define the range over which they are.

Goals for Today

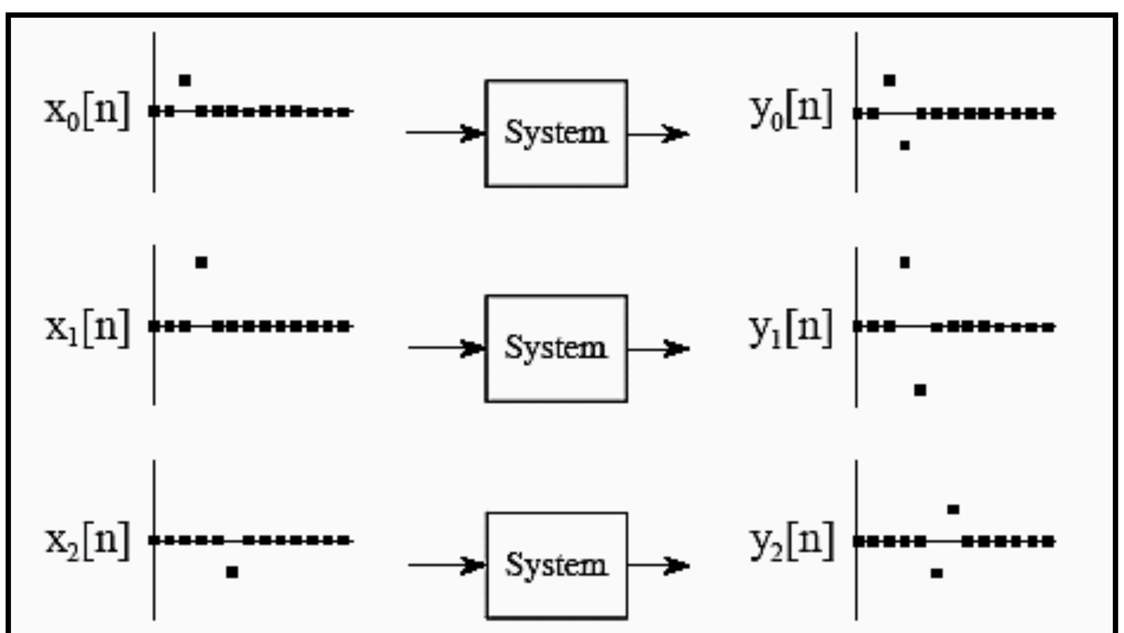
1. Formally define LTI systems
2. Convolution & impulse response
3. Introduce the frequency domain

Why LTI?

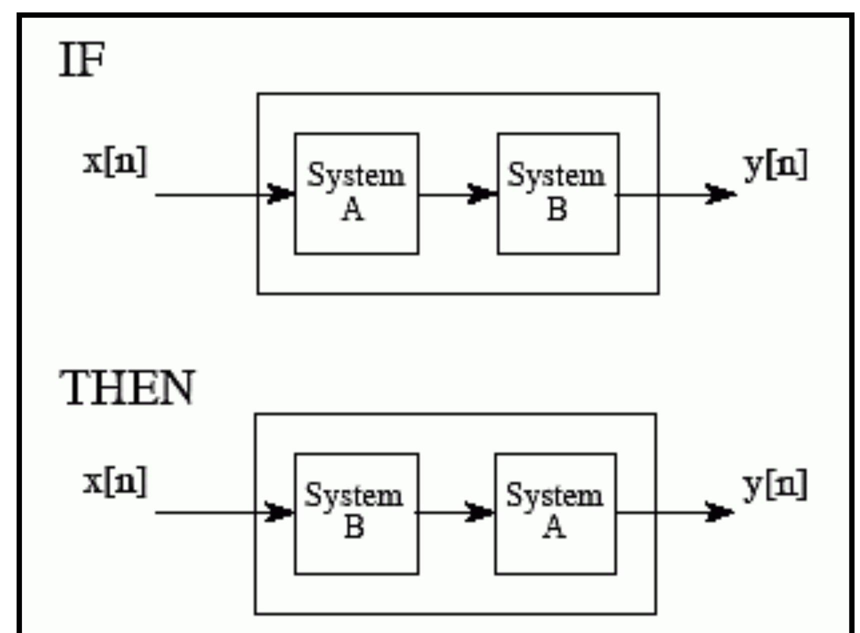
System are almost never LTI, so why bother **modeling** and **analyzing** them as such?

Answer: because it allows us to break up (decompose) complex transformations and signals, both **in components** and **in time**.

Additive



Commutative



Foundation of Digital Signal Processing

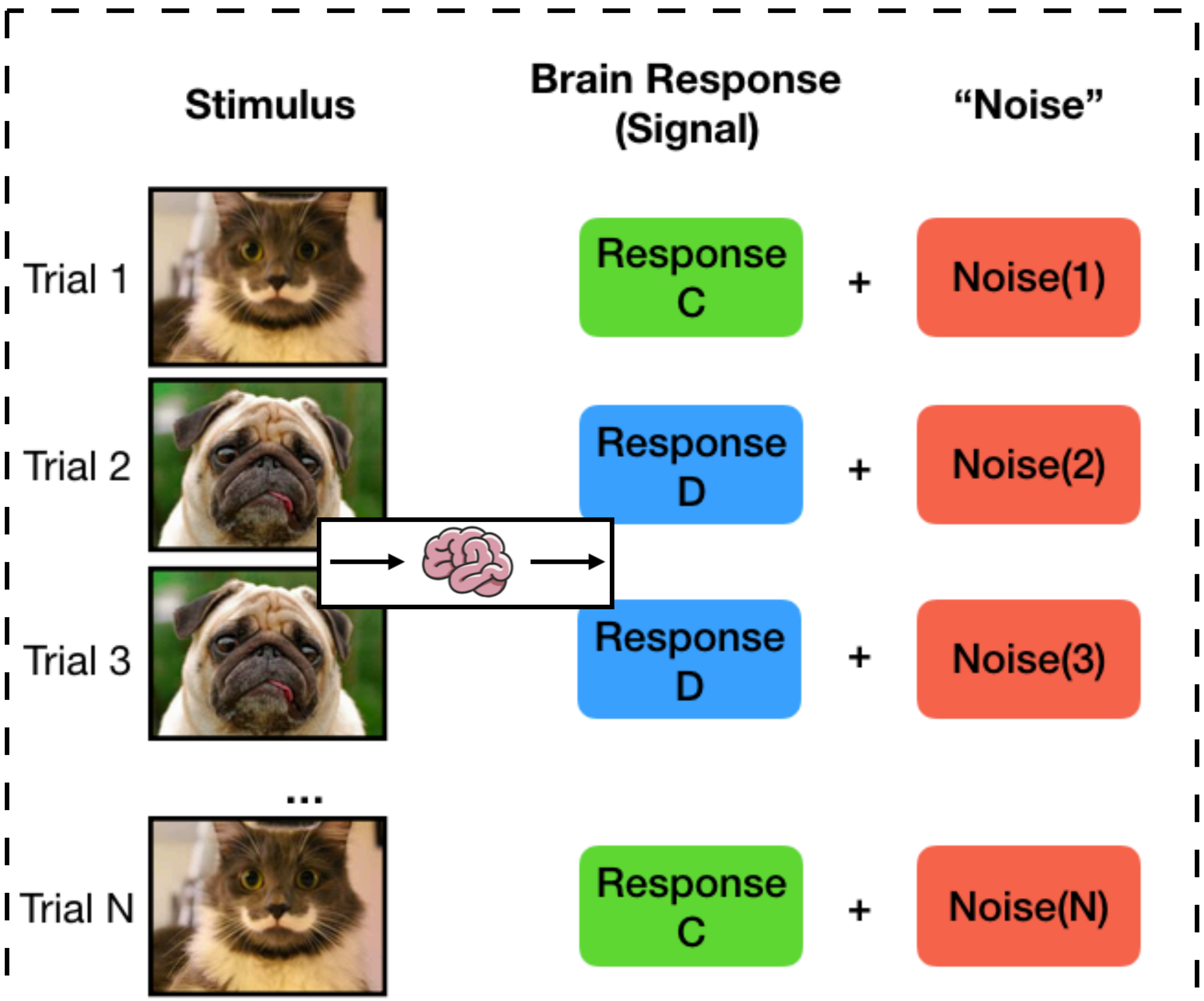
Intuition: we know exactly how your body responds when you eat various fruits, but we have no idea what drinking a smoothie would do. How can we infer your body's response?

