Algorithmic Trading

E4729

Information Based Models

COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Asymmetric Information and Sequential Trade Models

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Sequential Trade Model

- In part I of this lecture we will present an introduction to information-based models.
- We will begin with examination of a simple sequential trade model of transaction prices, in which both informed and uninformed investors exist, arriving in the market place singly, sequentially and independently to trade.
- Here we will examine consequences of this type of model in the nature of transaction prices and the market dynamics of market makers' bid-offer spreads.

Overview

- Recall that in the Roll framework, everyone possesses the *same* information.
- New public information can cause the quotes to change, but *trades* have *no impact,* and are therefore not *informative*.
- Because trades are not informative, there is no particular need for them to be reported and disseminated in a timely fashion.
- Trading is not a strategic problem because agents always face the *same spread*.
- The *costs* reflected in dealers' spreads reflect only *expenditures* for computers, communications, salaries, business fees, and so on, the sort of expenses that a wholesaler of any good or service might incur.

Overview

- Although some securities markets might function this way some of the time, most do not work this way.
- Trade reports are *valuable* information. Orders appear to move prices. Spreads vary across markets and with market conditions.
- Trading strategies must reflect these realities.
- Dropping the assumption of uniform information opens the door for more sensible economic explanations for these features of market behavior.
- The asymmetric information models described here take this direction.

General Features

These models have the following general features:

- The security payoff is usually of a *common value* component, reflecting terminal liquidation value that is the same for all holders.
- Generally, public information initially consists of *common knowledge* concerning the probability structure of the economy, in particular the unconditional *distribution* of *terminal security value* and the distribution of *types of agents*.
- But *private value* components also exist, reflecting diversification or risk exposure needs that are idiosyncratic to each agent.

General Features

- As trading unfolds, the most important updates to the public information set are market data, such as **bids**, **asks**, and the **prices** and **volumes** of trades.
- Most of the models make no provision for the arrival of non-trade public information (e.g., news announcements) during trading.
- Private information may consist of a *signal* about terminal security value, or, potentially, *perfect knowledge* of the terminal security value.
- The majority of the asymmetric information models in microstructure examine market dynamics subject to a single source of uncertainty, that is, a *single information event*. At the *end of trading*, the security payoff (terminal value) is *realized and known*.

Types of Models

Theoretical market microstructure has two main sorts of asymmetric information models:

- In the **sequential trade models**, randomly selected traders arrive at the market *singly*, *sequentially*, and *independently* (e.g. Glosten and Milgrom (1985)).

- The other class of models is **strategic trader models**, which usually feature a *single* informed agent who can trade at *multiple times* (e.g. O'Hara (1995), Kyle (1985)).

When an individual trader only participates in the market once (as in the sequential trade models), <u>there is no need for her to take into account the effect her actions might have on subsequent decisions of others</u>.

A trader who revisits the market, however, <u>*must*</u> make such calculations, and they involve considerations of *strategy.*

Types of Models and Features

- The *essential feature* of both models is that a trade reveals something about the agent's *private information*.
- For example, a "buy" from the dealer might result from a trader who has private positive information, but it won't originate from a trader who has private negative information.
- Rational, competitive market makers will set their bid and ask quotes accordingly. *Larger* information asymmetries lead to *wider* quotes.
- Trades will also engender a "permanent" impact on subsequent prices.
- The spread and trade-impact effects are the principal empirical implications of these models.
- We will begin with the sequential trade models.

A Simple Sequential Trade Model

The essential sequential trade model is a simple construct. The model presented here is a special case of Glosten and Milgrom (1985). It is also contained as a special case in many other analyses.

- There is one security with a value (payoff) V that is either *high* or *low*, $\overline{V}$ or $\underline{V}$.
- The *probability* of the *low* outcome is δ. The value is revealed after the market closes. It is *not*, however, *affected by trading*. It is determined, by a *random draw*, *prior* to the *market open*.
- The trading population (customers) consists of **informed** and **uninformed** traders. Informed traders (insiders) *know* the value outcome. The proportion of informed traders in the population is μ.
- A dealer posts bid and ask quotes, *B* and *A*.
- A *trader* is *drawn* at random from the population.
- If the trader is *informed*, she *buys* if V = $\overline{V}$ and *sells* if V = $\underline{V}$.
- If the trader is *uninformed*, he buys or sells randomly and with *equal probability*. The dealer *does not* know whether the trader is informed.
- The event tree for the first trade is given in Figure 1.

