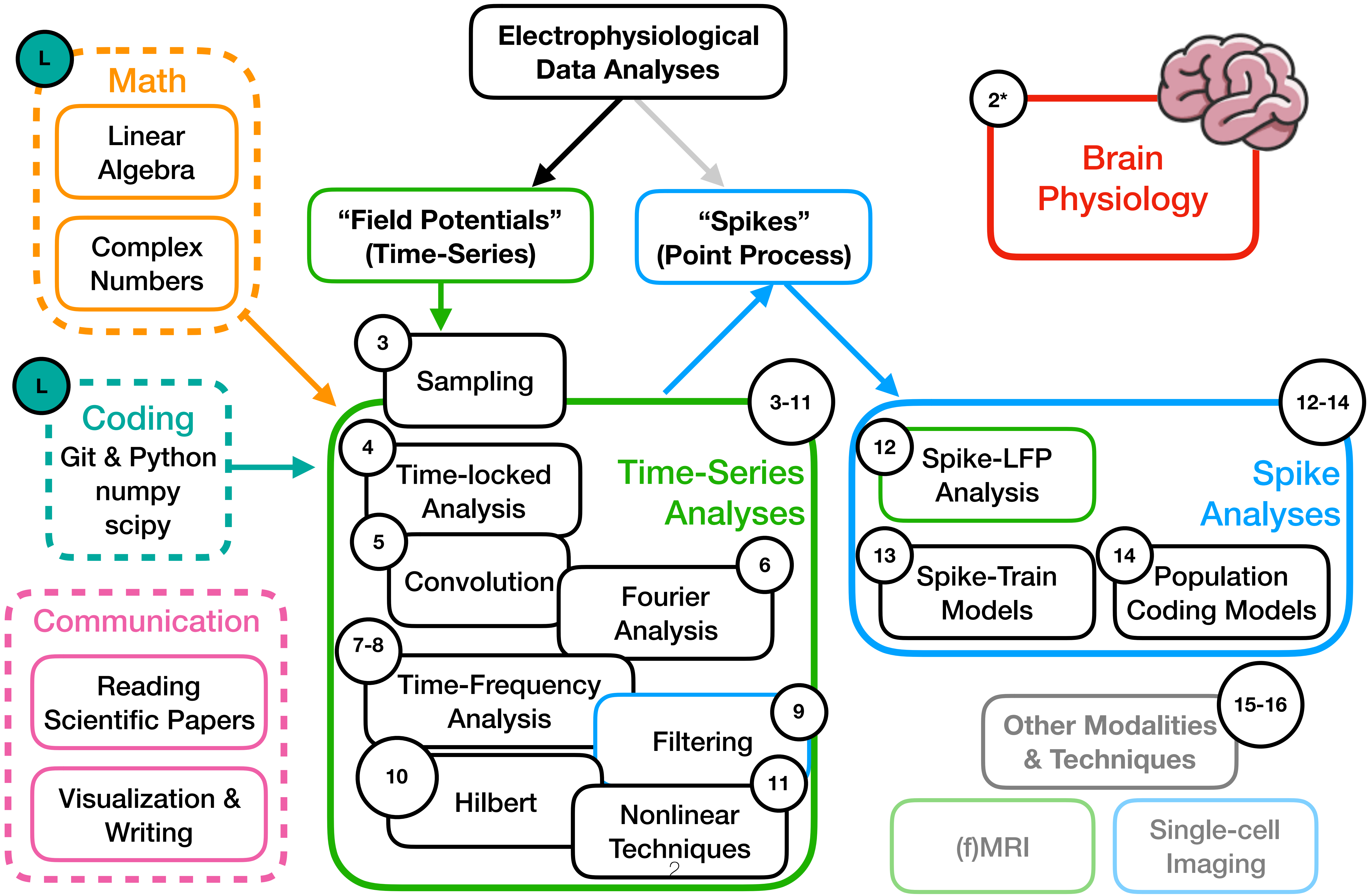
Analytic Signal & Hilbert Transform

Lecture 10

July 18, 2019

Course Outline: Road Map



Goals for Today

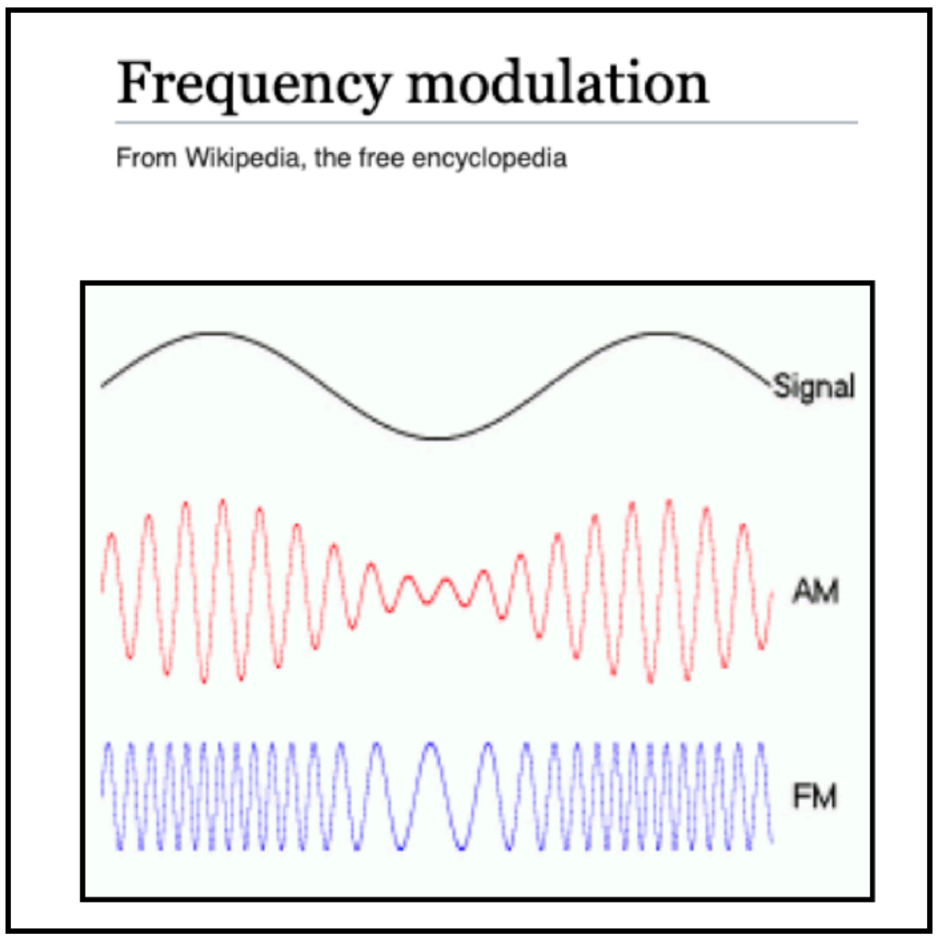
1. Motivate “analytic signals”

2. Understand & derive negative frequencies & DFT symmetry

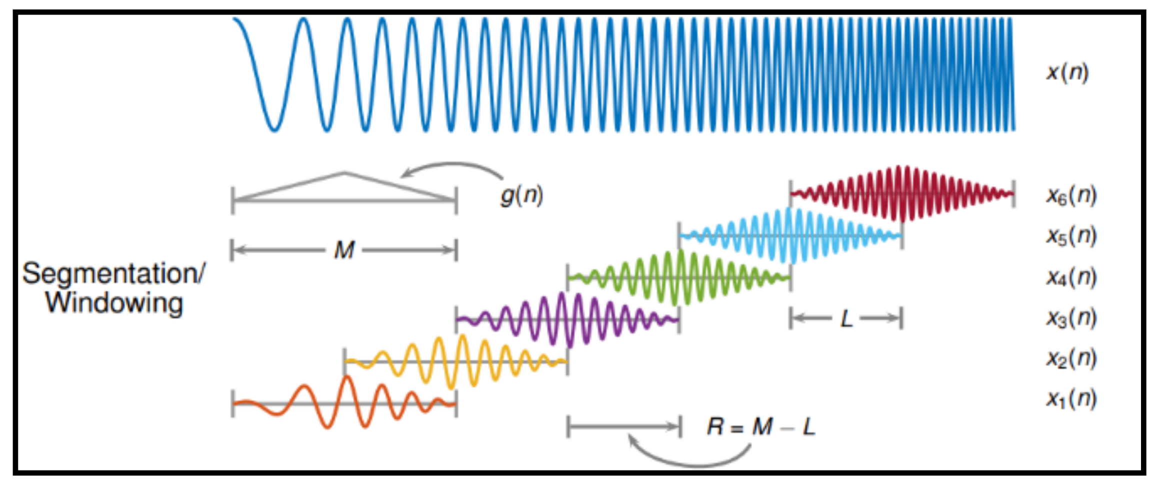
3. Conceptualize Hilbert Transform

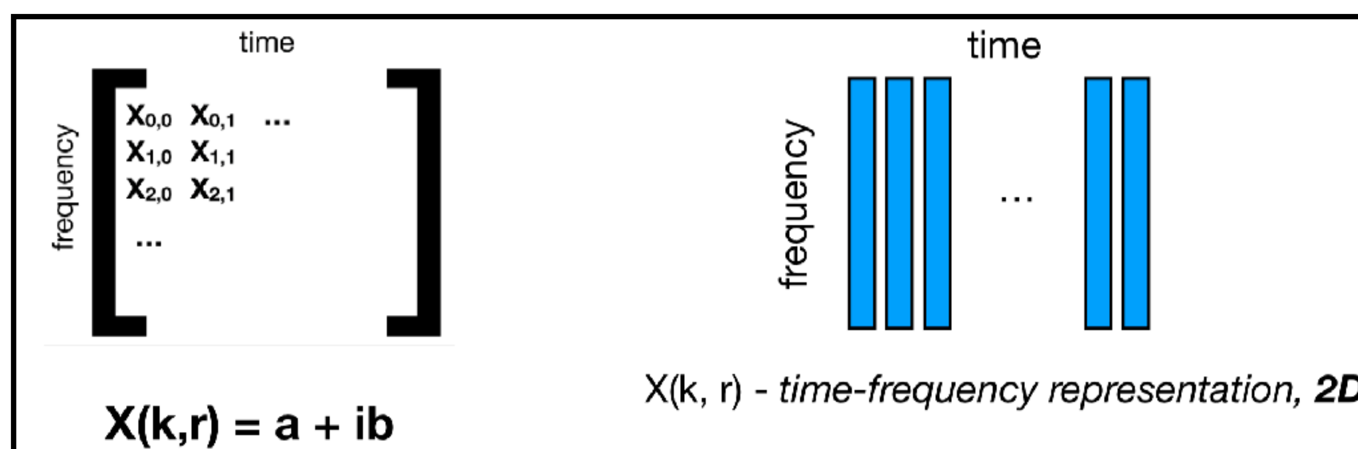
“Instantaneous” Power and Frequency

Amplitude (power) and frequency can change very quickly for non-stationary signals like AM & FM radio.