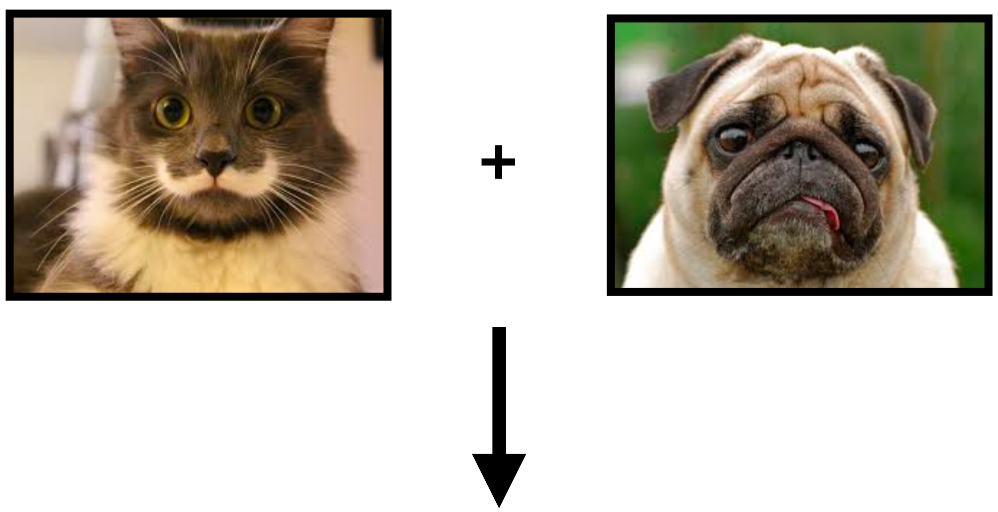
Digital Signal Processing (DSP):
divide (decomposition) and conquer (transformation)