A Simple Sequential Trade Model

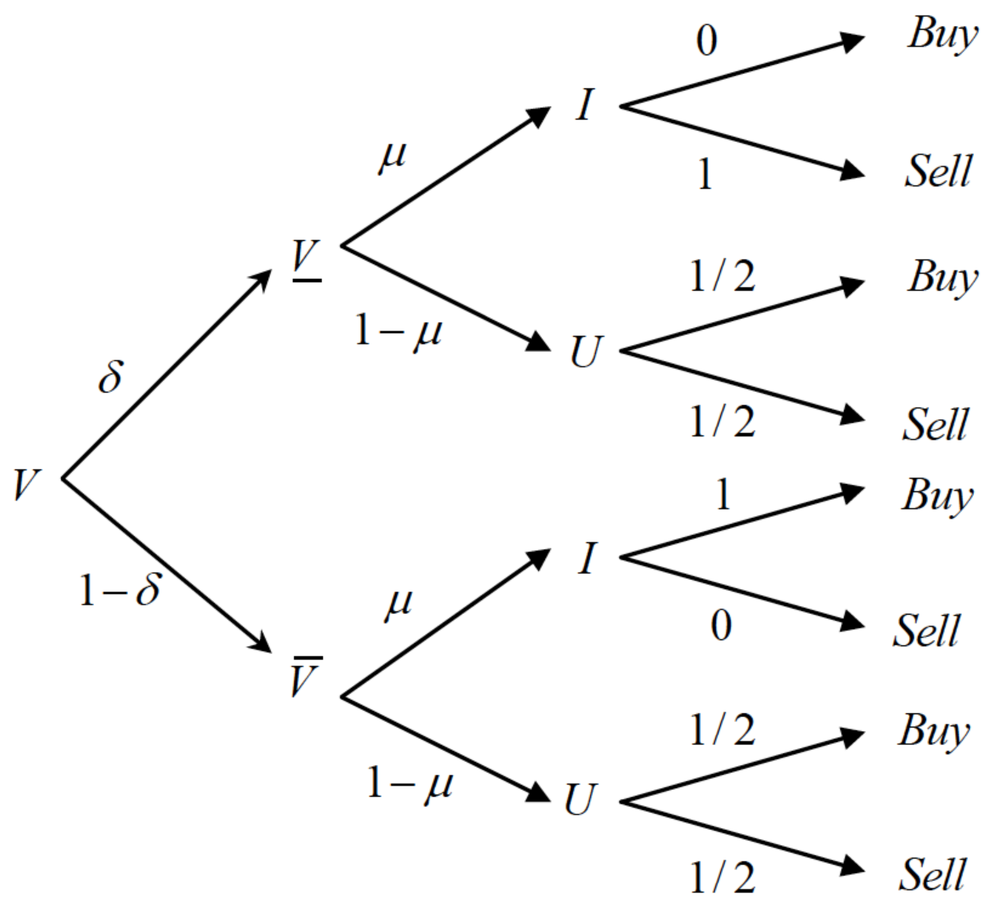


Figure 1: The Event Tree for the Basic Sequential Trade Model

A Simple Sequential Trade Model

In this figure, *I* and *U* denote the arrivals of informed and uninformed traders. A *buy* is a purchase by the customer at the dealer's *ask price*, *A*; a *sell* is a customer sale at the dealer's *bid price B*. The value attached to the arrow is the probability of the indicated transition. Total probabilities are obtained by multiplying along a *path*. For example, the probability of a low realization for *V*, followed by a the arrival of an uninformed trader who buys is $\delta(1-\mu)/2$.

The sum of the total probabilities over terminal *Buy* nodes gives the *unconditional* probability of a *buy*:

$$\Pr(Buy) = \left(1 + \mu(1-2\delta)\right)/2 \qquad \text{(EQ 1)}$$

Similarly, unconditional probability of a sell is

$$\Pr(Sell) = \left(1 - \mu(1-2\delta)\right)/2 \qquad \text{(EQ 2)}$$

In the case where $\delta = 1/2$ (equal probabilities of good and bad outcomes), the unconditional buy and sell probabilities are also *equal*.

Dealer's Perspective

Consider the dealer's situation.

- The purchases and sales of the uninformed traders are *not* sensitive to the quoted bid or ask.
  - So if the dealer is a **monopolist**, expected profits are maximized by setting the bid infinitely low and the ask infinitely high.
  - Obviously, at these prices, only the uninformed trade.
- In practice, the dealer's market power is constrained by competition (and regulation).
  - Competition arises from other dealers, but also and more generally from anyone who is setting a visible quote, such as a public customer using a limit order.
  - In some venues, regulation limits the dealers' power.
- We will assume that competition among dealers drives expected profits to *zero*. Furthermore, for the usual reasons, the dealer can't cross-subsidize buys with sells or vice versa. (If he were making a profit selling to customers, for example, another dealer would undercut his ask.) It thus suffices to consider buys and sells separately.