Our proposed solution so far:
Time-Frequency Analysis (STFT)





“Instantaneous” Power and Frequency

Our proposed solution so far:
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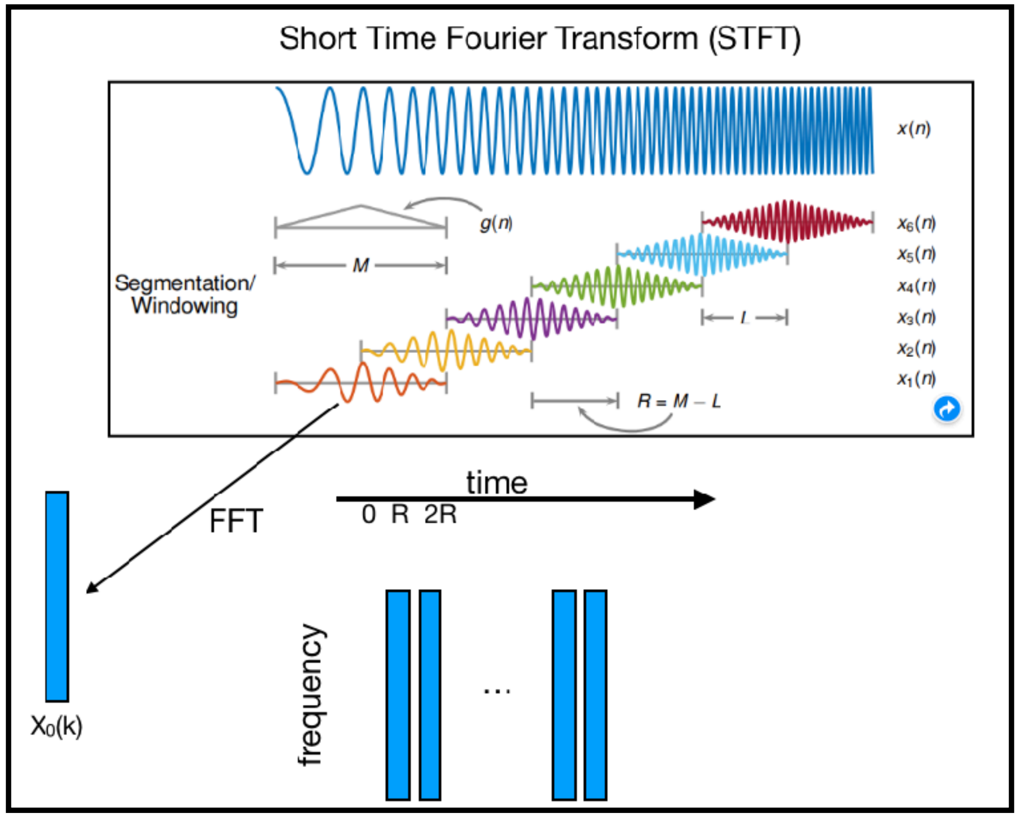
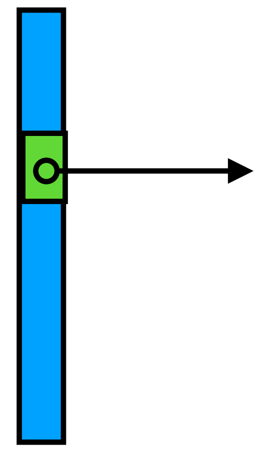
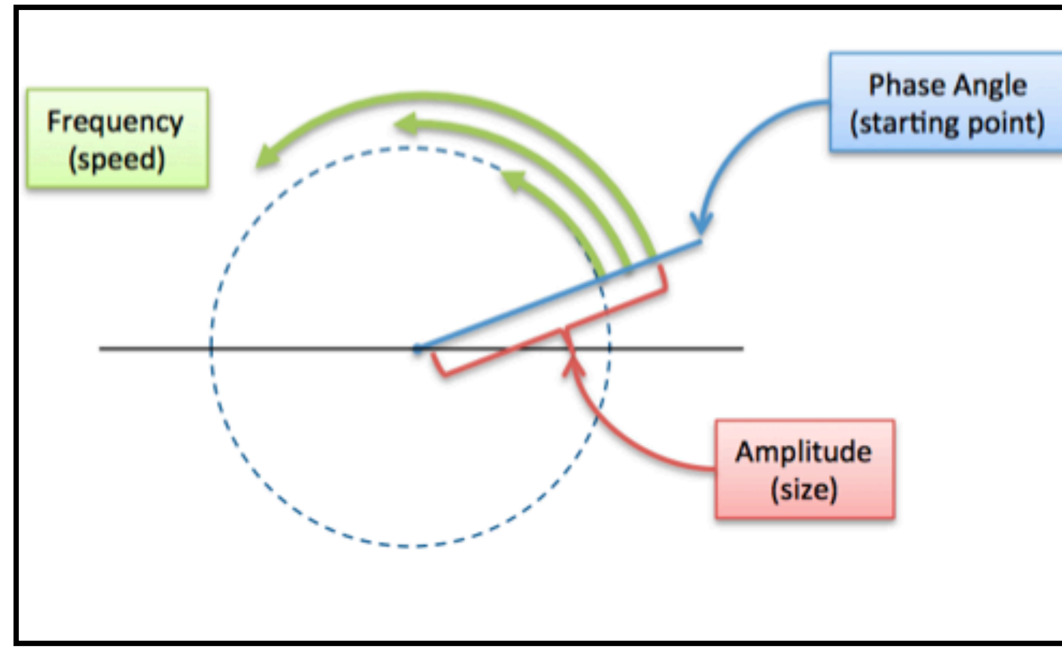




$X_0(k)$



average amplitude and phase of data inside the short window

STFT Parameters (Choice)
M = window length (s)
R = step length (s)
L = overlap length (s)
* **M = R + L**

We can only distinguish a frequency resolution of ____, at a time resolution of ____

How can we improve both?

“Instantaneous” Power and Frequency

STFT Parameters (Choice)

M = window length (s)

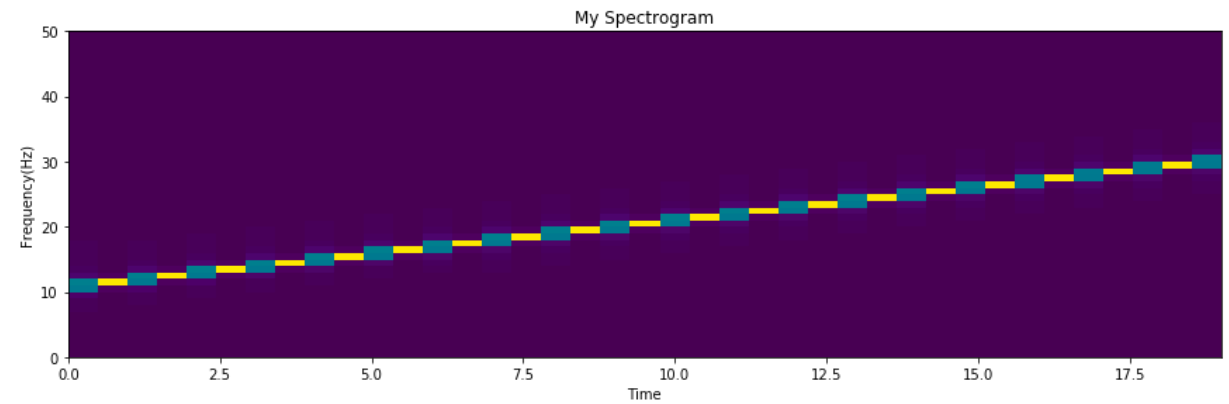
R = step length (s)

L = overlap length (s)

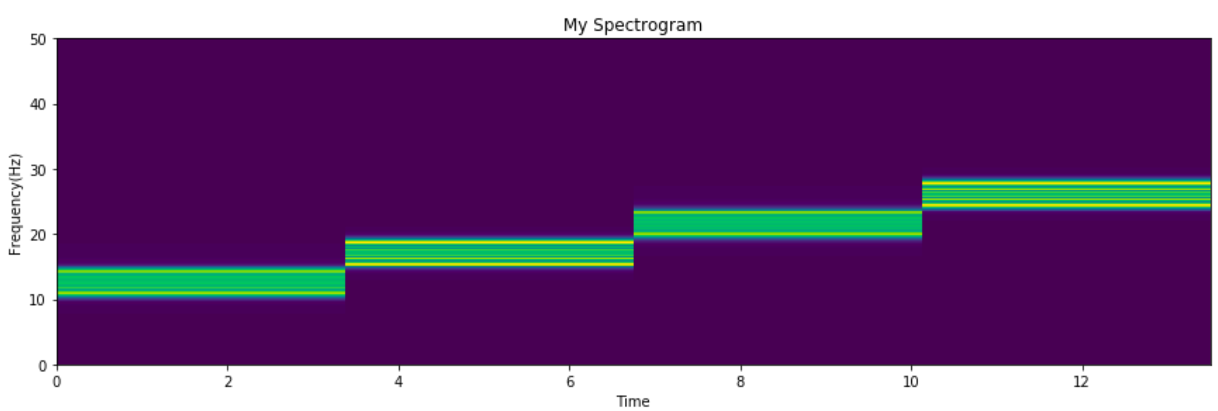
*** M = R + L**

How can we improve both?