Impulse Response

Frequency Decomposition

Impulse Response



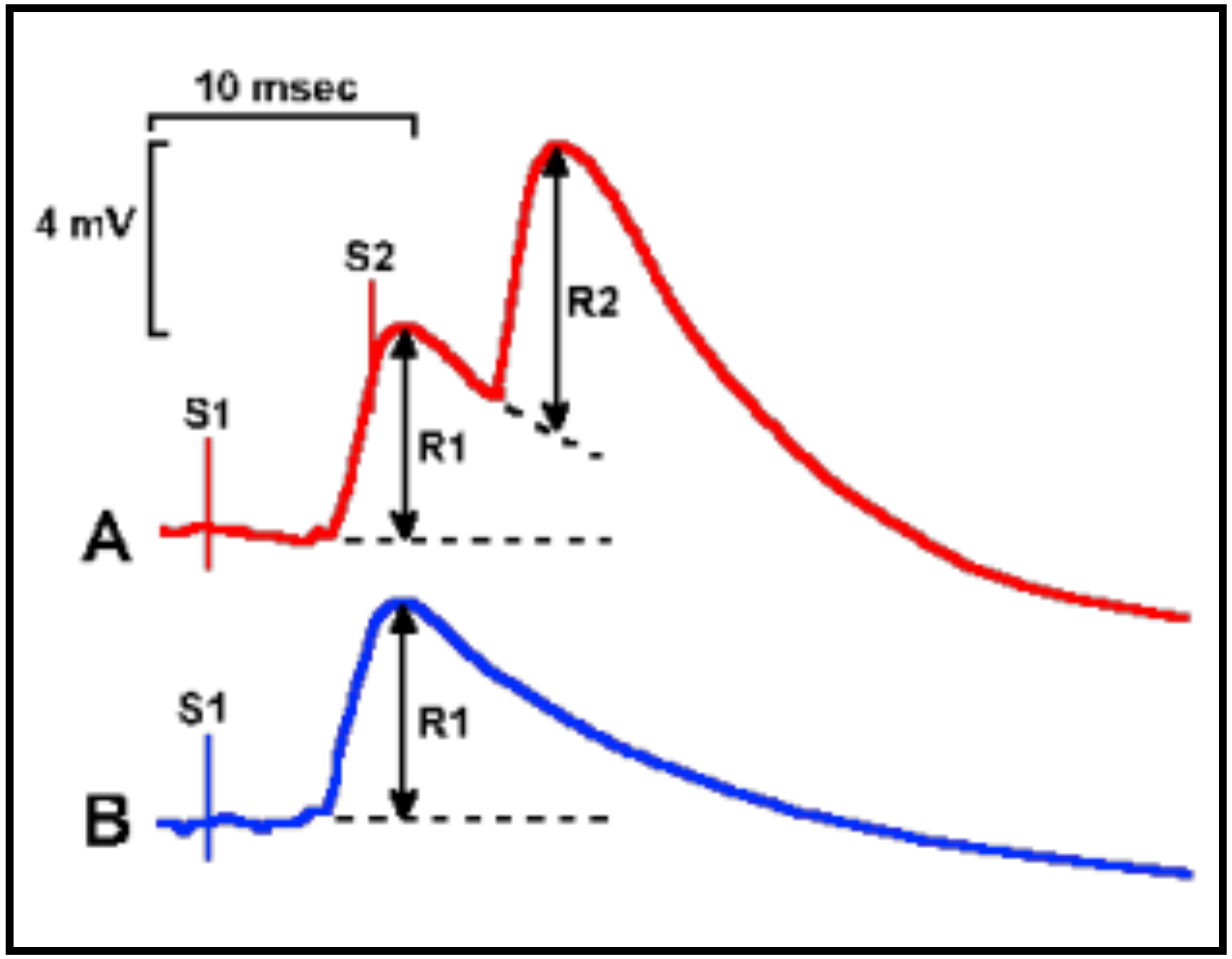


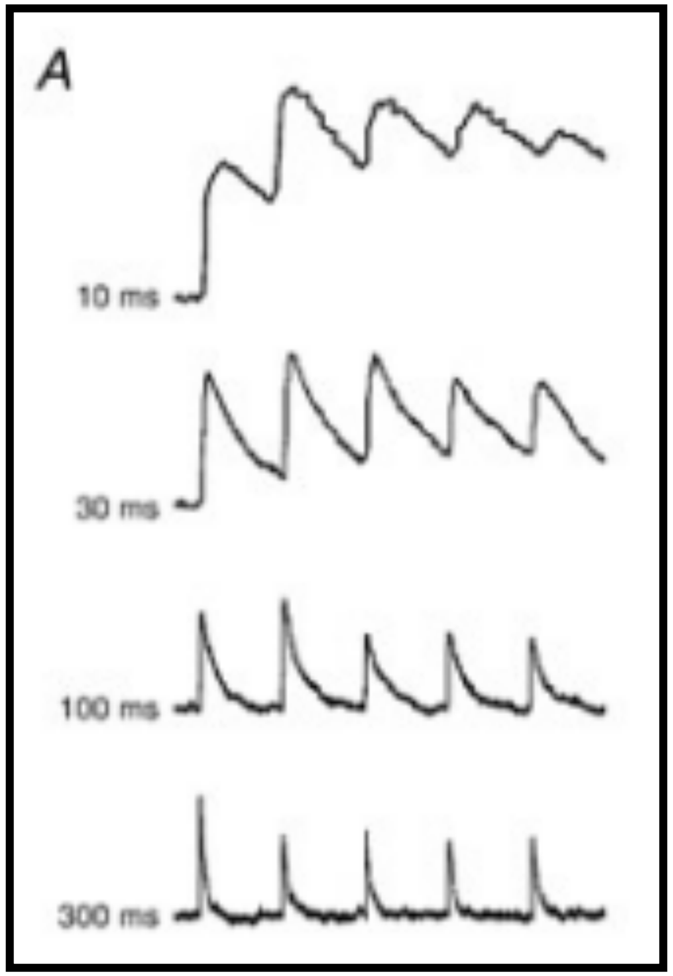
What will the combined brain response be?

Synaptic Response as Impulse Response

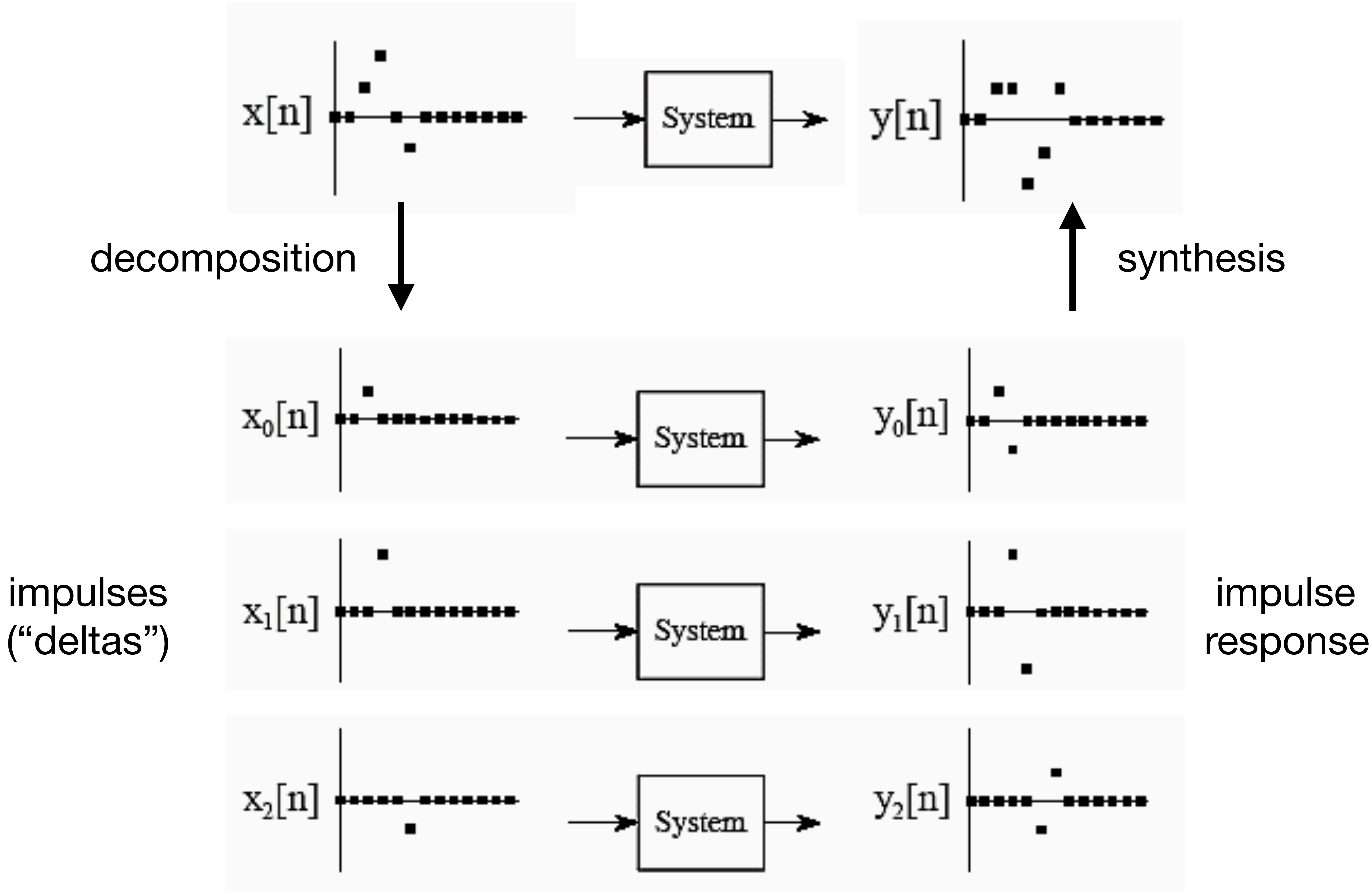
The same synaptic potential is triggered by an action potential, no matter when, and are summed over time.