Dealer's Revision of Beliefs

Dealer's Revisions of Beliefs - The dealer's *inference* given that the first trade is a buy or sell can be summarized by her *revised beliefs* about the *probability* of a *low* outcome, *given* that the customer *buys*, denoted $\delta_1(.)$, this probability is

$$\delta_1(Buy) = \Pr(\underline{V} \mid Buy) = \frac{\Pr(\underline{V}, Buy)}{\Pr(Buy)} = \frac{\delta(1-\mu)}{1+\mu(1-2\delta)} \qquad \text{(EQ 3)}$$

Because μ and δ are between zero and one, $\partial\delta_1(Buy)/\partial\mu < 0$: The revision in beliefs is *stronger* when there are *more* informed traders in the population.

At the end of the day, the dealer's *realized profit* on the transaction is $\pi = A - V$. Immediately after the trade, the dealer's expectation of this profit is

$$E[\pi \mid Buy] = A - E[V \mid Buy] \qquad \text{(EQ 4)}$$

Dealer's Revision of Beliefs

Where

$$E[V \mid Buy] = \delta_1(Buy)\underline{V} + (1 - \delta_1(Buy))\overline{V} \qquad \text{(EQ 5)}$$

If competition drives this expected profit to zero, then,

$$A = E[V \mid Buy] = \frac{\underline{V}(1-\mu)\delta + \overline{V}(1-\delta)(1+\mu)}{1+\mu(1-2\delta)} \qquad \text{(EQ 6)}$$

A dealer's quote is essentially a proposal of *terms of trade*. When the *bid is hit* or the *offer is lifted*, this proposal has been *accepted*. In the context of this model ask is simply what the dealer believes the security to be *worth* and the quote is *regret-free* (unlike some situations in which ex post 'regret' or 'winner's curse' can occur).

Analysis of Ask Price

Analysis of Ask Price - The ask quote strikes a *balance* between *informed* and *uninformed* traders. The conditional expectation of value can be decomposed as

$$E[V \mid Buy] = E[V \mid U, Buy]\Pr(U \mid Buy) + E[V \mid I, Buy]\Pr(I \mid Buy) \qquad \text{(EQ 7)}$$

Substituting this into the zero-expected profit condition

$$A = E[V \mid Buy] \qquad \text{(EQ 8)}$$

and rearranging gives:

$$\underbrace{(A - E[V \mid U, Buy])\Pr(U \mid Buy)}_{\text{Gain from an uninformed trader}} = -\underbrace{(A - E[V \mid I, Buy])\Pr(I \mid Buy)}_{\text{Loss to an informed trader}} \qquad \text{(EQ 9)}$$

The expected gains from uninformed traders are balanced by the losses to informed traders. In this model, therefore, there is a net wealth transfer from uninformed to informed traders.

Analysis of Bid Price

Analysis of Bid Price - The analysis for the dealer's bid is similar. Following a sale to the dealer,

$$\delta_1(Sell) = \Pr(\underline{V} \mid Sell) = \frac{\Pr(\underline{V}, Sell)}{\Pr(Sell)} = \frac{\delta(1+\mu)}{1-(1-2\delta)\mu} \qquad \text{(EQ 10)}$$

For δ and μ between zero and one, $\delta_1(Sell) > \delta_1(Buy)$. $\underline{V}$ is *less likely* if the customer *bought*, reasons the dealer, because an informed customer who knew $V = \underline{V}$ would have sold. Furthermore $\partial\delta_1(Sell)/\partial\mu > 0$. The dealer's bid is set as

$$B = E[V \mid Sell] = \frac{\underline{V}(1+\mu)\delta + \overline{V}(1-\mu)(1-\delta)}{1-\mu(1-2\delta)} \qquad \text{(EQ 11)}$$

Bid-Ask Spread

Bid-Ask Spread - The *bid-ask spread* is

$$A - B = \frac{4(1-\delta)\delta\mu\left(\overline{V} - \underline{V}\right)}{1-(1-2\delta)^2\mu^2} \qquad \text{(EQ 12)}$$

In the *symmetric* case of $\delta = 1/2$, $A - B = \left(\overline{V} - \underline{V}\right)\mu$. Although in many situations the *midpoint* of the bid and ask is taken as a *proxy* for what the security is worth *absent* transaction costs, here the midpoint is *equal* to the *unconditional expectation* $E(V)$ *only* in the *symmetric* case $\delta = 1/2$. More generally, the bid and ask are *not set symmetrically* about the efficient price.