M = 1s, R=0.5s

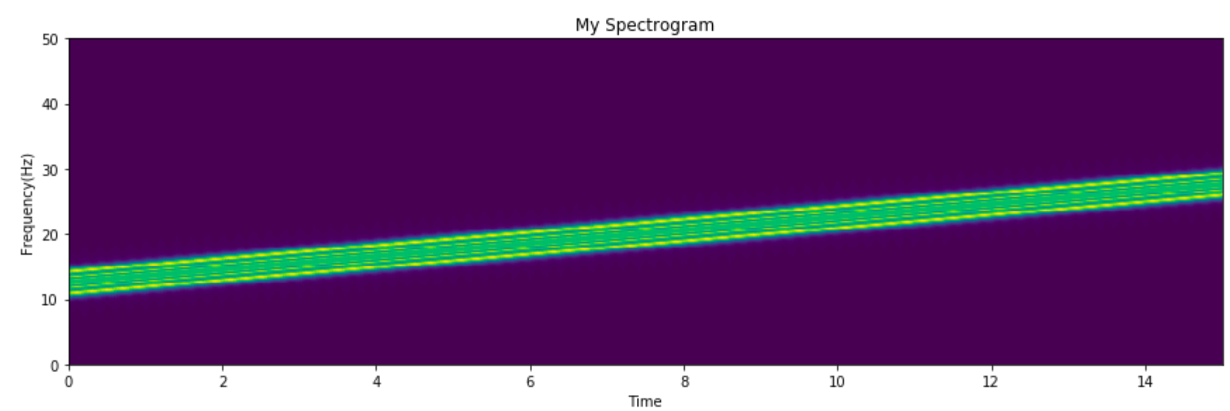


M = 5s, R=0.5s



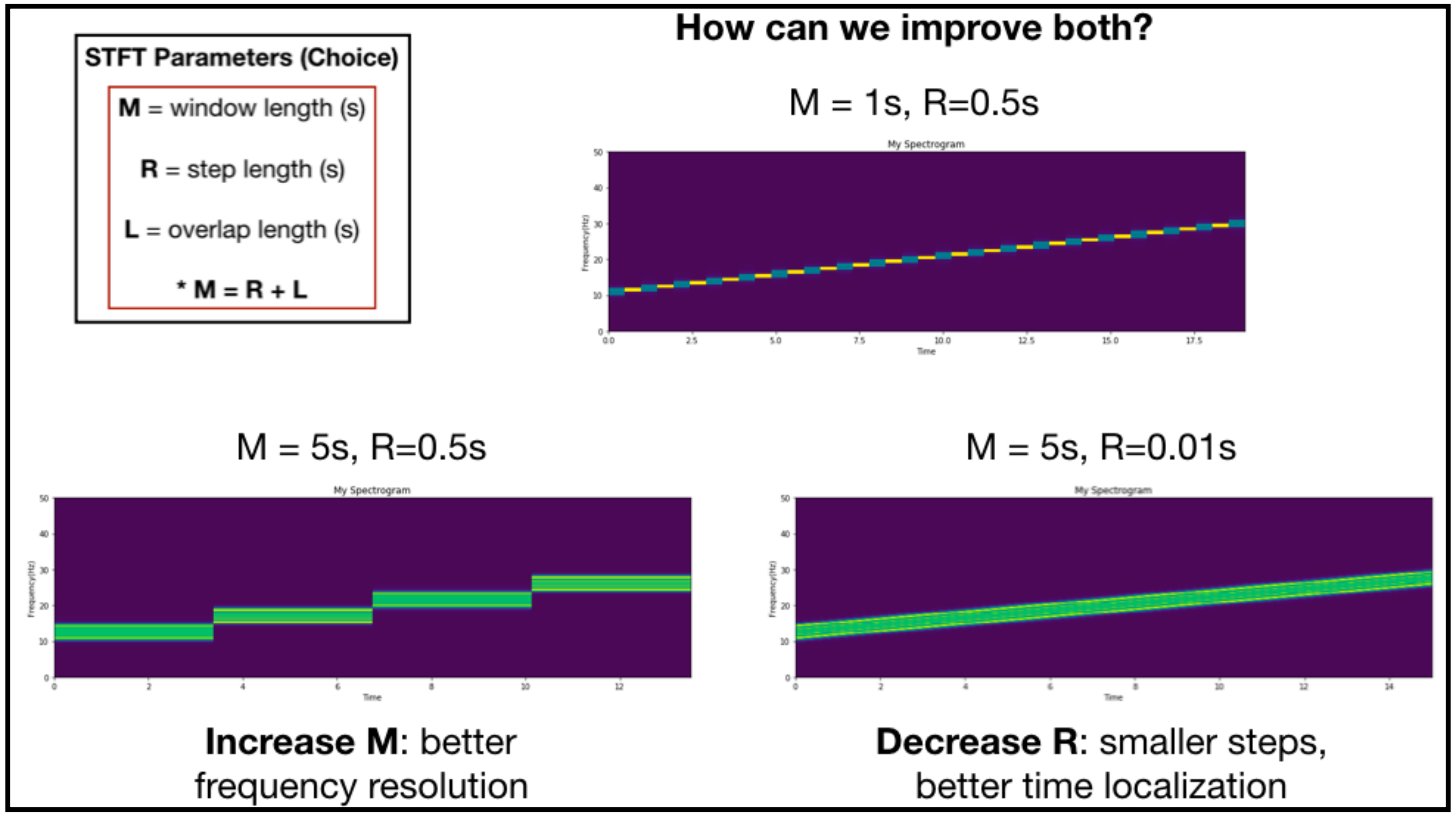
Increase M: better frequency resolution

M = 5s, R=0.01s

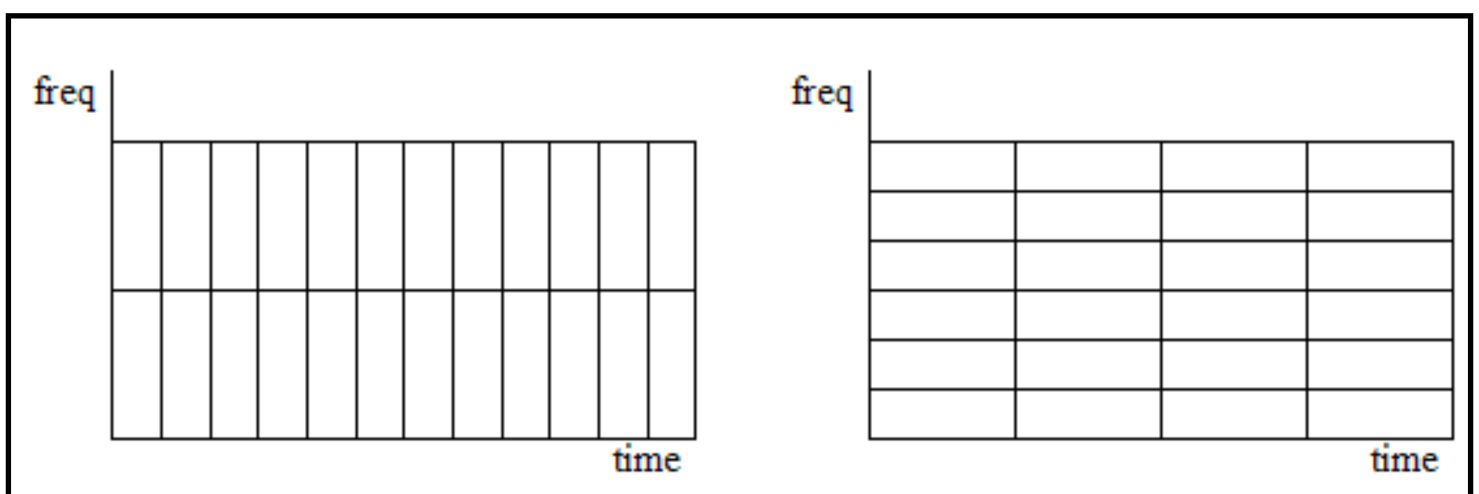


Decrease R: smaller steps, better time localization

“Instantaneous” Power and Frequency



But there is a limit on both, and it’s a huge waste.



In the Brain: Neural Signals

Neural oscillations are never stationary:
power and frequency change very quickly.

nature COMMUNICATIONS

Article | OPEN | Published: 12 December 2017

Fluctuations in instantaneous frequency predict alpha amplitude during visual perception

Stephanie Nelli, Sirawaj Itthipuripat, Ramesh Srinivasan & John T. Serences

Frequency modulation of neural oscillations according to visual task demands

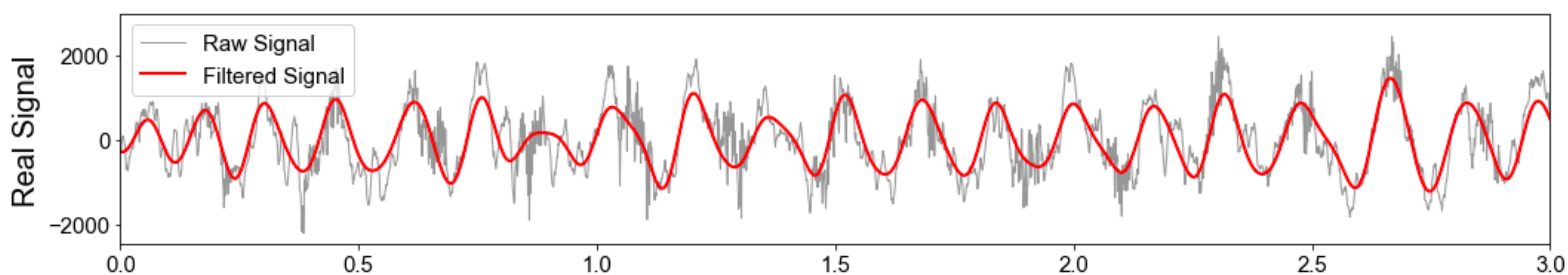
Andreas Wutz, David Melcher, and Jason Samaha

Fluctuations in Oscillation Frequency Control Spike Timing and Coordinate Neural Networks

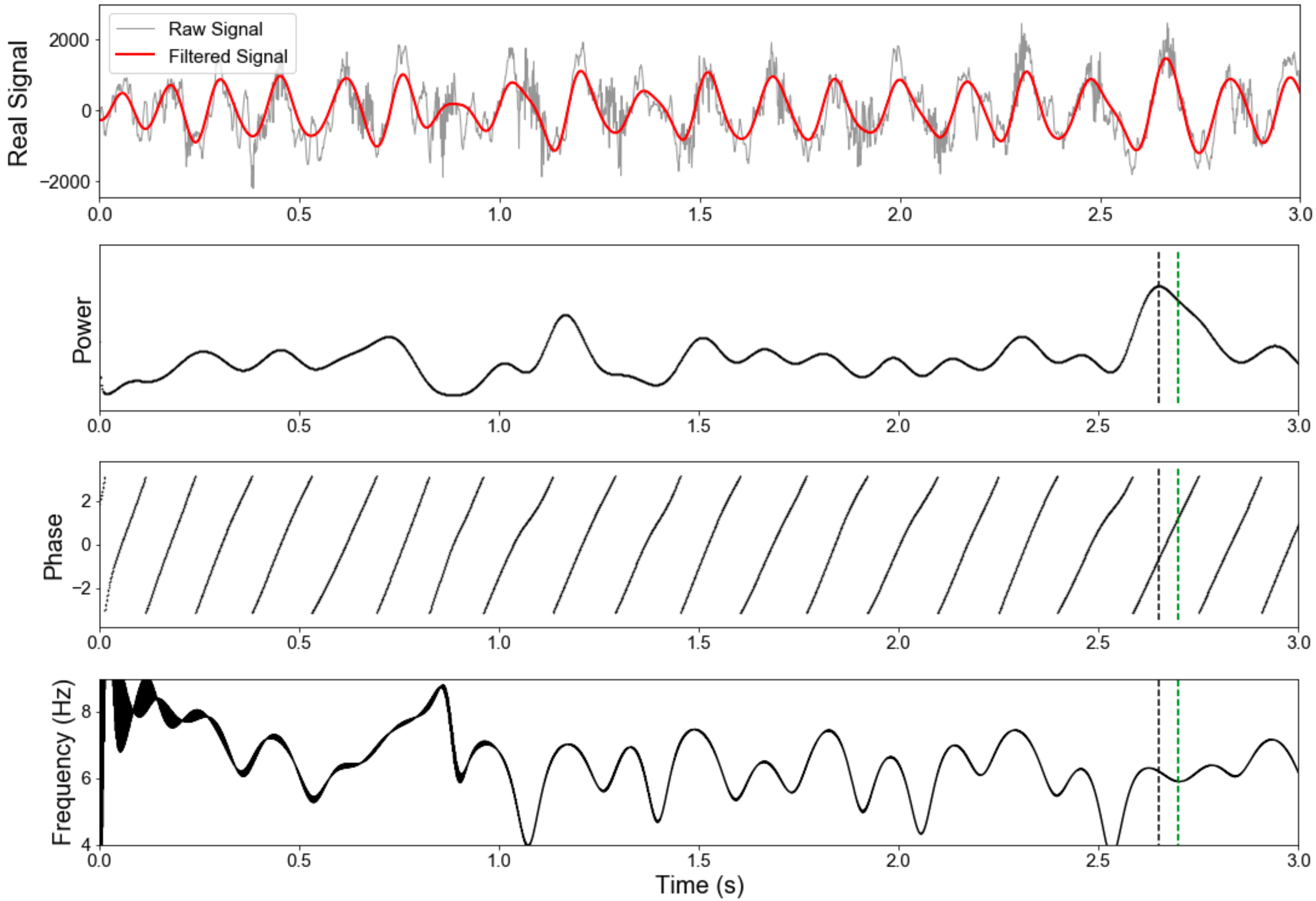
Michael X Cohen

Journal of Neuroscience 2 July 2014, 34 (27) 8988-8998; DOI: https://doi.org/10.1523/JNEUROSCI.0261-14.2014

rat hippocampus



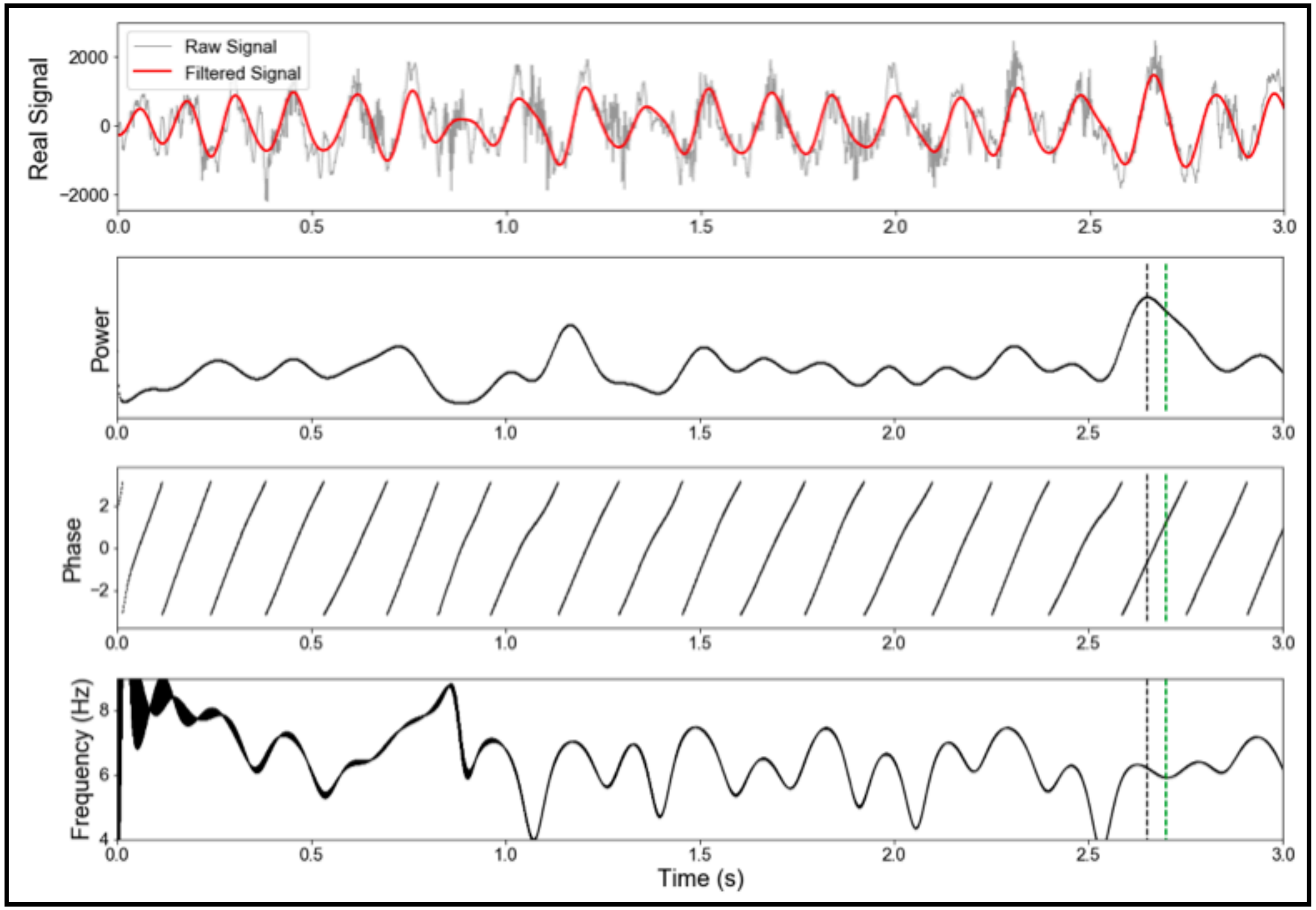
In Neural Signals



instantaneous frequency Φ: derivative of phase θ (rate of phase change)