** This is an ideal approximation





Impulse Response



Delta Function & Impulse

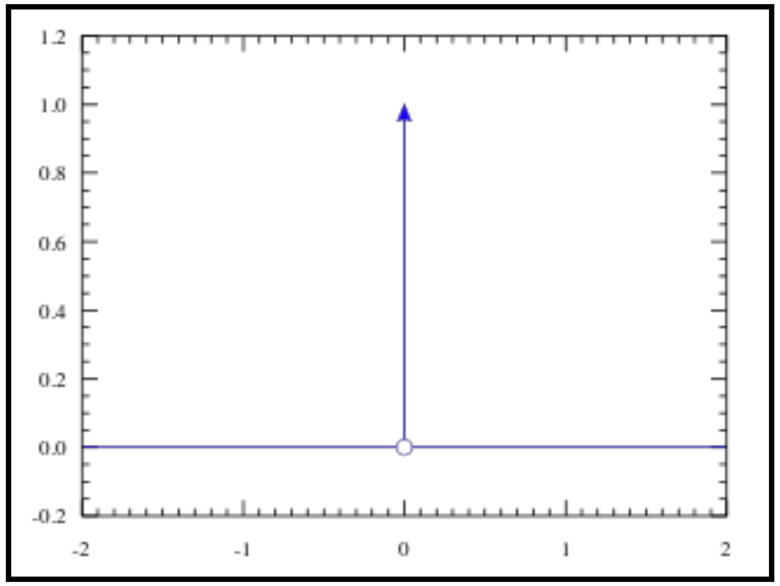
Continuous Time: Dirac's Delta
(Approximation at the Limit)

$$\delta(t)$$

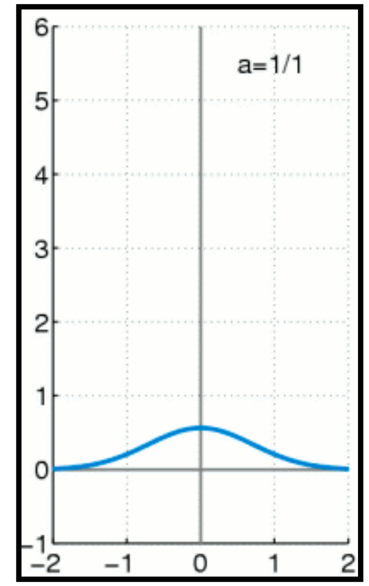
$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

(x is t here)



time

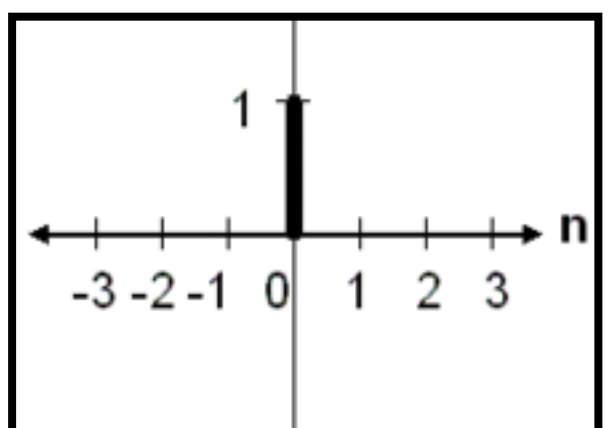


Very hairy!

Discrete Time (Kronecker Delta)

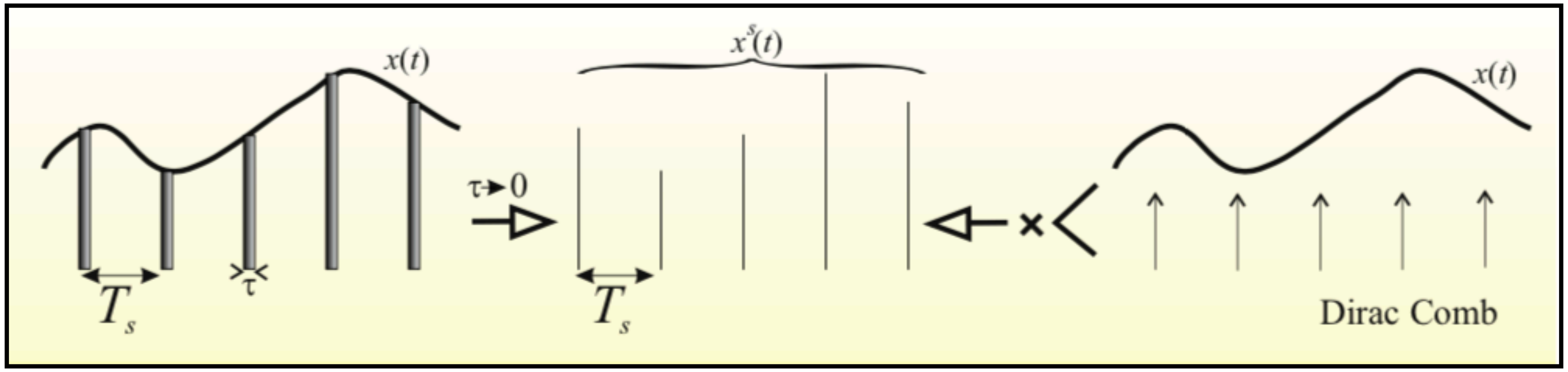
$$\delta(n)$$

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0. \end{cases}$$



Nice!

Delta Function & Digital Sampling



WvD Figure 2.4

Sampling in time is multiplying the continuous time function with a Dirac comb (or delta train), spaced at integer multiples of dt (1/fs).

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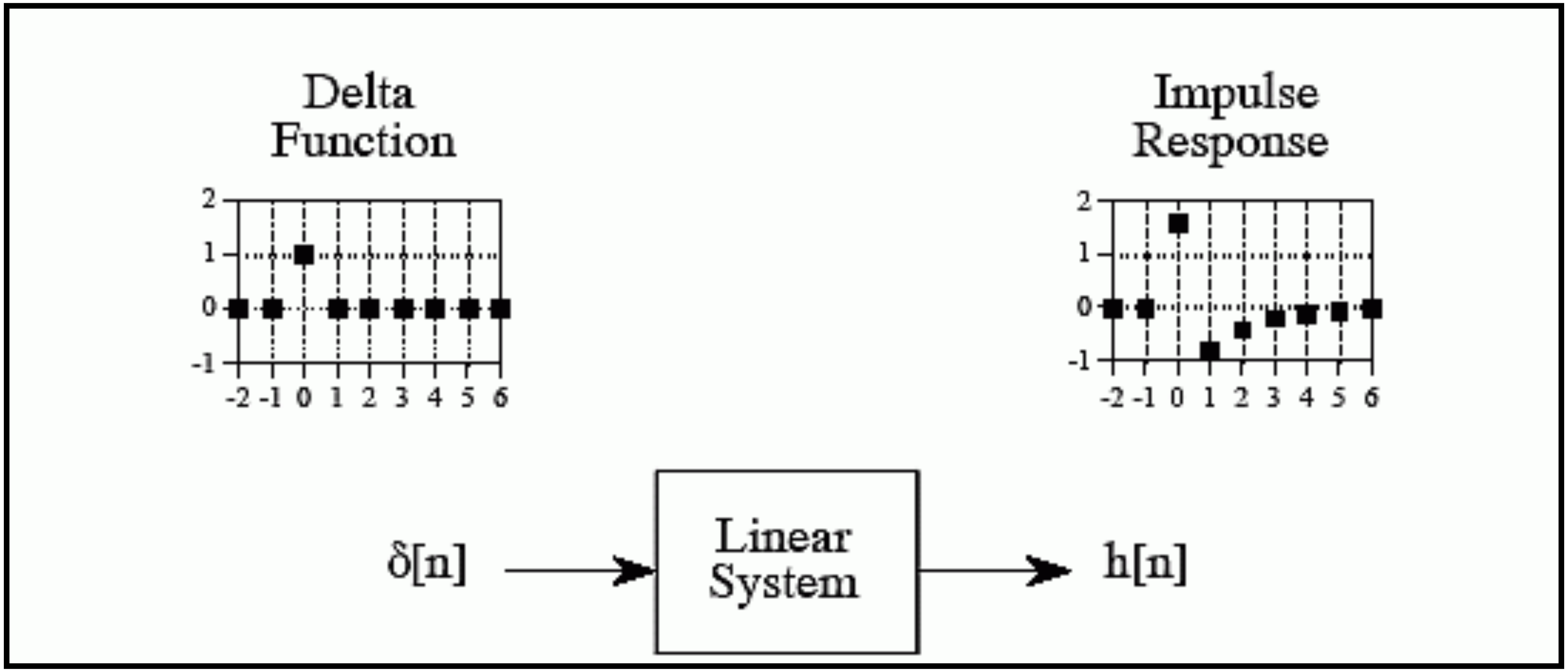
Impulse Response Function (IRF)

