Security Price Dynamics

Modeling Security Price Dynamics

In this lecture we'll discuss microstructure perspectives of price dynamics and examine issues related to specification of the dynamics of transaction price, leading to the analysis of the Roll model of trade prices.

Overview

- In taking a microstructure perspective on security price dynamics, we shift focus from monthly or daily characteristics down to the features that come into play at horizons of a minute or less.
- The figures below illustrate this transition for a particular stock (Citigroup Inc., Symbol C).
- Figures 1 – 3 depict the sequence of actual trade prices over January 2011, then in the transaction prices on a particular day (January 19), and the transaction prices corresponding to a certain hours of day.
- In Figures 2 and 3, trade prices are augmented by plots of bid and ask quotes.

Overview

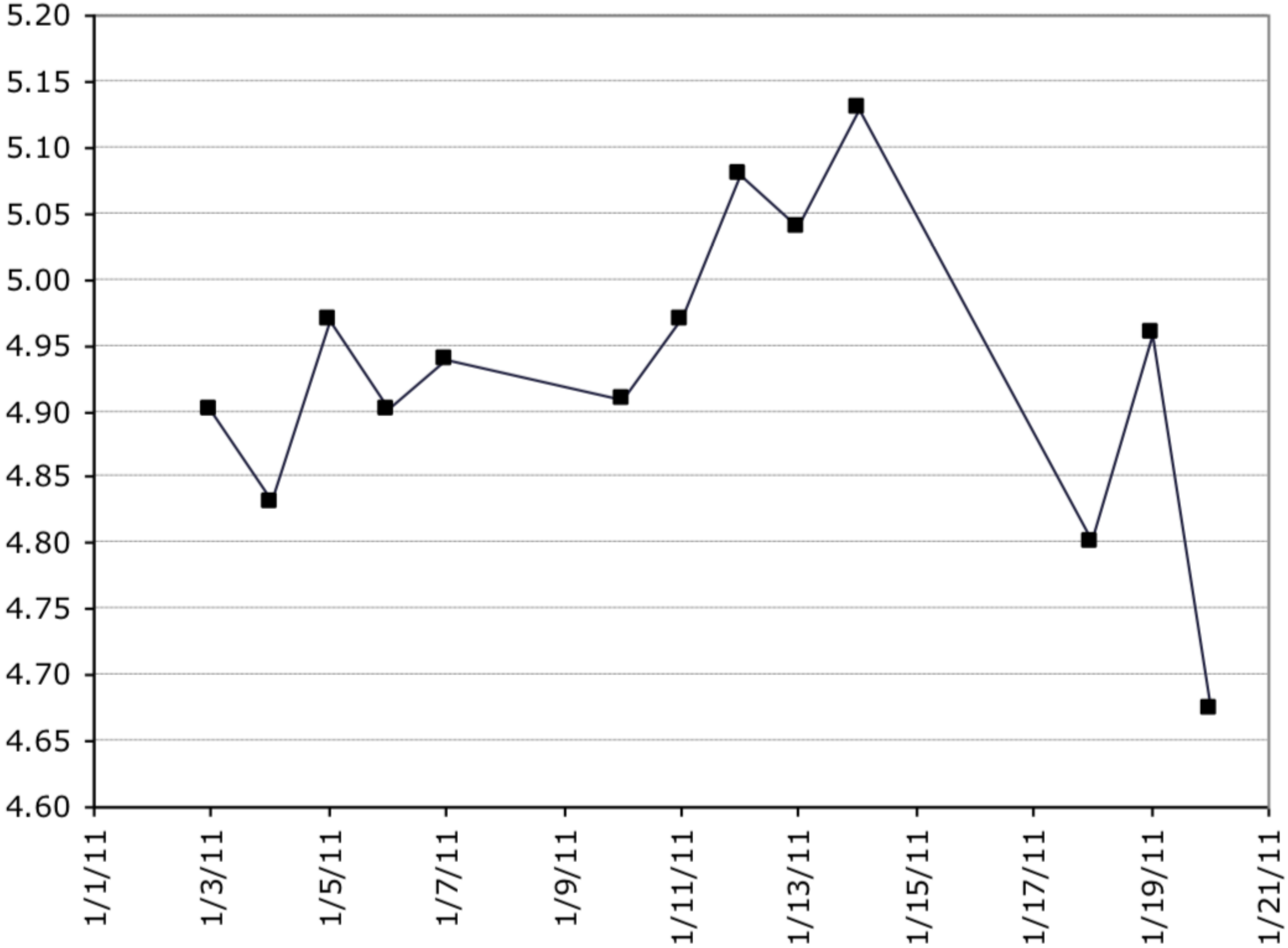

Figure 1: Daily Transaction Prices of Citigroup Inc. (January 2011)