$$\phi(t) = \frac{d\theta(t)}{dt}.$$

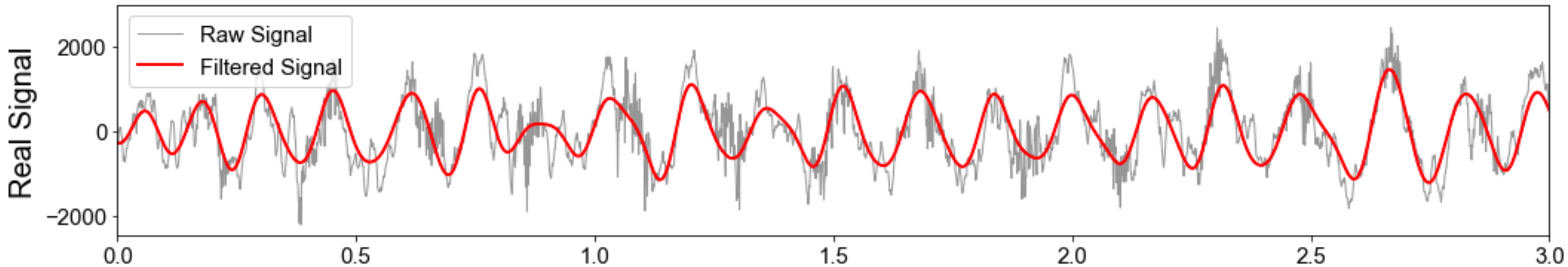
How to Define/Compute?



How do we actually compute these quantities?

What are their definitions?

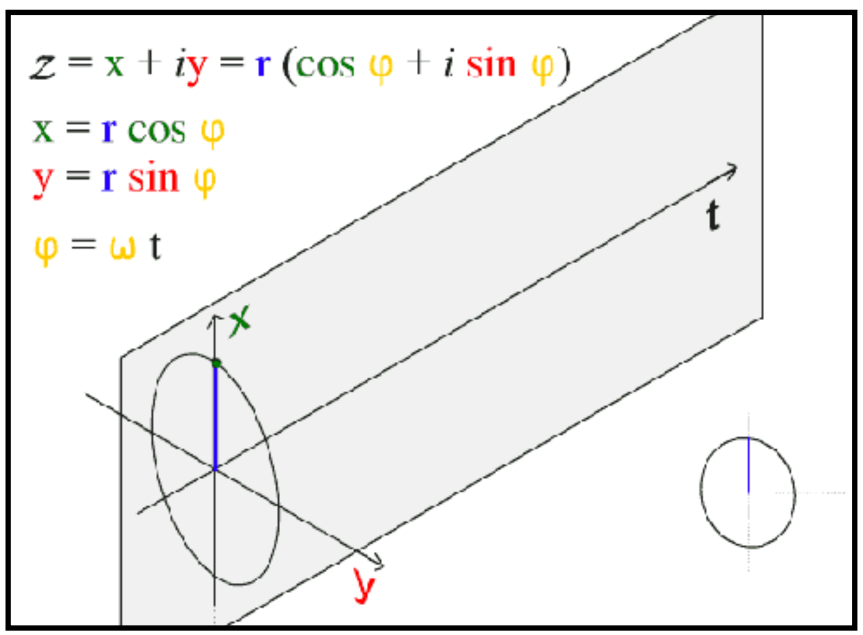
Analytic Signal



This real-valued signal does not have defined **instantaneous measures.**

Intuition: if I gave you a single time point, can you know its amplitude & phase?

Amplitude & phase both change, so you need 2 pieces of information (numbers)

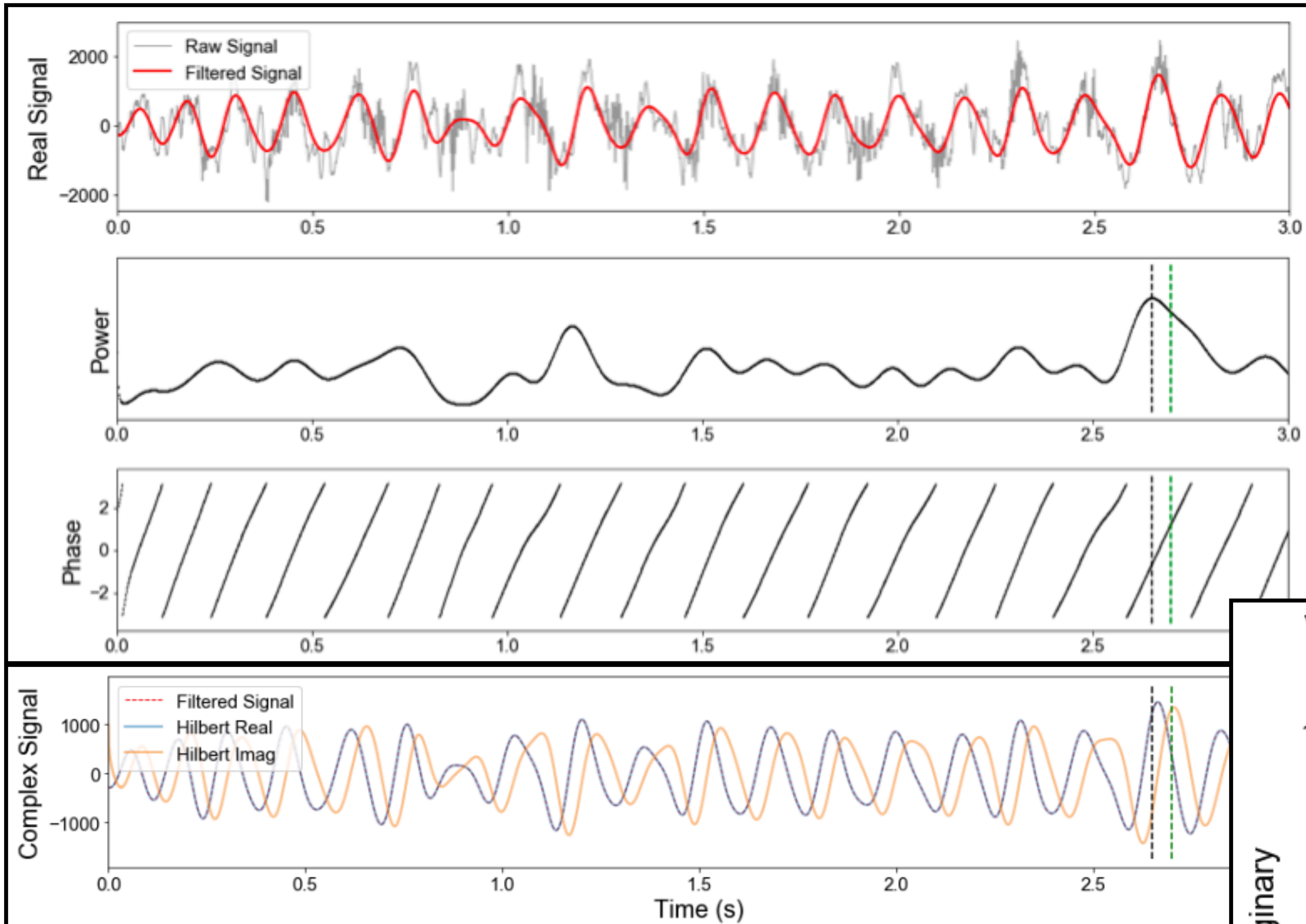


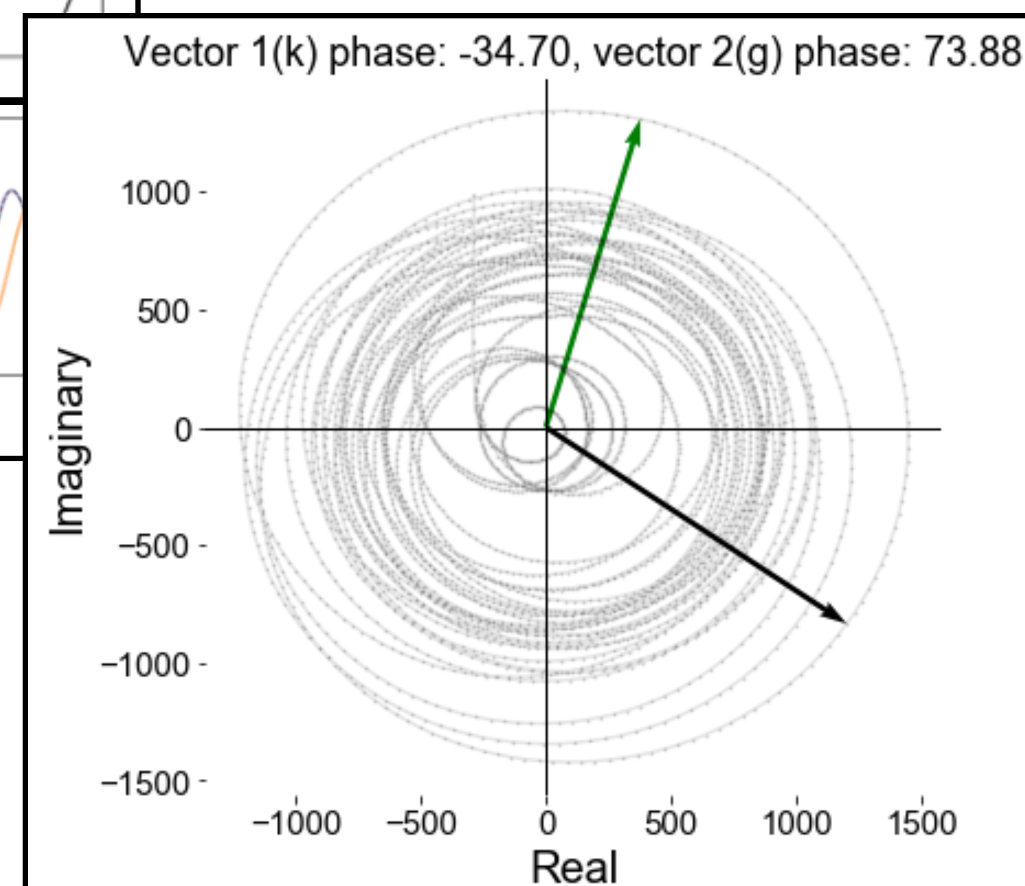
Analytic Signal: Complex Time-Series

Has well-defined instantaneous time-varying amplitude and phase (and rate of phase change)

Amplitude: magnitude
Phase: angle

How to Define/Compute?



