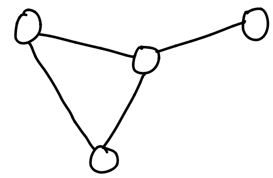
graph coloring IP formulation



Recall : input an undirected graph $G=(V,E)$

want : a color for each node s.t.

if $(u,v) \in E$, then u,v have different colors. Want # colors as small as possible

IP formulation : decision variables :

- x_{ij} for $i \in V$, $j \in [n]$, $n = |V|$

  $x_{ij} = 1$ means node i is assigned color j (max # colors $= n$)

- y_j for $j \in [n]$

  $y_j = 1$ means color j is used

objective function :

$$\min \sum_{j=1}^{n} y_j$$

(i.e. min # used colors)

constraints :

- x_{ij}, y_j all binary
- each node is assigned one color

  $$\sum_{j=1}^{n} x_{ij} = 1 \quad \forall\, i \in V$$

- if any node is assigned color j, then $y_j = 1$

  $$\equiv \sum_{i \in V} x_{ij} > 0 \Rightarrow y_j = 1$$

  $$\equiv n \cdot y_j \geq \sum_{i \in V} x_{ij} \qquad \text{or} \qquad x_{ij} \leq y_j \quad \forall\, i \in V, j \in [n]$$

- no two adjacent node has the same color

  $$x_{uj} + x_{vj} \leq 1 \qquad \forall\, (u,v) \in E,\ j \in [n]$$

$$x_{uj} + x_{vj} \leq y_j$$

$\forall\, (u,v) \in E$

$j \in [n]$

Symmetry of the LP:

if an optimal solution uses 3 colors

then there are $n \times (n-1) \times (n-2)$ "equivalent" solutions

(like color $1, 1, 5$
color $3, 4, 6$)

this slows down the solver $\Rightarrow$ break the symmetry by adding constraint

- $y_j \geq y_{j+1}$ (always prefer colors w/ smaller indices)
- $\sum_{i \in V} x_{ij} \geq \sum_{i \in V} x_{i(j+1)}$ (always prefers to use smaller indices for colors most used)