Overview

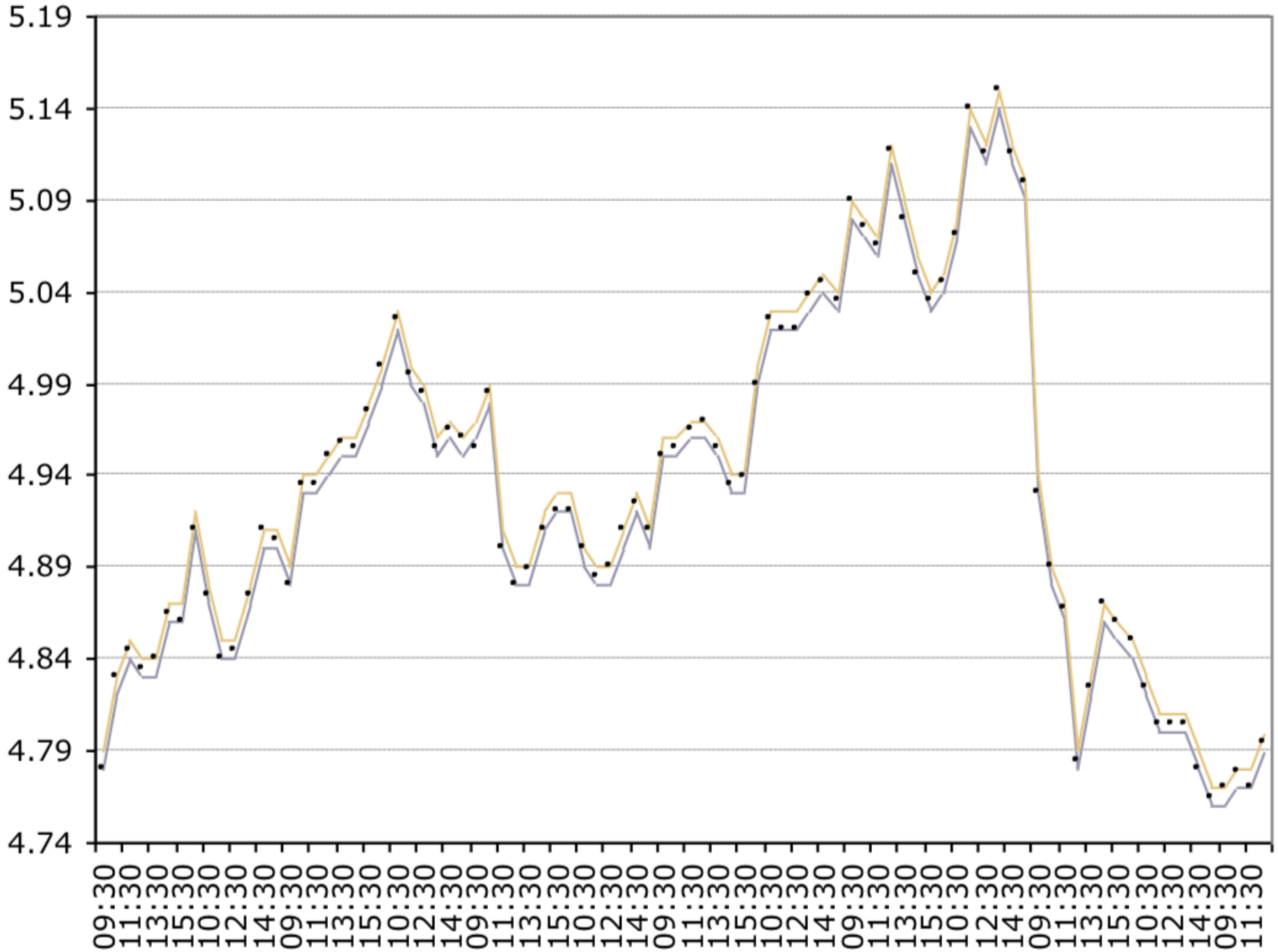

Figure 2: Hourly Transaction Prices of Citigroup Inc. (January 2011)

Overview

- The most detailed figure hints at the extent of the microstructure *complexities*.
- The three prices (bid, ask, and trade) *differ*. They all have jumps, and the *bid and ask* are *continual* in the sense that they always have values.
- *Trade prices* are more *discrete*, occurring as a sequence of well-defined points. The three prices tend to move together but certainly not in lockstep.
- The bid and ask sometimes change and then quickly revert. Trade prices occur at the posted bid and ask prices, but not always.