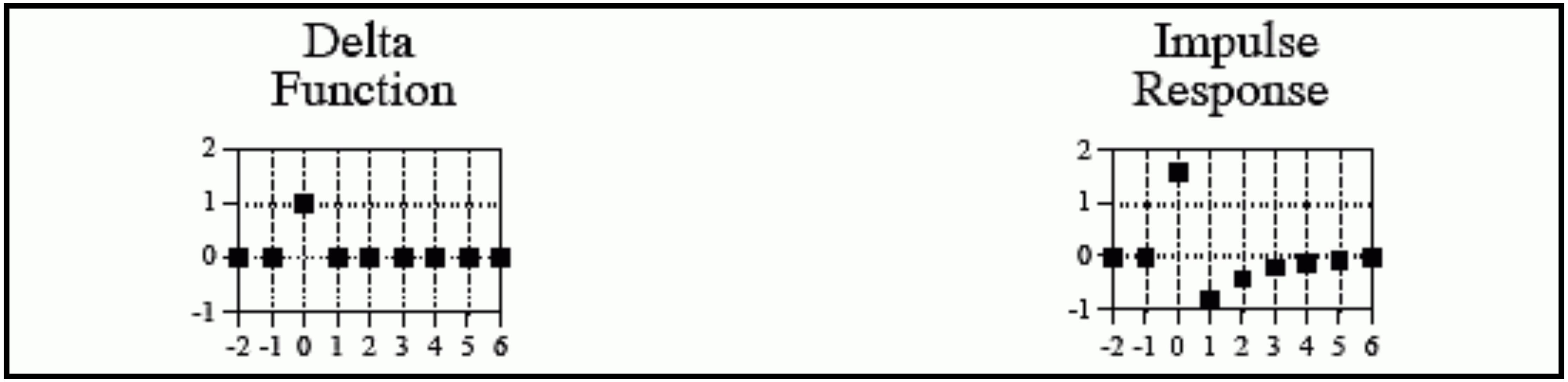


We know what the system does to a single delta.

We assume the system is LTI.

Therefore, we can reconstruct what the system will do for all signals, as applied to a series of **scaled** and **time-delayed** deltas.

Impulse Response Function (IRF)



Input: x(n) = [1,0,3,7,2,5]

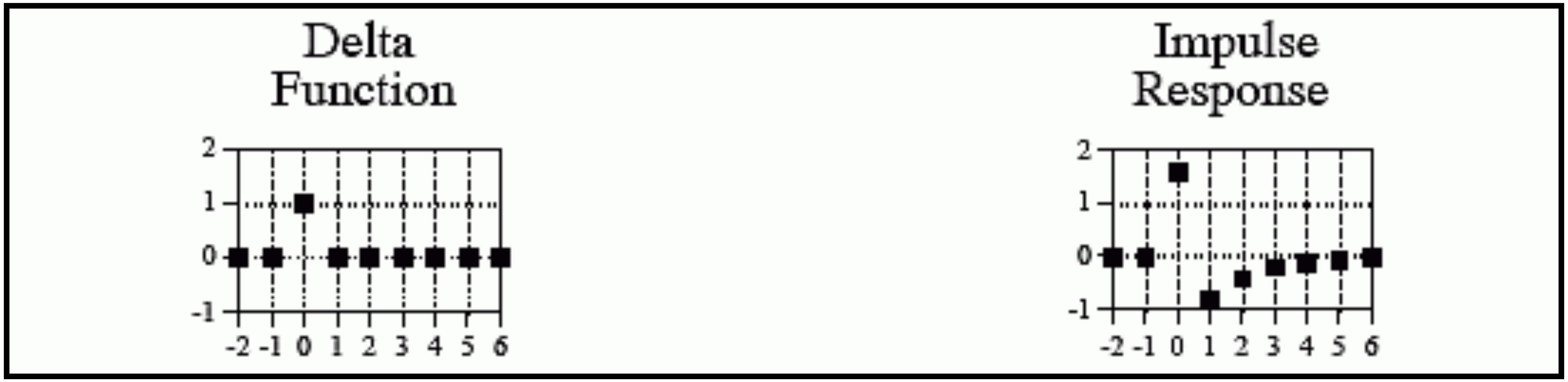
IRF: [2,-1,-1,0]

First, deconstruct the signal into a series of scaled & delayed deltas.

Apply the IRF to each delta.

Add the result together.

Impulse Response Function (IRF)



IRF: [2,-1,-1,0]

Input:
x(n) = [1,0,3,7,2,5]

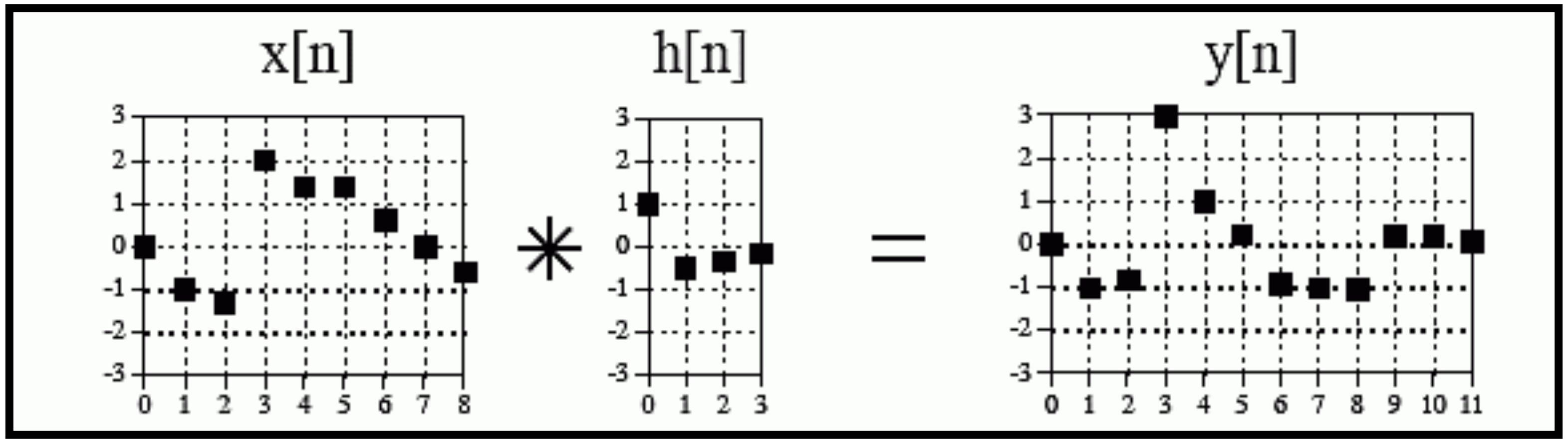
deltas

n =	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	2	-1	-1	0	0	0	0	0	0	0	0
0	0	0	0	0	0						
3	0	0	6	-3	-3	0					
7	0	0	0	14	-7	-7	0				
2	0	0	0	0	4	-2	-2	0			
5	0	0	0	0	0	10	-5	-5	0	0	
	2	-1	5	11	-6	1	-7	-5	0	0	0

Convolution

The computation you just performed is called a **convolution**.