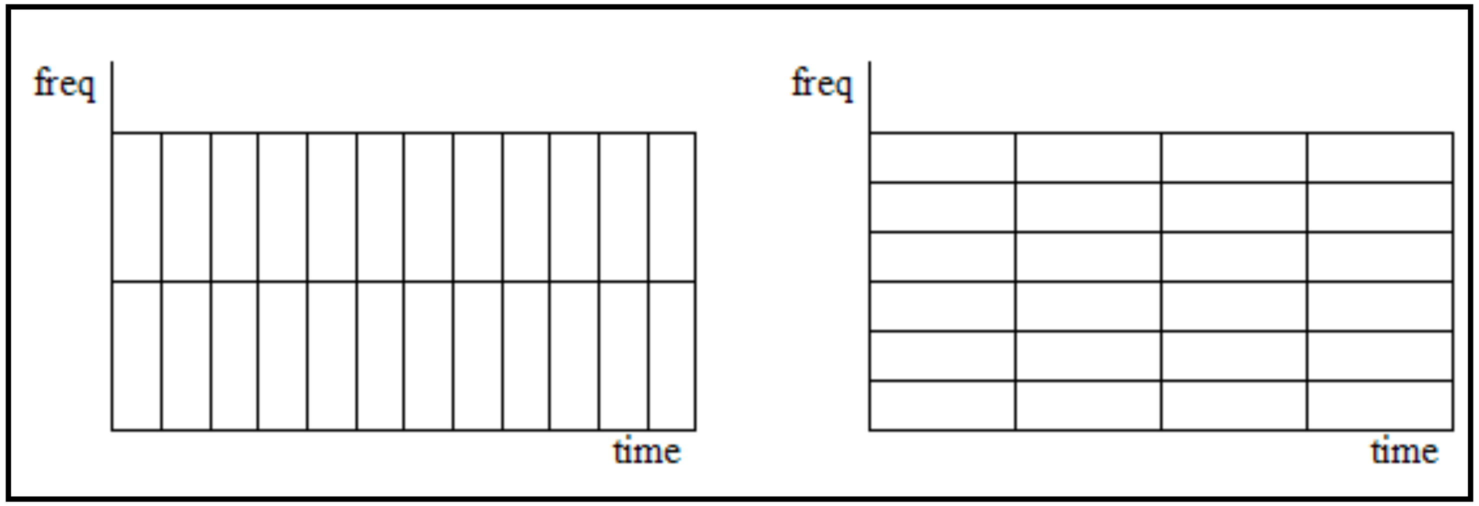


Analytic Signal: Complex Time-Series

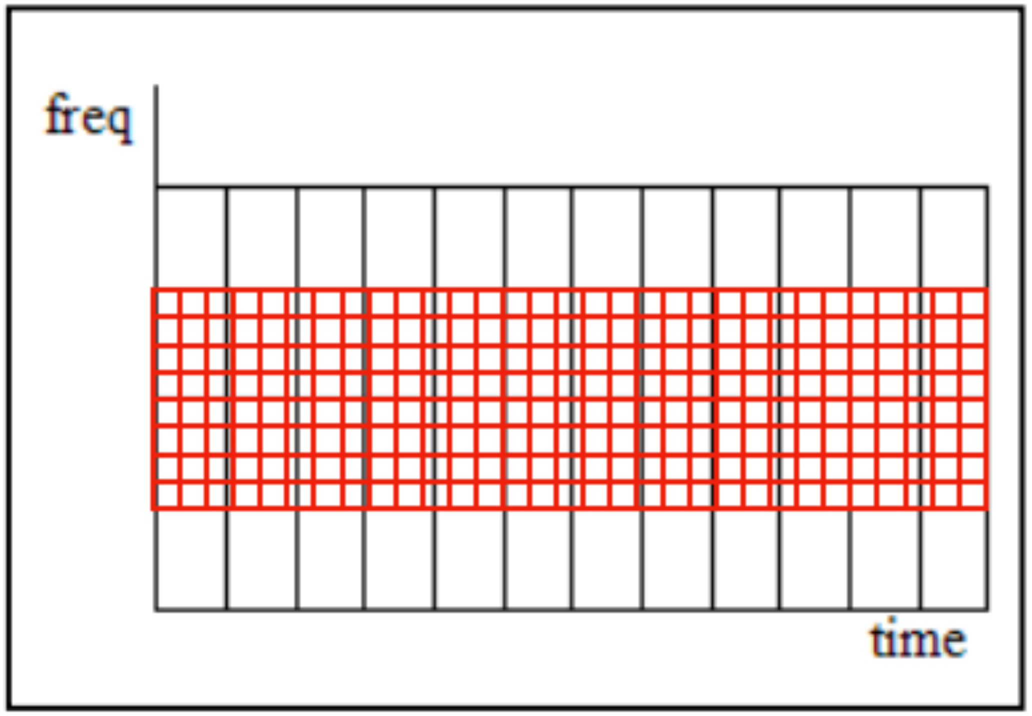
Has well-defined instantaneous time-varying amplitude and phase (and rate of phase change)

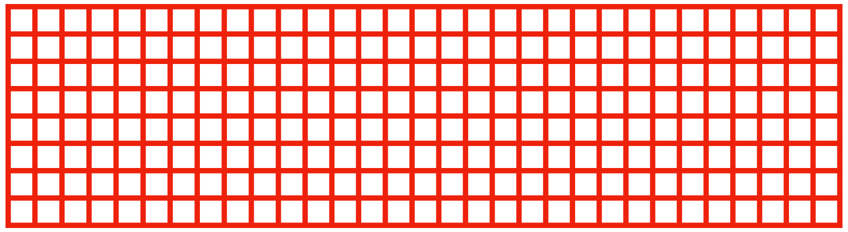
Better Time **and** Frequency Resolution

STFT



Analytic Signal





time resolution: **dt**
frequency resolution: **df**

For a “**well-behaved**” signal:
(usually) narrowband & denoised

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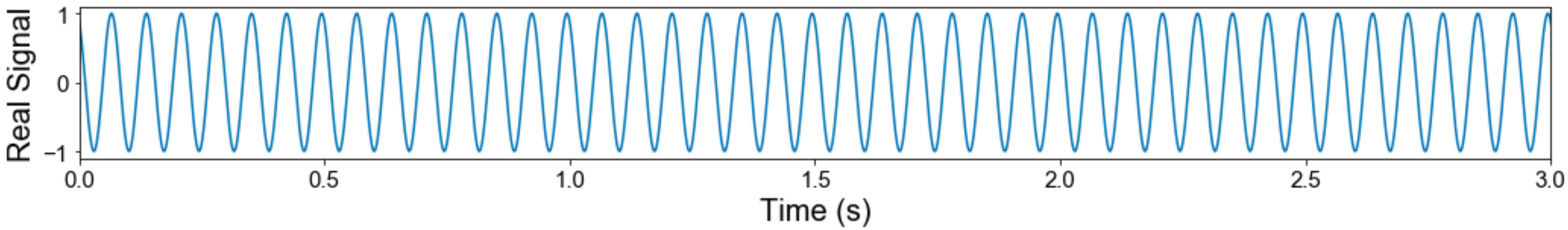
DFT: Negative Frequencies



$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$
$$= \sum_{n=0}^{N-1} x_n \cdot \left[\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)\right], \quad \textbf{\textit{(Eq.1)}}$$

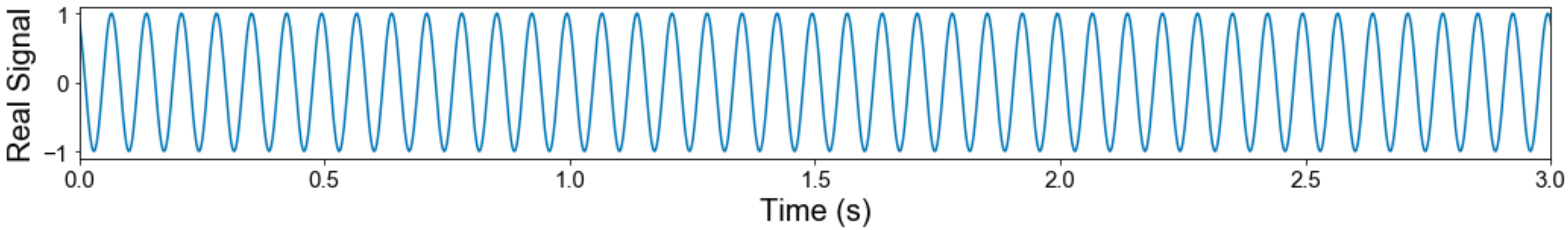
$$x(n) = cos(2\pi 14n/N + \theta)$$

If you do FT on the first 1s of data, what are the X_k's (which ones are not 0)?
hint: orthogonality

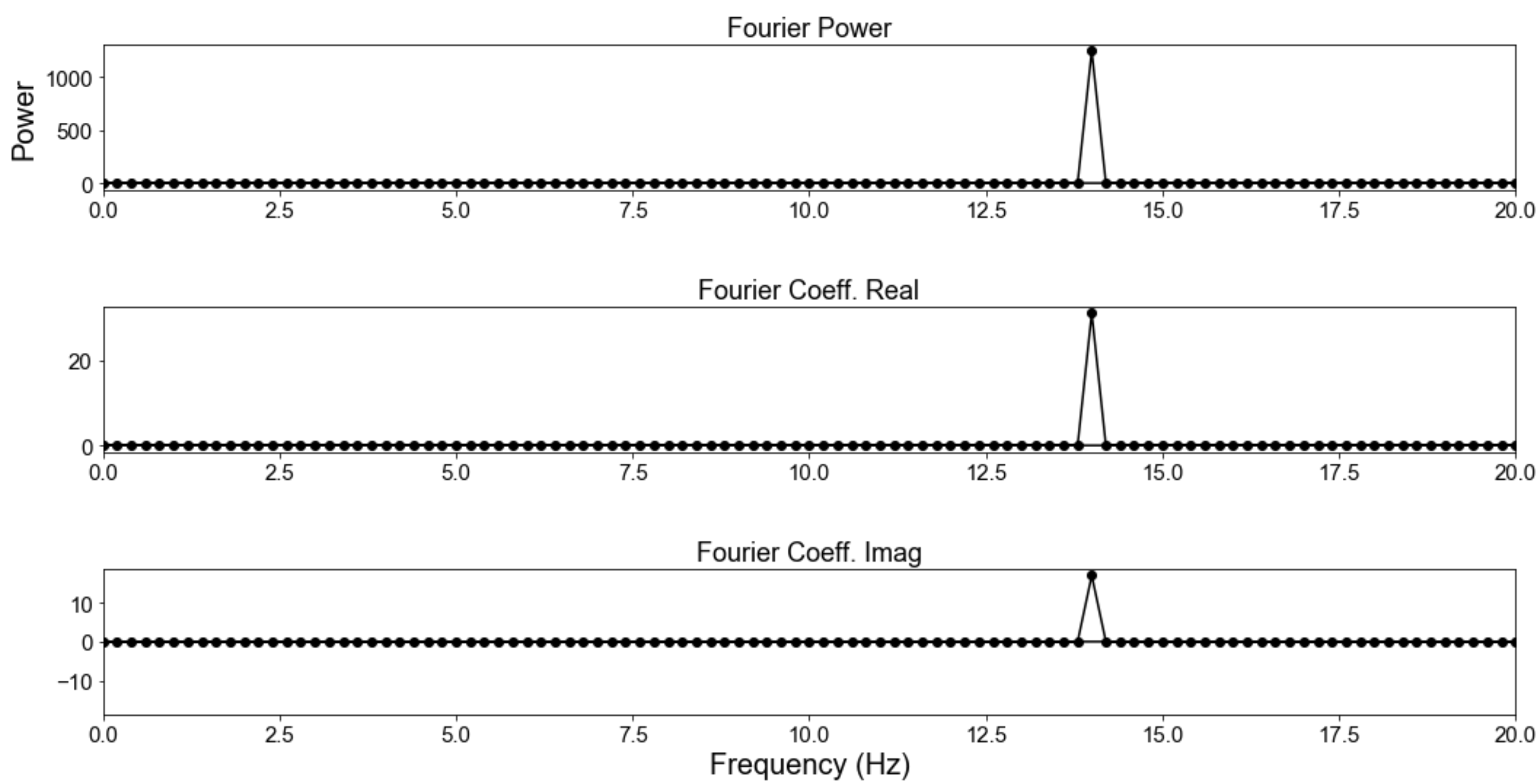
X_{14} = a - bi, rest are 0s…

or are they?