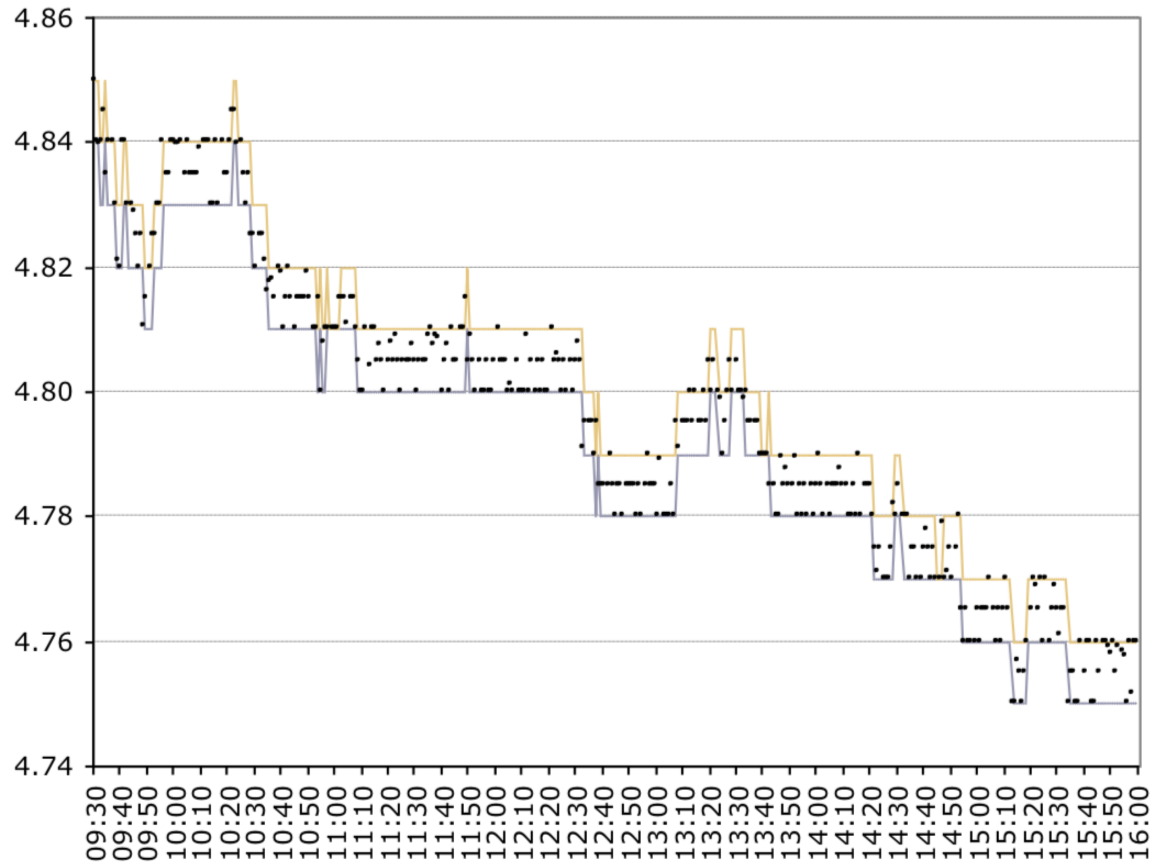

Figure 3: Ten-Minute Bar Prices of Citigroup Inc. (January 19, 2011)

The Random Walk Model of Security Prices

- Before financial economists began to concentrate on the trading prices the *standard* statistical model for a security prices was the *random walk*.
- The random walk model is no longer considered to be a complete and valid description of short-term price dynamics, but it nevertheless retains an important role as a model for the fundamental security value.
- Security valuation, of course, arises from information about security cash flows, which are long-lasting, when compared to the effects attributable to the market organization and trading process which are transient.
- Let p_t denote the transaction price at time t, where t indexes regular points of the real calendar (clock) time, for example, end-of-day, end-of-hour, and so on.
- Because it is unlikely that trade occur exactly at these times, we will approximate these observations by using prices of the last (most recent) trade, for example, the day's *closing prices*. This assumption will hold true unless we are dealing with actual tick data.