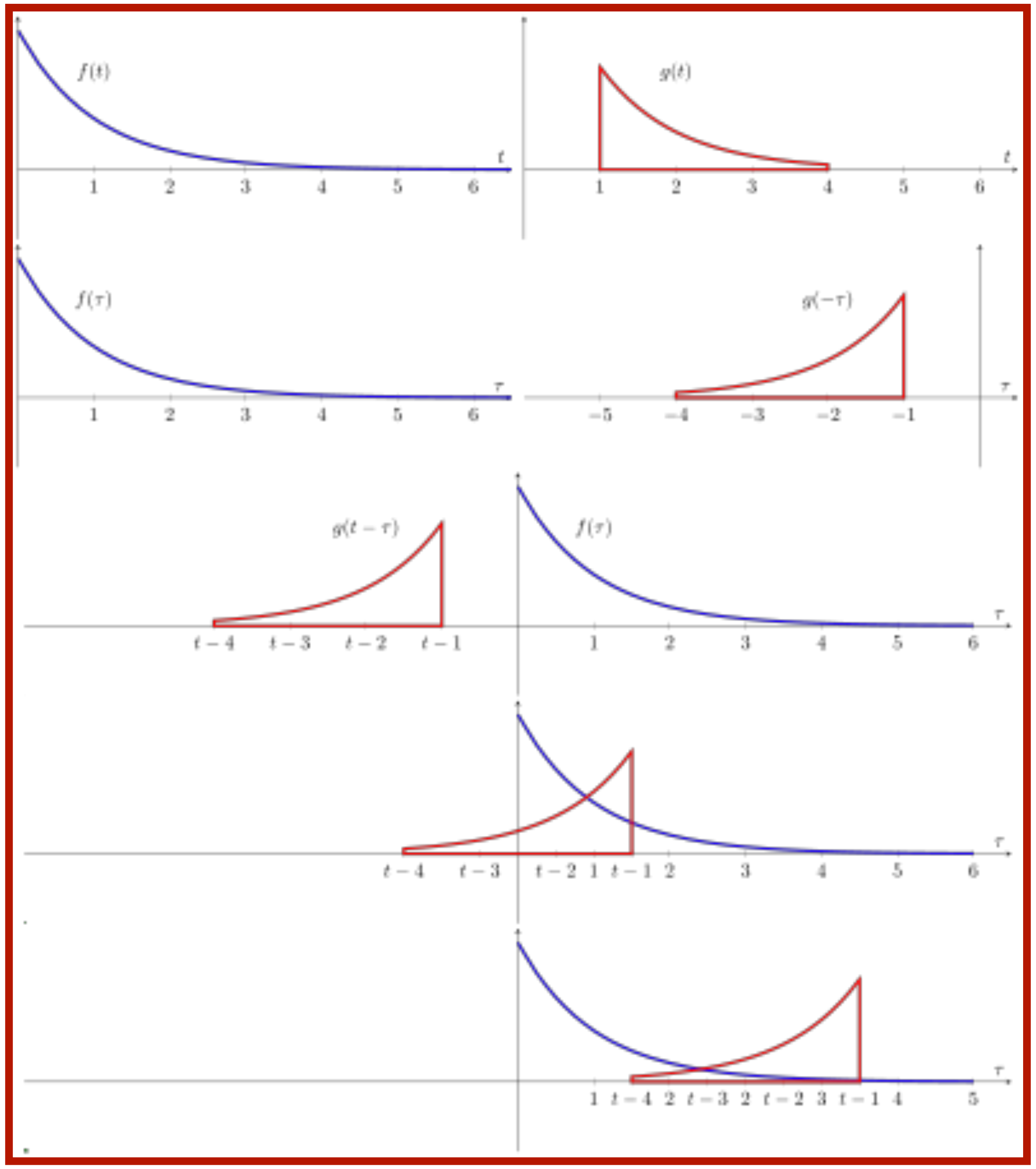
The output of a LTI system is the **convolution** of the input signal with the impulse response function, or the signal **convolved** with the IRF.



$$(f * g)(t) \triangleq \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau.$$

$$\boxed{(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m] g[n-m]}$$

$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m]g[n-m]$$



Input: x(n) = [1,0,3,7,2,5]

IRF: [2,-1,-1,0]

IRF[-n]: [0, -1, -1, 2]