DFT: Negative Frequencies

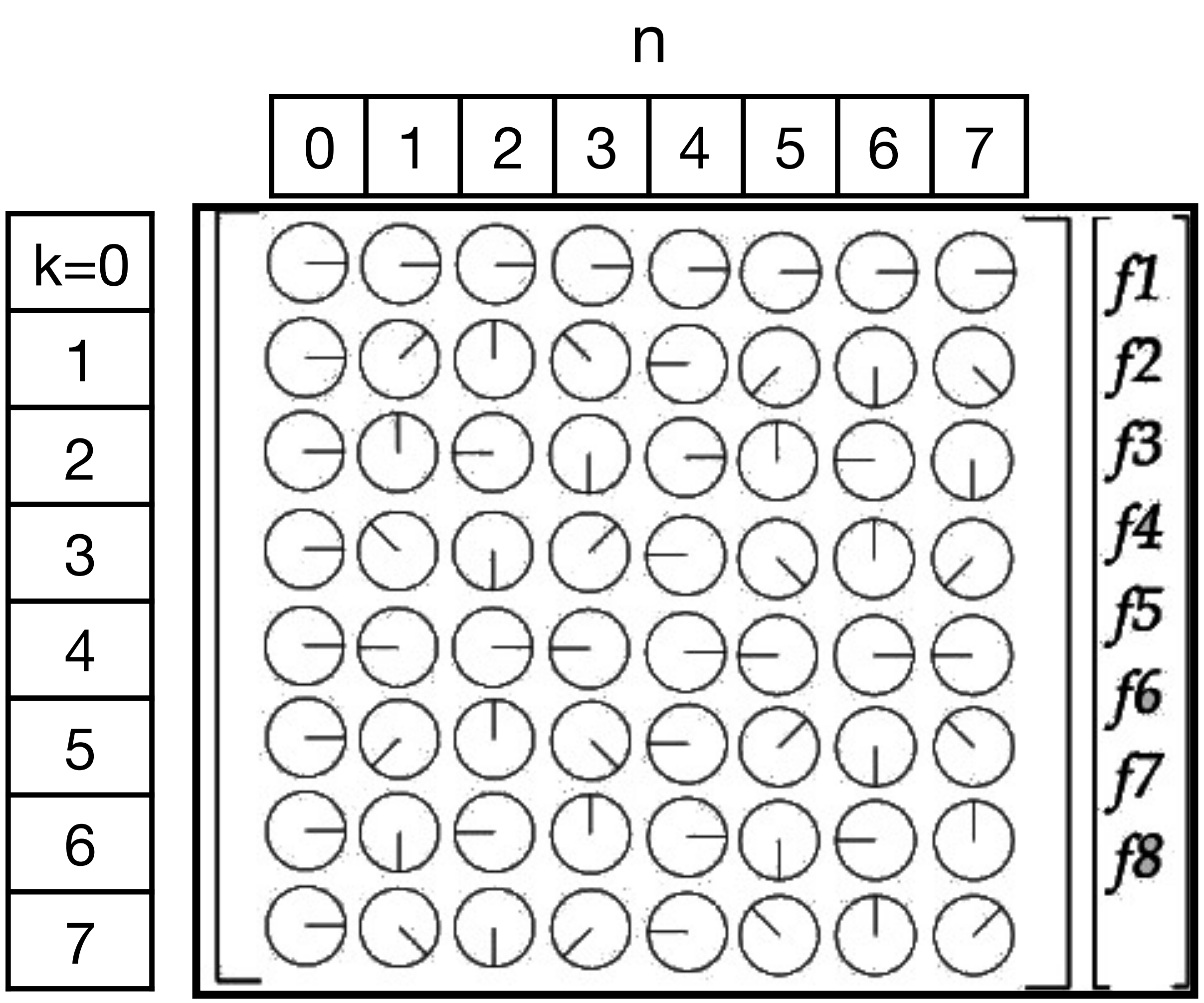


What we've been looking at: **positive frequencies**



$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$
$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)], \quad \textbf{(Eq.1)}$$

k = 0, 1, 2, … N-1



$$cos(2\pi - \theta) =$$
$$sin(2\pi - \theta) =$$

$\mathbf{X_{N-k} = X_{-k}}$

k = N/2 is the end
freq(k) = ?

High School Trig Gymnastics

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$
$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)], \quad \textbf{(Eq.1)}$$

k = 0, 1, 2, … N-1

if x(n) **real**, and

cos(2*pi*k*n/N) = cos(-2*pi*k*n/N) = cos(2*pi*k*n/N)

X_k = a - bi

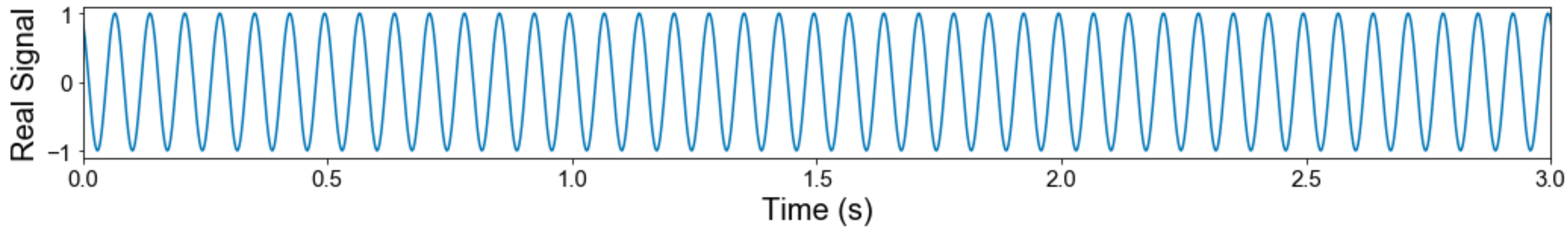
sin(2*pi*k*n/N) = sin(-2*pi*k*n/N) = -sin(2*pi*k*n/N)

X_{N-k} = ?

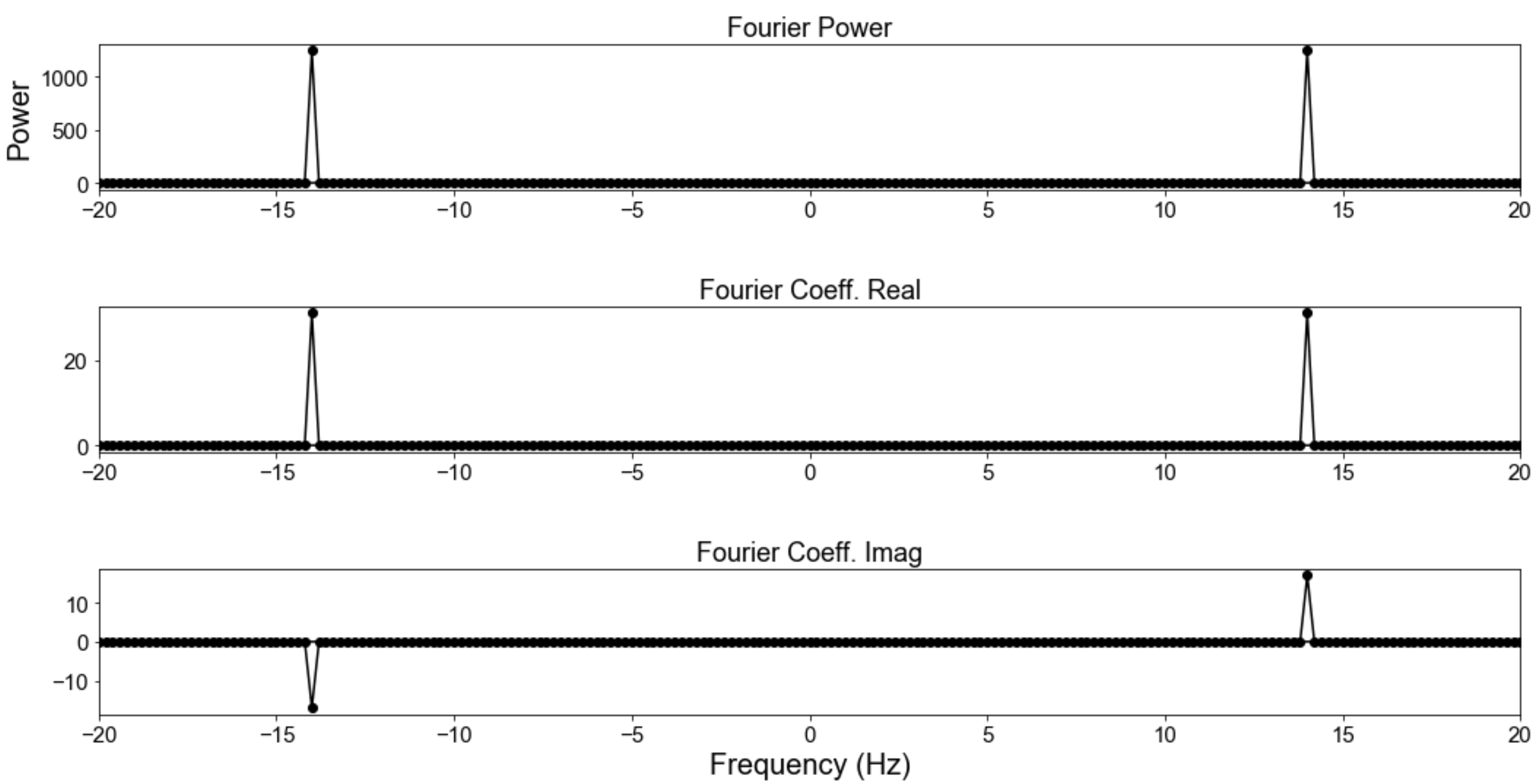
a + bi

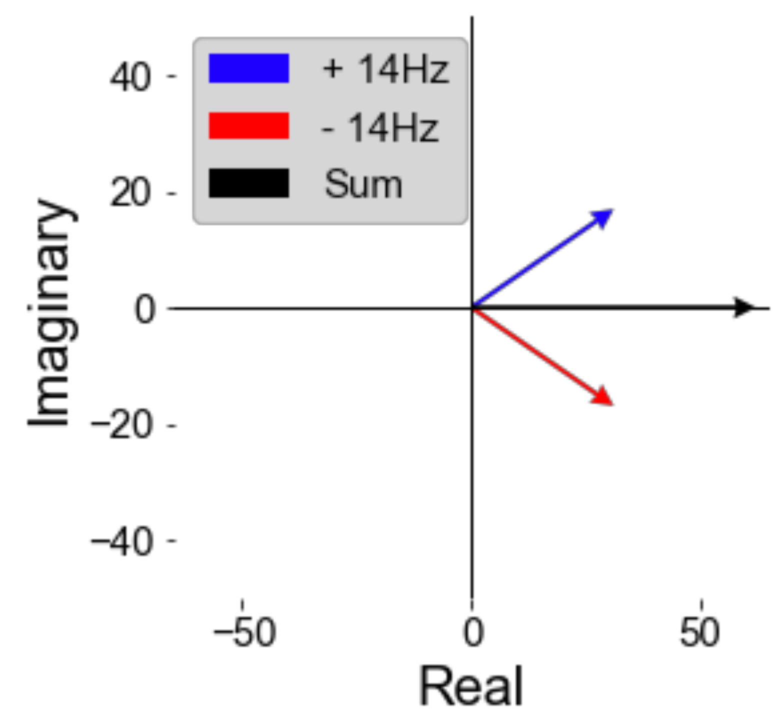
X_k and X_{-k} are **complex conjugates.**