The Random Walk Model of Security Prices

Random Walk Model of Transaction (Trade) Prices - The random walk model (with drift) is given by:

$$p_t = p_{t-1} + \mu + u_t \qquad \text{(EQ 1)}$$

- where the u_t, at t = 0, 1, 2,... are *independently* and *identically distributed* (i.i.d.) random variables. Intuitively, they arise from *new information* that bears on the security value.
- μ is the expected price change or drift (assumed for now to be a constant).
- The units of p_t are either *levels* (e.g. dollars per share) or *logarithms*.

The logarithmic form is sometimes more convenient because price changes can be interpreted as continuously compounded returns. Some phenomena, however, require level (price) representation. Price discreteness, for example, reflects a tick size (minimum price increment) that is generally set in a level unit.

The Random Walk Model of Security Prices

In most microstructure analyses the drift is assumed to be zero and is dropped (see below). When μ = 0, p_t cannot be forecast beyond its most recent value, that is:

$$E[p_{t+1} \mid p_t, p_{t-1}, p_{t-2}, \ldots] = p_t \qquad \text{(EQ2)}$$

A process with this property is generally described as a *martingale.*

Definition (Martingale): A discrete time stochastic process $\{x_t\}$ is a *martingale* if $E|x_t| < \infty$ for all t, and $E[x_{t+1} \mid x_t, x_{t-1}, \ldots] = x_t$.

It follows immediately from the law of iterated expectations that:

Corollary (k-period Forecast): A discrete time stochastic process $\{x_t\}$ is a martingale if and only if $E|x_t| < \infty$ for all t, and $E[x_{t+k} \mid x_t, x_{t-1}, \ldots] = x_t$ **for all $\boldsymbol{k} > \mathbf{0}$**.

The Random Walk Model of Security Prices

This corollary can be used as an alternative definition of a martingale process. The following theorem establishes the connection between (discrete) transaction (trade) prices of securities and (discrete time) martingale processes:

Theorem (Trade Prices and Martingales):
Under the assumption of frictionless markets and the absence of arbitrage, the transaction prices of a (non-dividend paying) security can be represented by a discrete-time martingale process.

- This martingale behavior of asset prices is a classic result arising in many economic models with individual optimization, absence of arbitrage, or security market equilibrium.

- The result is generally *contingent*, however, on the assumption of frictionless trading opportunities, which are *not* appropriate in most microstructure applications.

The Random Walk Model of Security Prices

The expectation shown above are conditioned on lagged values of x_t, that is, the *history* of the process. A more general definition involves conditioning on *broader* information sets:

Definition (Martingale with Respect to an Information Set):

The process $\{x_t\}$ is a martingale with respect to a (possibly multi-dimensional) process $\{z_t\}$ if $E|x_t| < \infty$ for all t, and $E[x_{t+1} \,|z_t, z_{t-1}, \ldots] = x_t$.

- In particular, suppose that at some terminal time the cash value or payoff of a security is a random variable v.
- Traders form a sequence of beliefs based on a sequence of information sets $\Phi_1, \Phi_2, \ldots$.
- This sequence does not contract; something know at time t is know at time $\tau > t$, such that the conditional expectations of the security's terminal payoff $x_t = E[v \,|\Phi_t]$ will be a martingale with respect to the information sets $\{\Phi_t\}$.

The Random Walk Model of Security Prices

Theorem (Conditional Expectation as Martingales):

Given the sequence of information sets $\Phi_1, \Phi_2, \ldots$ and a security with terminal payoff ν, the *conditional expectation* $x_t = E[\nu \,|\Phi_t]$, is a martingale with respect to the information sets $\{\Phi_t\}$.

- When the conditioning information is all public information, the conditional expectation is sometimes called the fundamental value, the efficient price or the “fair value” of the security.
- It is the starting point for many microstructure models. One of the basic goals of microstructure analysis is a detailed and realistic view of how informational efficiency arises, that is, the processes by which new information comes to be impounded or reflected in prices.

In microstructure analysis, transaction prices are usually not martingales.