[0, -1, -1, 2]
[1,0,3,7,2,5] 0

[0, -1, -1, 2]
[1,0,3,7,2,5] 2

[0, -1, -1, 2]
[1,0,3,7,2,5] -1

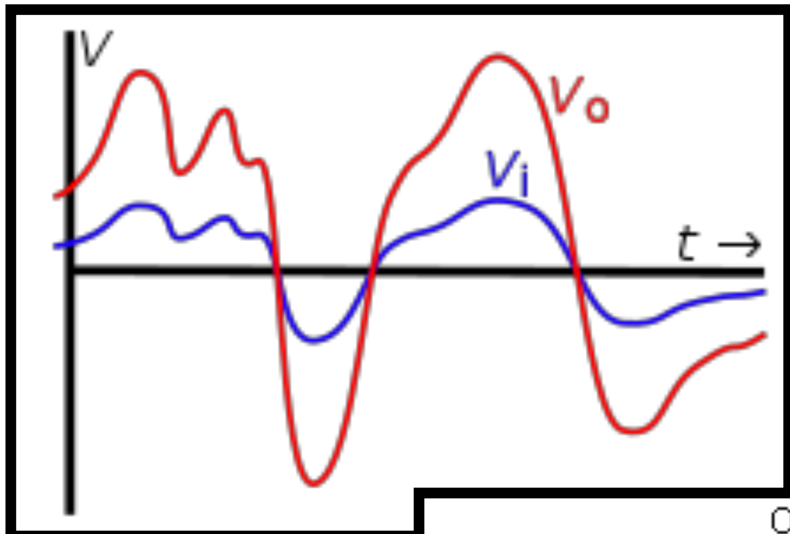
[0,-1,-1,2]
[1,0, 3, 7,2,5] 5

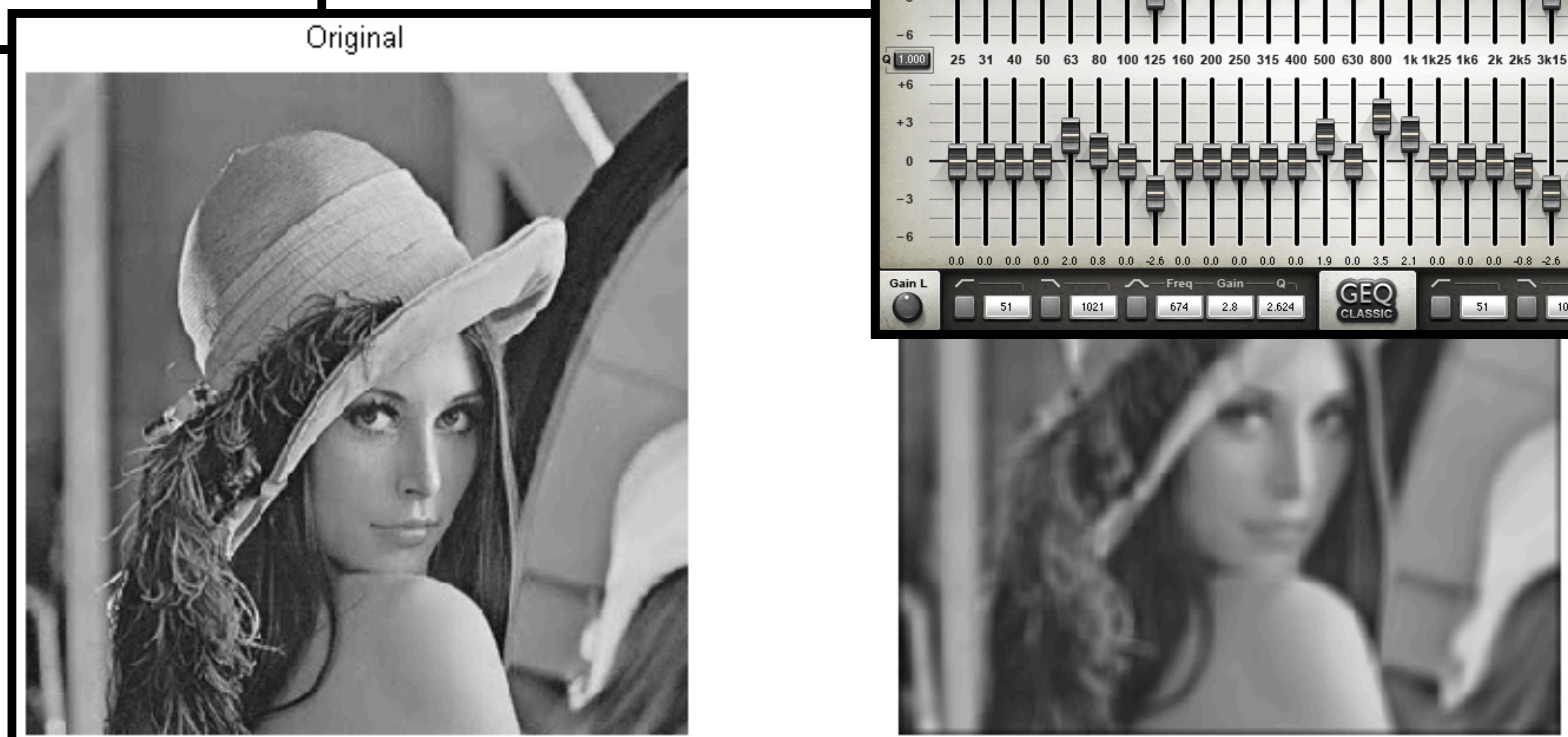
[0,-1,-1,2]
[1,0, 3, 7,2,5] -5

Amplification

Image Smoothing

Filtering





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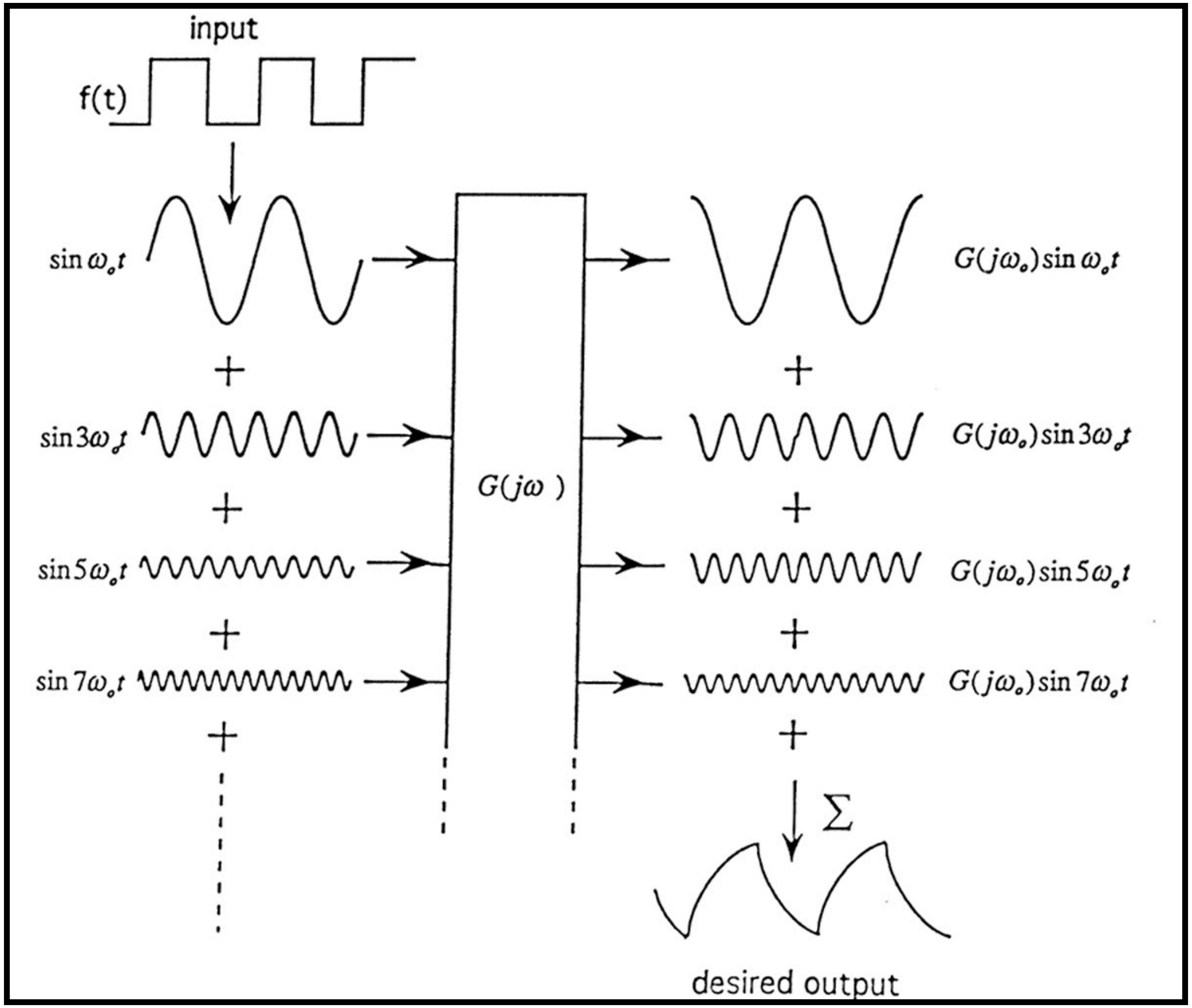