DFT: Negative Frequencies



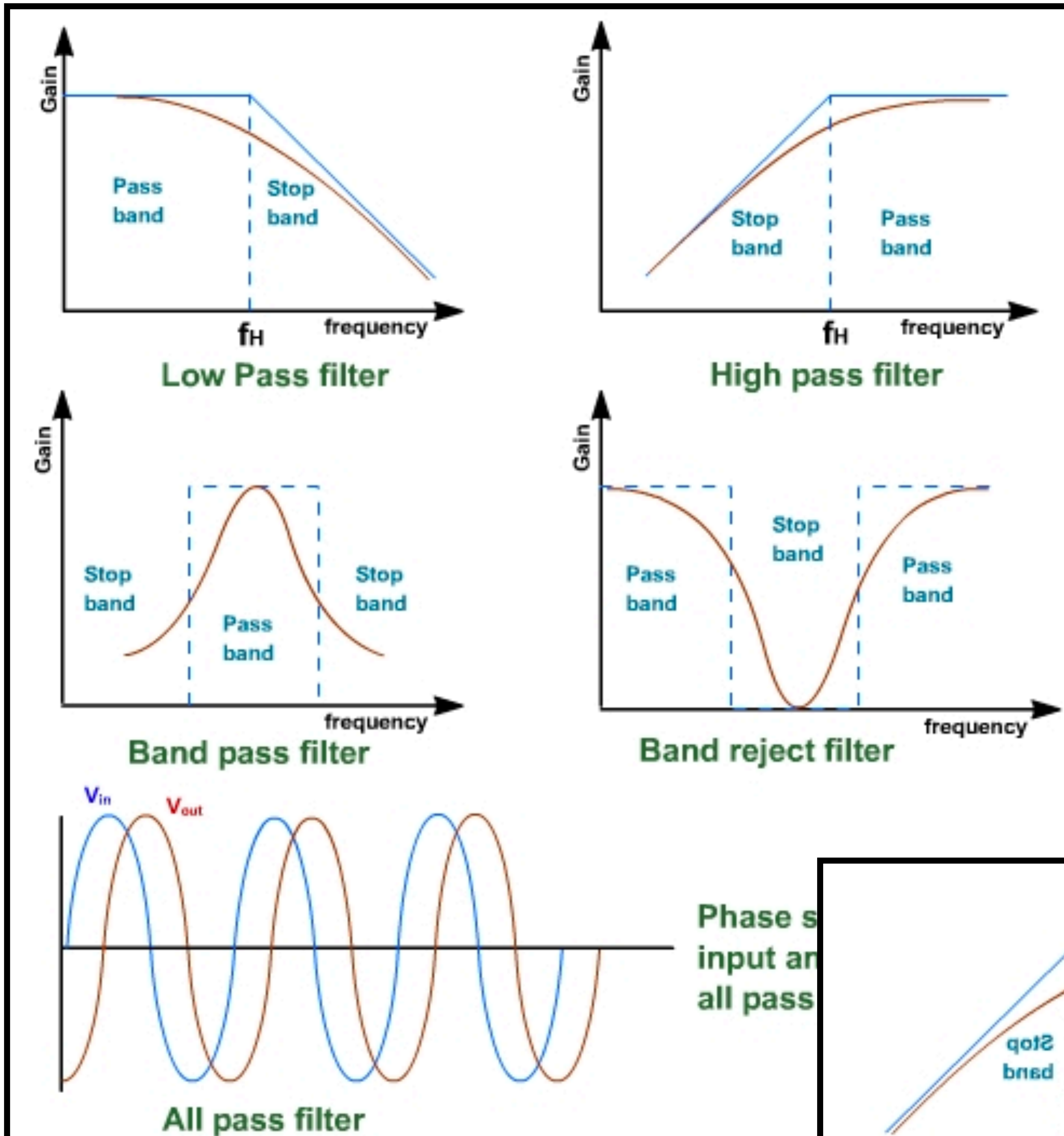
Here's the **full** spectrum: with negative frequencies

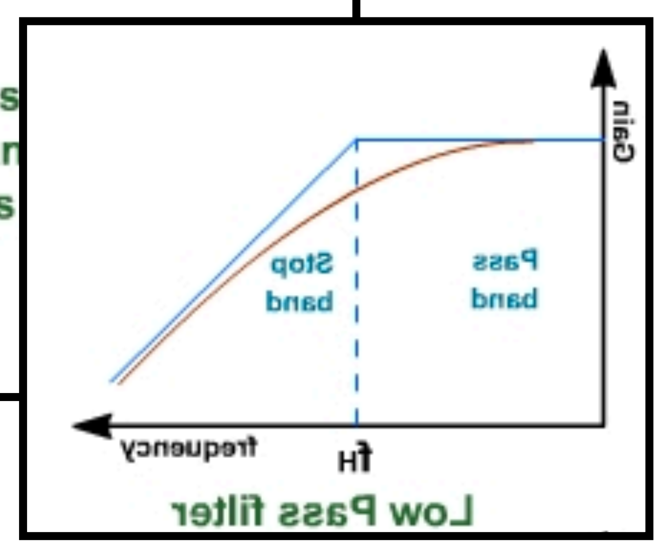


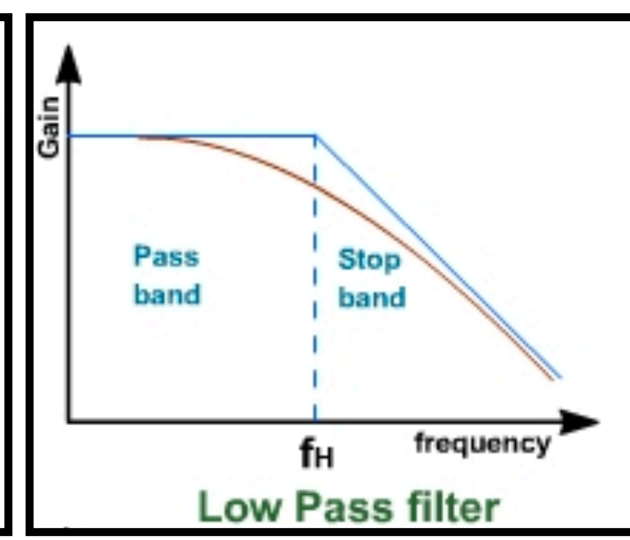


$$x(t) = 2\cos(\omega_0 t) = e^{j\omega_0 t} + e^{-j\omega_0 t}.$$

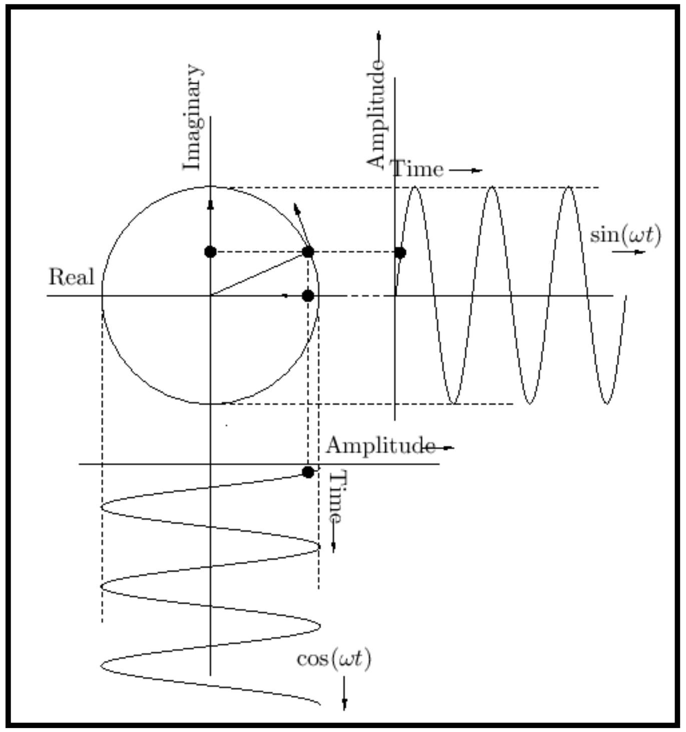
Filter Response is Symmetric Too



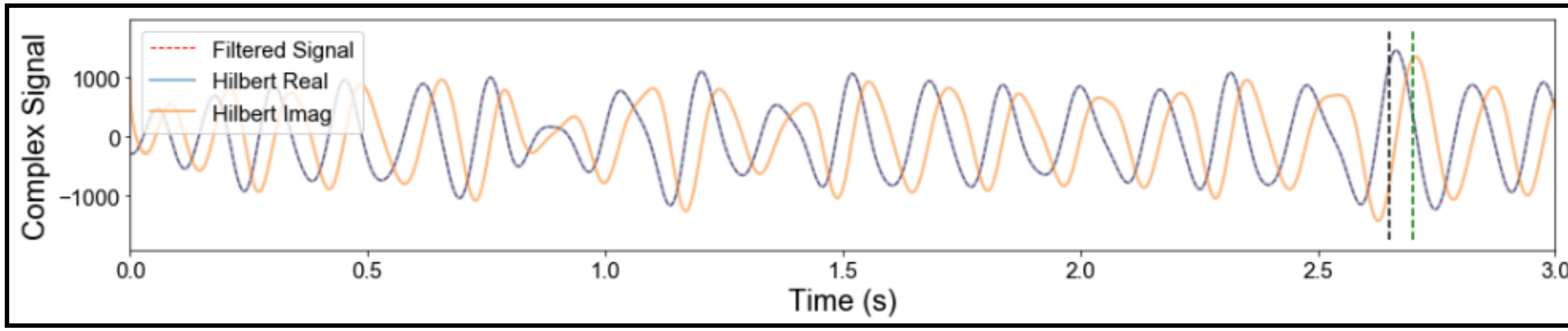




Cancellation of Imaginary Component



Analytic (Complex) Signal: a single complex exponential



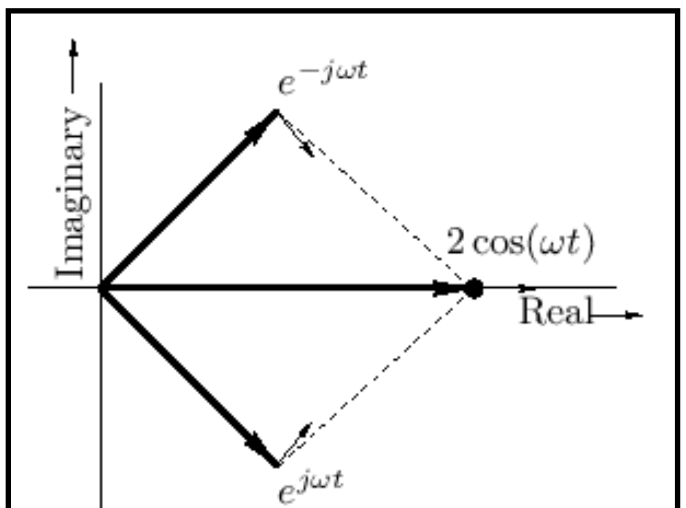
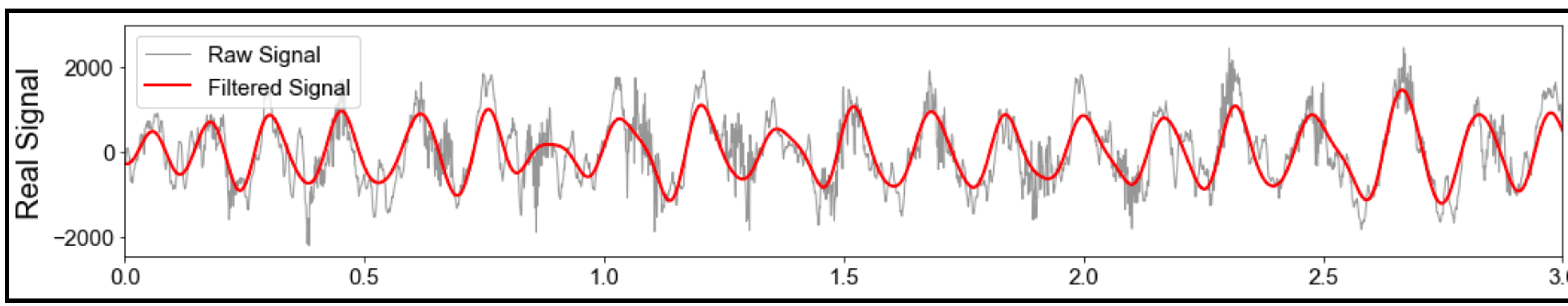


Figure 1.9: Opposite circular motions add to give real sinusoidal motion,
$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$.

Real Signal: sum of 2 complex exponential conjugates



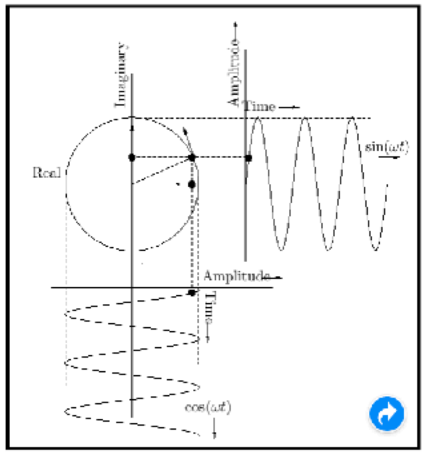
How to retrieve the analytic signal from the real signal?