- Sometimes it is not even the case that the public information includes the history of transaction prices (in dealer markets, trades are often not reported).
- In general, however, short term dynamics of security prices reflect bid / ask spread and other microstructure effects that will not be represented in martingales.
- By imposing economic or statistical structure, however, it is often possible to identify a martingale component of the prices (with respect to a given information set).

The Random Walk Model of Security Prices

A special case of martingale is the random walk which is constructed as the sum of independently and identically distributed transom variables with zero mean:

Definition (Random Walk):

A random walk is a process whose increments are independently and identically distributed (i.i.d.) zero-mean random variables.

- The transaction price in Equation 1 for example accumulates the random variable u_t. Because u_t are i.i.d., the price process is time-homogenous; that is it exhibits the same behavior whenever in time we sample it.
- This is only sensible if the economic process underlying the security is also time-homogenous.
- Stocks, for example, are claims on ongoing economic activities (of the firm, sector overall economy, etc.) and are therefore plausibly approximated in the long run by random walk.

Statistical Analysis of Security Price Series

Statistical inference in the random-walk model is straightforward. Suppose that we have a sample price series $\{p_1, p_2, \ldots, p_T\}$, generated in accordance with Equation 1. Because the u_t are i.i.d., the price changes $\Delta p_t = p_t - p_{t-1}$ should be i.i.d. with mean μ and variance $Var(u_t) = \sigma_u^2$, for which we can compute the usual estimate. When we analyze actual data samples, however, we often encounter features that suggest wariness in the interpretation and subsequent use of the estimates.

Short-run security price changes typically exhibit:

- *Means* very close to zero (zero drift assumption)
- Extreme *dispersion*(fat tails)
- *Dependence* between successive observations (autocorrelations)

We will elaborate on each of these points further below.

Near-Zero Mean Returns

In microstructure data samples μ is usually *small* relative to the *estimation error* of its usual estimate, the logarithmic mean. For this reason it is often preferable to drop the mean return from the model, implicitly setting μ to *zero*. This is illustrated in Figure 4 below.

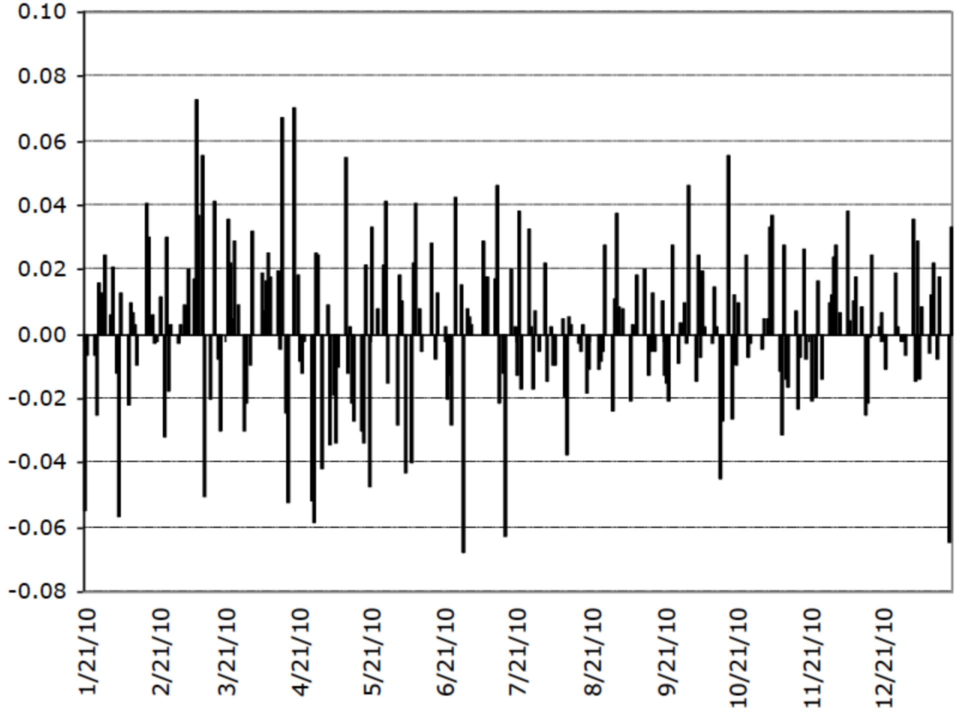

Figure 4: Daily Returns of Citigroup Inc. Stock during January 2011

Near-Zero Mean Returns

Zero, is of course, a *biased* estimate of μ, but its *estimation error* will generally be *lower* than that of arithmetic mean. For example, suppose that t indexes *days*. Consider the properties of the annual log price change implied by the *log random-walk* model:

$$p_{365} - p_0 = \sum_{t=1}^{365} \Delta p_t = \mu_{Annual} + \sum_{t=1}^{365} u_t \qquad \text{(EQ 3)}$$

where $\mu_{Annual} = 365\mu$. The *annual variance* is

$$\sigma^2_{Annual} := Var\left(p_{365} - p_0\right) = 365\sigma^2_u \qquad \text{(EQ 4)}$$