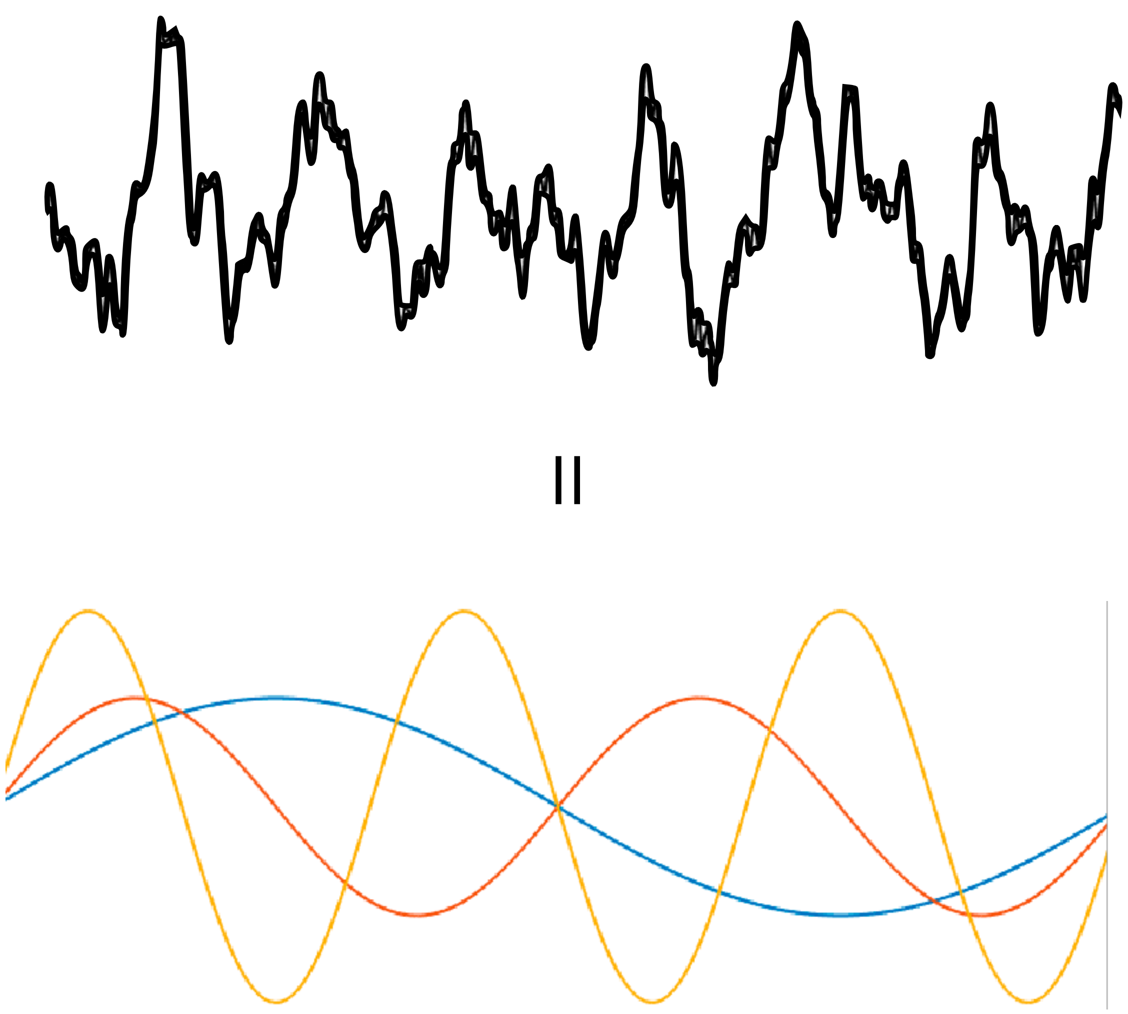
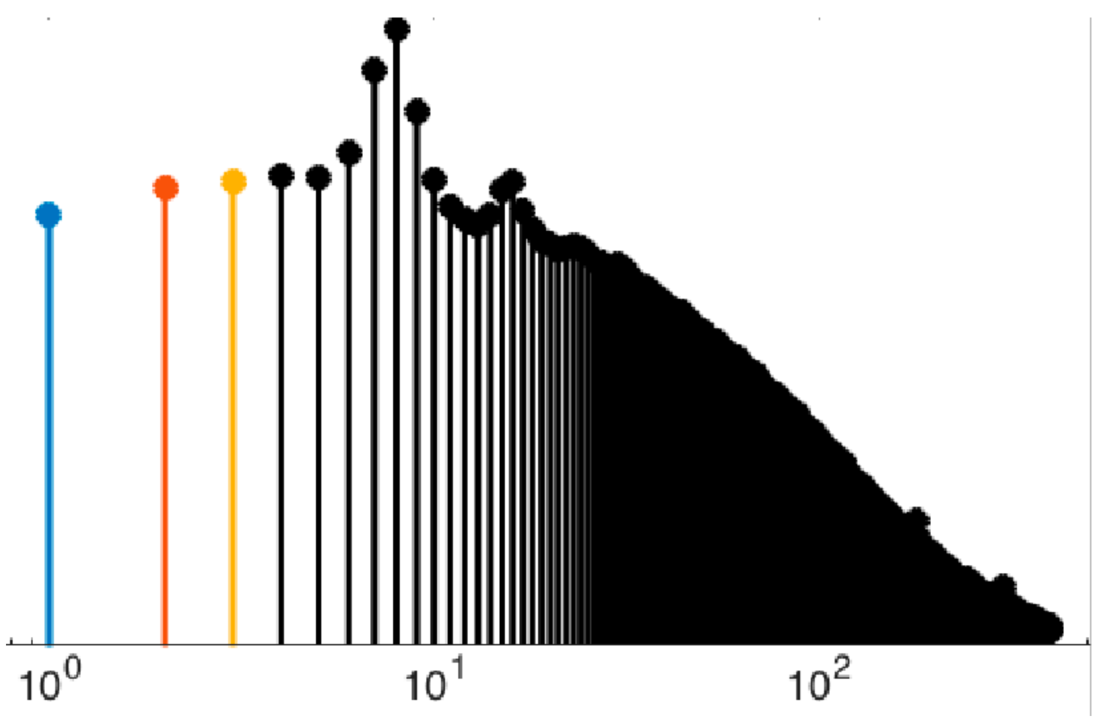


Frequency Decomposition

Time Domain



Frequency Domain



time (s)

frequency (Hz)

Convenient Duality

Time	⟷	Frequency
Convolution	⟷	Multiplication
Sine	⟷	Delta
Hairy!		**Nice!**

Summary

1. Formally define LTI systems

2. Convolution & impulse response

3. Introduce the frequency domain

https://tinyurl.com/cogs118c-att