1. Motivate “analytic signals”

2. Understand & derive negative frequencies & DFT symmetry

3. Conceptualize Hilbert Transform

Hilbert Transform



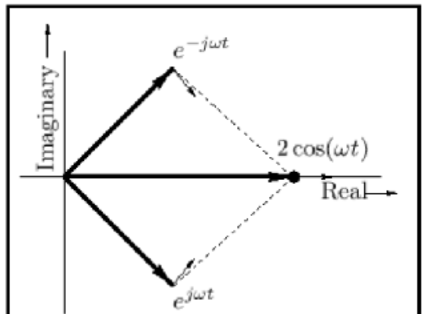
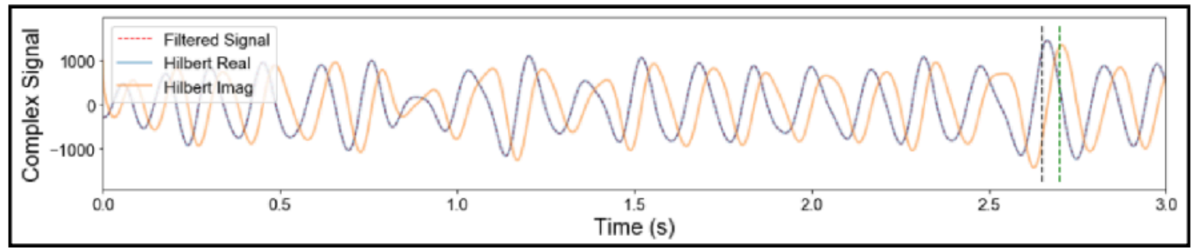
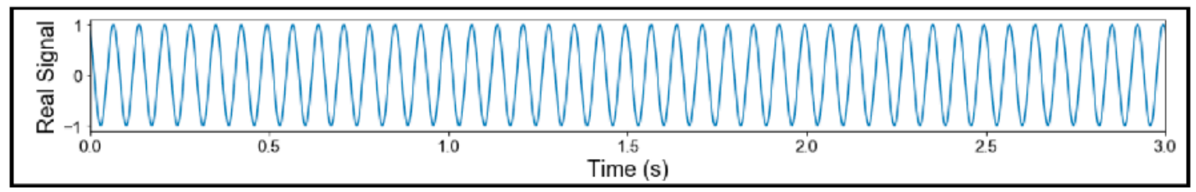


Figure 1.9: Opposite circular motions add to give real sinusoidal motion, $e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$.

Analytic (Complex) Signal: a single complex exponential



Real Signal: sum of 2 complex exponential conjugates



How to retrieve the analytic signal from the real signal?

scipy.signal.hilbert

scipy.signal.**hilbert**(*x*, *N=None*, *axis=-1*)

Compute the analytic signal, using the Hilbert transform.

The transformation is done along the last axis by default.

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scipy.signal.**hilbert**(*x*, *N=None*, *axis=-1*)

Compute the analytic signal, using the Hilbert transform.

The transformation is done along the last axis by default.

Parameters:

x : *array_like*
Signal data. Must be real.

N : *int, optional*
Number of Fourier components. Default: `x.shape[axis]`

axis : *int, optional*
Axis along which to do the transformation. Default: -1.

Returns:

xa : *ndarray*
Analytic signal of *x*, of each 1-D array along *axis*

See also:

scipy.fftpack.hilbert Return Hilbert transform of a periodic sequence x.

Notes

The analytic signal `x_a(t)` of signal `x(t)` is:

$$x_a = F^{-1}(F(x)2U) = x + iy$$

where *F* is the Fourier transform, *U* the unit step function, and *y* the Hilbert transform of *x*. [1]

Hilbert Transform

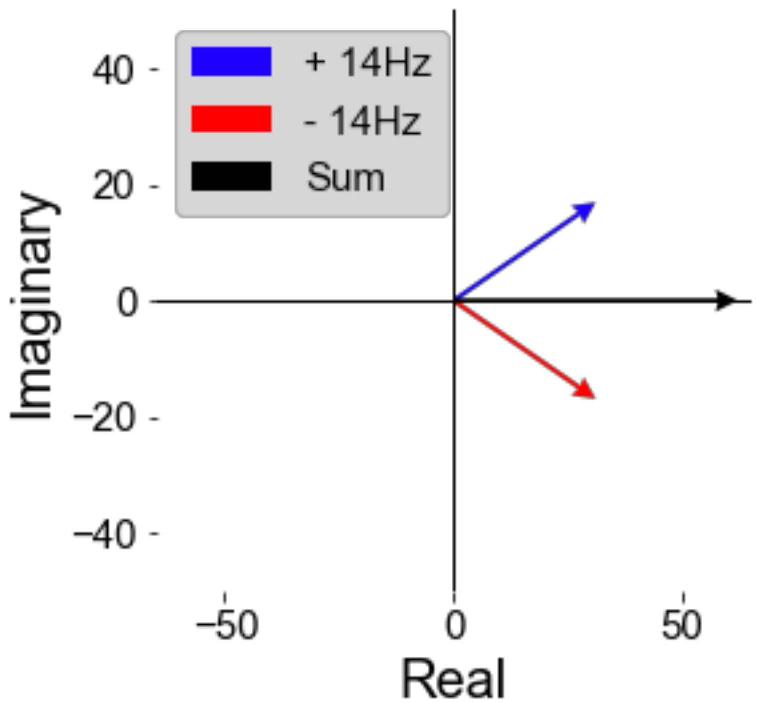
The analytic signal `x_a(t)` of signal `x(t)` is:

$$x_a = F^{-1}(F(x)2U) = x + iy$$

where F is the Fourier transform, U the unit step function, and y the Hilbert transform of x. [1]

U is the unit step function: U(x) = 1 for x>0, 0 for x<0

analytic signal: FT -> lose the negative spectrum -> inverse FT



y is the "Hilbert Transform" of x.

"Hilbert Transform" is also the operation.

You might also hear "Hilbert Transform" to mean the analytic signal.

Hilbert Transform

The analytic signal `x_a(t)` of signal `x(t)` is:

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analytic signal: FT -> lose the negative spectrum -> inverse FT

Time Domain Convolution

$$H(u)(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau}\, d\tau,$$

$$1/(\pi t)$$

Frequency Domain Multiply

$$\mathcal{F}(H(u))(\omega) = (-i\,\mathrm{sgn}(\omega)) \cdot \mathcal{F}(u)(\omega)$$

$$\sigma_H(\omega) = \begin{cases} i = e^{+\frac{i\pi}{2}}, & \text{for } \omega < 0 \\ 0, & \text{for } \omega = 0 \\ -i = e^{-\frac{i\pi}{2}}, & \text{for } \omega > 0 \end{cases}$$

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Analytic Signals and Hilbert Transform Filters

A signal which has no negative-frequency components is called an *analytic signal*.[4.12] Therefore, in continuous time, every analytic signal $z(t)$ can be represented as

$$z(t) = \frac{1}{2\pi}\int_0^\infty Z(\omega)e^{j\omega t}d\omega$$

where $Z(\omega)$ is the complex coefficient (setting the amplitude and phase) of the positive-frequency complex sinusoid $\exp(j\omega t)$ at frequency ω .

Any real sinusoid $A\cos(\omega t + \phi)$ may be converted to a positive-frequency complex sinusoid $A\exp[j(\omega t + \phi)]$ by simply generating a phase-quadrature component $A\sin(\omega t + \phi)$ to serve as the ``imaginary part'':

$$Ae^{j(\omega t+\phi)} = A\cos(\omega t + \phi) + jA\sin(\omega t + \phi)$$

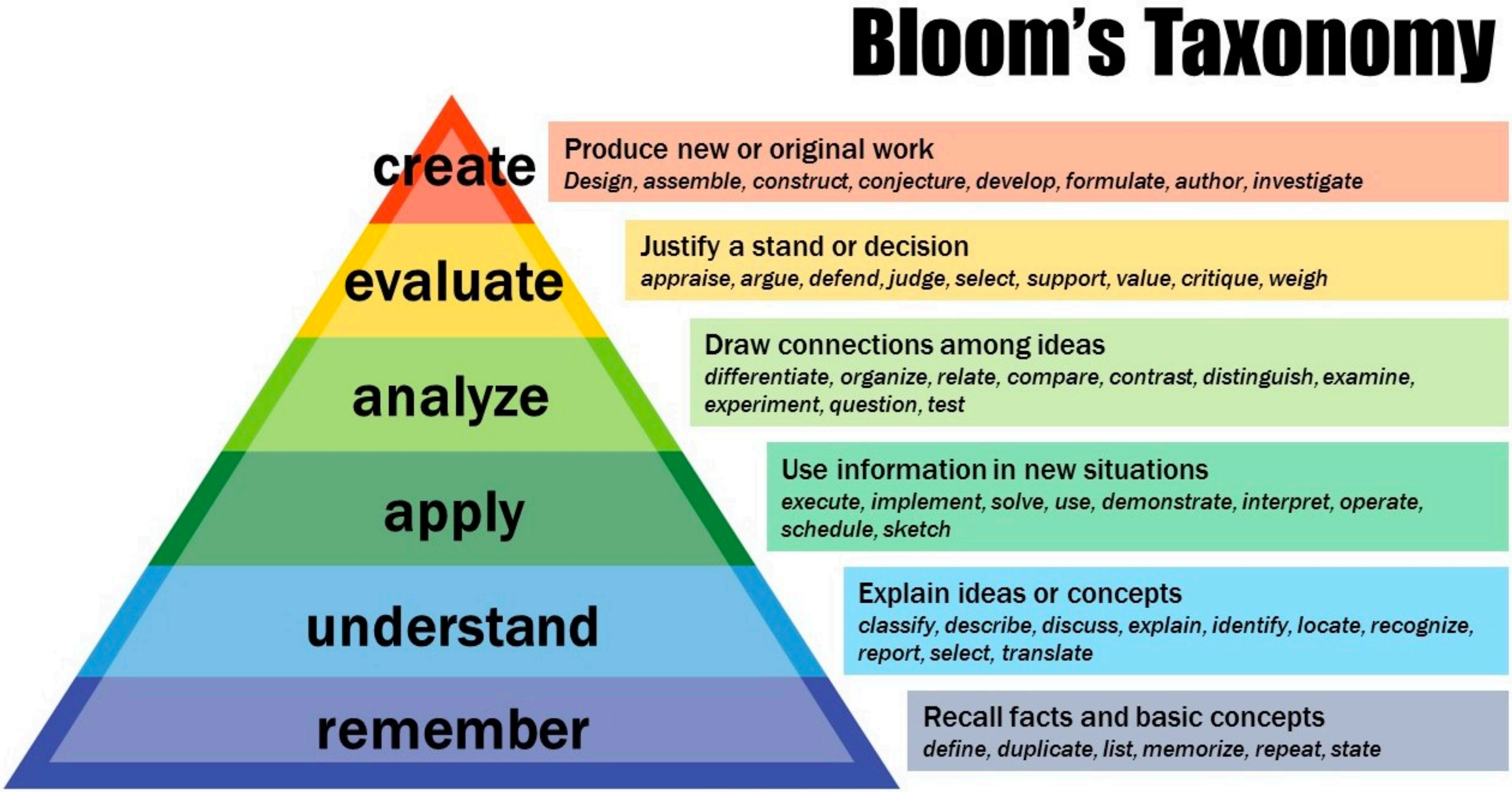
Summary

1. Motivate “analytic signals”

2. Understand & derive negative frequencies & DFT symmetry

3. Conceptualize Hilbert Transform

https://tinyurl.com/cogs118c-att


Bloom's Taxonomy
create
Produce new or original work
Design, assemble, construct, conjecture, develop, formulate, author, investigate
evaluate
Justify a stand or decision
appraise, argue, defend, judge, select, support, value, critique, weigh
analyze
Draw connections among ideas
differentiate, organize, relate, compare, contrast, distinguish, examine, experiment, question, test
apply
Use information in new situations
execute, implement, solve, use, demonstrate, interpret, operate, schedule, sketch
understand
Explain ideas or concepts
classify, describe, discuss, explain, identify, locate, recognize, report, select, translate
remember
Recall facts and basic concepts
define, duplicate, list, memorize, repeat, state