A typical U.S. stock might have an annual expected return of $\mu_{Annual} = 0.1$ (or 10%) and an annual variance of $\sigma^2_{Annual} = 0.25^2$. The implied *daily expected return* is

$$\mu_{Daily} = \mu_{Annual} / 365 = 0.1/365 = 0.000274 \qquad \text{(EQ 5)}$$

and the implied *daily variance* is given by:

$$\sigma^2_{Daily} = \sigma^2_{Annual} / 365 = 0.25^2 / 365 = 0.000171 \qquad \text{(EQ 7)}$$

Near-Zero Mean Returns

With $n = 365$ daily observations, the *standard error* of estimate for the sample *mean* is

$$Stderr\left(\mu_{Daily}\right) = \sqrt{\sigma^2_{Daily} / 365} = \sqrt{\left(\sigma^2_{Annual} / 365\right) / 365} = 0.000685 \qquad \text{(EQ 8)}$$

This is about two and half times the true mean. An estimate of zero is clearly biased downward, but the standard error of estimate, being 0.000274, is a relatively small number. At the cost of a little bias, we can greatly reduce the estimation error by making the assumption that the mean is zero.

As we refine the frequency of observations from annually to monthly to daily and so on, the *number of observations* increases. More numerous observations usually enhance the precision of the estimate, but often only if it also increases the *calendar span* of the sample. Here, though, the increase in observations is not accompanied by any increase in the calendar span of the sample. Merton (1980) has shown that estimate of *second moments* (variances and covariances) are *improved* by more frequent sampling. Estimates of the mean *returns* are *not*. For this reason, the expected return will often be dropped from our microstructure models.

Extreme Dispersion

Statistical analysis of speculative price changes at all horizons generally encounters *sample distributions* with *fat tails*. The incidence of extreme values is so great as to raise *doubt* whether population parameters like kurtosis, skewness, or *even* the variance of the underlying distributions are finite. This behavior is depicted in Figure 5 below:

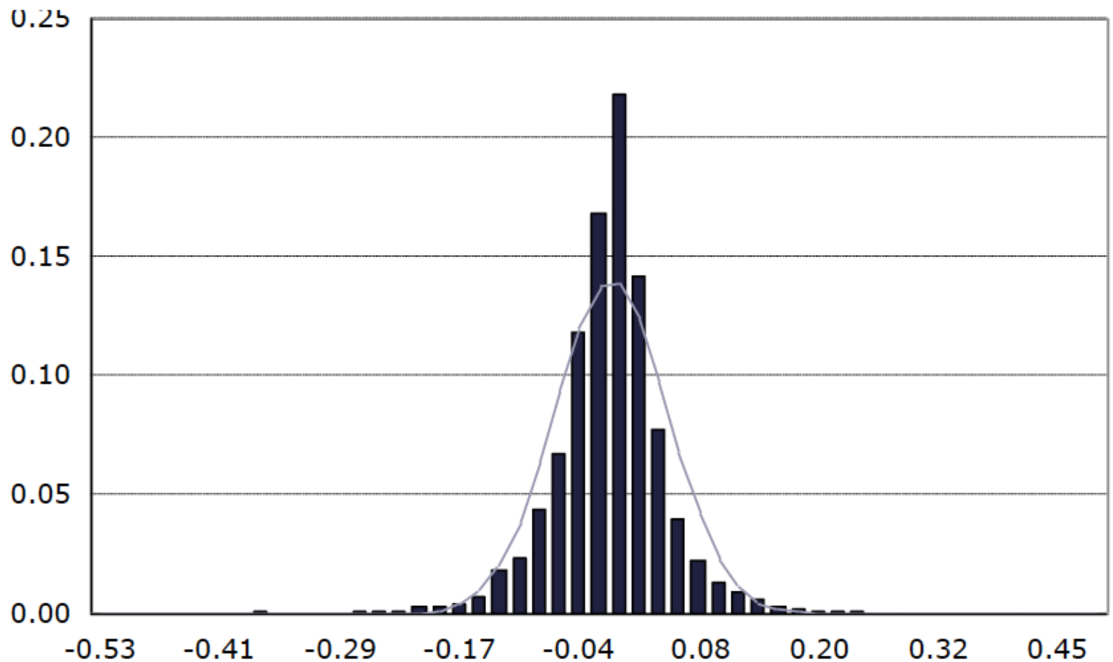

Figure 5: Fat-tailed Distribution of Daily Stock Returns

COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Extreme Dispersion

The conventional assumption that price changes are *normally distributed* is routinely *violated*. For example, from July 7, 1962 to December 31, 2004 (10,698 observations), the average daily return on the Standard & Poor's S&P 500 index is about 0.0003 (0.03%) and the standard deviation is about 0.0094. Letting $\Phi(z)$ denote the standard normal distribution function, if returns are normally distributed, then the number of days with returns below -5% is expected to be $10{,}698\, x\, \Phi\left(\left(-0.05-0.0003\right)/0.0094\right) \approx 0.0005$, that is considerably less than 1. In fact, there are eight such realizations (with minimum of -20.5% occurring on October 19,1987).

Statistical analysis of this sort of dispersion falls under the rubric of *extreme value analysis*. For a random variable X the population moment of order α is defined as $E\left(X^{\alpha}\right)$. The normal probability density possesses *finite* moments of *all orders*. In other distributions, though, a moment may be infinite because as X goes to $\pm\infty$ the quantity X^{α} grows faster than the probability density declines.

Extreme Dispersion

Theorem (Asymptotically Consistent Estimates of the Mean): Let $\alpha > 0$. If $E(X^{\alpha})$ is finite, then the *sample estimates* $\sum_{t=1}^{T} X_t^{\alpha} / T$, where T is the sample size, is an asymptotically *consistent* estimate of $E(X^{\alpha})$.

Proof follows from the application of the law of large numbers.

This theorem states that under the assuming $E(X^{\alpha}) < \infty$ we have *consistent sample estimates* of the mean $E(X^{\alpha})$. *Hypothesis testing*, however, often requires existence of the *asymptotic variance* of the *sample estimate*, which *requires* existence of moments of order 2α. To get the standard error of the mean, for example, we need a consistent estimate of the variance.

Extreme Dispersion

A recent study (Gabaix et al. 2003) suggests that finite moments of daily equity return exist only up to order 3 and for daily trading volume only up to order 1.5. These findings, if correct, impose substantive *restrictions* on the sorts of models that can be sensibly estimated.

Why should one be concerned about *convergence* failures in infinite samples? The answer is that whatever one's beliefs about the properties of the distribution generating the data, the existence of extreme values in finite samples is an irrefutable fact leading to many *practical* consequences. Sample estimates may be *dominated* by a few *extreme* values. *Increasing* sample size does *not* increase precision as much as we would expect. Estimate parameters are *sensitive* to *model specification*. Finally, conclusions drawn from the model are *fragile*, which is especially disturbing in trading applications.

Dependence of Successive Observations

Time series data are *ordered*, and statistical analysis must at least allow for the probability that there is *dependence* among the ordered data. The most important summary measures of time-series dependence are *autocovariances* and *autocorrelations*.

For a real-valued time series $\{x_t\}$ the autocorrelations and autocovariance (of order k) are defined as $Cov(x_t, x_{t-k})$ and $Corr(x_t, x_{t-k})$ for $k = 0, 1, 2\ldots$

Under the assumptions above (stationarity of time series), these quantities depend *only* on k, the *lag* (separation) between the component terms. Accordingly they can be expressed as $\gamma_k = Cov(x_t, x_{t-k})$ and $\rho_k = Corr(x_t, x_{t-k})$. When the *mean* of the series is *zero*, these quantities can be *estimated* using the sample average cross-product as

$$\hat{\gamma}_k = \sum_{t=k+1}^{T} x_t x_{t-k} /(T-k) \text{ and } \hat{\rho}_k = \hat{\gamma}_k / \hat{\gamma}_0 \qquad \text{(EQ 9)}$$

The incremental changes in a *random walk* are *uncorrelated*. So we would expect to find $\hat{\rho}_k \approx 0$ for $k \neq 0$. In actual samples, however, the first-order autocorrelations of short run speculative price changes are usually *negative*. The economic interpretation for this finding is the main contribution of the Roll model.

The Roll Model

The Roll Model of Bid, Ask and Transaction Prices

In returning to the economic perspective, we keep the random-walk assumption, but now apply it to the (martingale) efficient price instead of the actual transaction prices. Denoting this efficient price by $m_{t,}$ we assume that

$$m_t = m_{t-1} + u_t \qquad \text{(EQ 10)}$$

Where, as before, the u_t are zero-mean i.i.d. random variables.

- We also suppose that all trades are conducted through dealers.
- The best dealer quotes are bid and ask (offer) prices, b_t and a_t . If a customer wants to buy, he or she must pay the dealer's ask price (thereby "lifting the ask"). If a customer wants to sell, he or she receives the dealer's bid price (or "hitting the bid").
- Dealers incur a cost of c per trade. This charge reflects the costs like clearing fees, and per-trade allocation of fixed costs, such as computers, telephones, etc. These costs are non-informational, in the sense that they are not related to the dynamics of m_t .
- If the dealers compete to the point where the costs are just covered, the bid and ask prices are $b_t = m_t - c$ and $a_t = m_t + c$ respectively. The bid-ask spread is:

$$bid - ask = a_t - b_t = 2c \qquad \text{(EQ 11)}$$

The Roll Model of Bid, Ask and Transaction Prices

At time t, there is a trade at transaction price p_t which may be expressed as:

$$p_t = m_t + q_t c \qquad \text{(EQ 12)}$$

Where:

- q_t is a trade direction indicator set to +1 if the customer is buying and -1 if the customer is selling.
- We also assume that
  - the buys and sells are equally likely, serially independent (a buy in this period doesn't change the probability of a buy in the next period),
  - and that agents buy or sell independently of u_t (a customer buy or sell is unrelated to the evolution of m_t).
- This model was most clearly analyzed by Roll (1984), though certain elements of the analysis were first discussed by others (Niederhoffer and Osborne (1966)), so from here on out we'll refer to it as the Roll model.

The Roll Model of Bid, Ask and Transaction Prices

The Roll model has two parameters, c and σ_u^2. These are most conveniently estimated from the variance and first-order autocovariances of the price changes Δp_t. The variance is given by:

$$\begin{aligned}\gamma_0 &= Var(\Delta p_t) = E(\Delta p_t)^2 \\ &= \mathrm{E}[q_{t-1}^2 c^2 + q_t^2 c^2 \ - 2q_{t-1}q_t c^2 \ - 2q_{t-1}u_t c + 2q_t u_t c + u_t^2] \\ &= 2c^2 + \sigma_u^2\end{aligned} \qquad \text{(EQ 13)}$$

The last equality follows because in expectation, all the cross-products vanish except for those involving q_t, $q_{t-1}$and u_t^2. The first order autocovariance is given by:

$$\begin{aligned}\gamma_1 &= Cov(\Delta p_{t-1}, \Delta p_t) = E(\Delta p_{t-1}\Delta p_t) \\ &= \mathrm{E}[c^2(q_{t-2}q_{t-1} - q_{t-1}^2 - q_{t-2}q_t + q_{t-1}q_t) + c(q_t u_{t-1} - q_{t-1}u_t + u_t q_{t-1} - u_{t-1}q_{t-2})] \\ &= -c^2\end{aligned} \qquad \text{(EQ 14)}$$

The Roll Model of Bid, Ask and Transaction Prices

It is easily verified that all autocovariances of order 2 or higher are zero:

$\gamma_k = 0$ for k = 2, 3, …. (EQ 15)

From the above it is clear that $c = \sqrt{-\gamma_1}$ and $\sigma_u^2 = \gamma_0 + 2\gamma_1$. Given a sample of data, it is sensible to estimate γ_0 and γ_1 and apply these transformations to obtain estimates of the model parameters.

It should be noted that the validity of the assumptions underlying this model is questionable:

- The plot in the one-hour segment of Figure 2 suggests that the bid-ask spread is varying.
- Contrary to the assumption of serial independence, the correlation between q_1 and q_{t-1} is generally positive; that is, buys tend to follow buys and sells tend to follow sells.
- Contrary to the assumption of independence between q_t and u_t, changes in the quote midpoint are positively correlated with the most recent trade direction.

These are important violations of the Roll model. Future lectures will investigate modifications to the model that will account for these effects.