Market Dynamics: Bid and Ask Quotes over Time

After the initial trade, the dealer *updates* his beliefs and *posts* new quotes. The next trader arrives, and the process repeats. This *recurrence* is clearest in the expressions for $\delta(Buy)$ and $\delta(Sell)$ in Equations 3, 10. These equations *map* a *prior probability* (the δ in the right-hand side) into a *posterior probability* based on the direction of the trade. Let δ_k denote the probability of a low outcome *given* δ_{k-1} and the direction of the kth trade, with the original (unconditional) probability being $\delta_0 \equiv \delta$. Then Equations 3, 10 can be generalized as

$$\delta_k\left(Buy_k;\delta_{k-1}\right)=\frac{\delta_{k-1}(1-\mu)}{1+\mu(1-2\delta_{k-1})} \quad \text{and}$$

$$\delta_k\left(Sell_k;\delta_{k-1}\right)=\frac{\delta_{k-1}(1+\mu)}{1-\mu(1-2\delta_{k-1})} \qquad \text{(EQ 13)}$$

The updating recursion can be expressed in general form because all probabilities in the event tree except δ are constant over time.

Features of Market Dynamics

Features of Market Dynamics - Market dynamics have the following features:

- The *trade price* series is a *martingale*. Recall from the foregoing analysis that $B_k = E[V \mid Sell_k]$ and $A_k = E[V \mid Buy_k]$. Because the trade occurs at one or the other of these prices, the sequence of trade prices $\{P_k\}$ is a sequence of *conditional expectations* $E[V \mid \Phi_k]$ where Φ_k is the information set consisting of the history (including the *k*th trade) of the buy/sell directions. A sequence of expectations conditioned on *expanding* information sets is a martingale.

- The *order flow* is *not symmetric*. Using q_k to denote the trade direction as we did in the Roll model (+1 for a buy, −1 for a sell), $E[q_k \mid \Phi_{k-1}]$ is in general *nonzero*.

Features of Market Dynamics

- The spread *declines over time*. Knowing the *long-run* proportion of buys and sells in the order flow is tantamount to knowing the outcome. With each trade, the dealer can estimate this proportion *more precisely*, and hence uncertainty is reduced.

- The orders are *serially correlated*. Although the *agents* are *drawn independently*, one *subset* of the population (the informed traders) always trades in the *same direction*.

- There is a *price impact* of trades. For any given pattern of buys and sells through trade k, a buy on the $k+1$ trade causes a *downward revision* in the *conditional probability* of a low outcome, and a consequent *increase* in the bid and ask. The trade price impact is a particularly useful empirical implication of the model. It can be estimated from market data and is plausibly a useful *proxy* for *information asymmetry*.

Extensions

The sequential trade framework is a modeling platform that is easily extended to accommodate various features of trading. The following subsections describe some of these developments.

Quote Matching

- To this point the information asymmetries in the model have centered on fundamental value. This not the only sort of information differential that can arise. Most orders pass through brokers or other intermediaries, whose relationships with the order submitters *may convey* advantage. Brokers, for example, usually possess trading history, and other sorts of client data. If their *assessment* of the likelihood that the customer is *informed*, conditional on customer identity, is lower than the likelihood for other brokers, they may profitably trade against their own customers if this is permitted.
- This model can be considered at best a partial equilibrium. Why do broker *b*'s customers include more sheep and fewer wolves? Why can't other brokers attract a similar customer base? The model nevertheless illustrates a *nonfundamental* informational advantage. This is an important point because although information asymmetries related to value are quite plausible in some markets, they are more dubious in others. In equity markets, for example, advance knowledge of an earnings surprise, takeover announcements are obviously very advantageous.

Fixed Transaction Costs

Fixed Transaction Costs - Suppose that in addition to asymmetric information considerations, the dealer must pay a *transaction cost c* on each trade (as in the Roll model). The modification is straightforward. The ask and bid now are set to *recover c* as well as the information costs:

$$A = E[V \mid Buy] + c \text{ and } B = E[V \mid Sell] - c \qquad \text{(EQ 14)}$$

The ask quote sequence may still be expressed as a sequence of conditional expectations: $A_k = E[V \mid \Phi_k] + c$ where Φ_k is the information set that includes the direction of the *k*th trade. Therefore the *ask sequence* is a *martingale*. So, too, is the *bid sequence*. Because trades can occur at either the bid or the ask, however, the sequence of *trade prices* is *not* a *martingale* (due to the ±*c* asymmetry in the problem). In terms of the original Roll model, the effect of asymmetric information is to *break* the *independence* between trade direction q_t and the innovation to the efficient price u_t (Glosten and Milgrom (1985)).

Event Uncertainty

Event Uncertainty - In actual security markets, information and information asymmetries often arrive in a *lumpy* fashion. Long periods with no new information and steady or sluggish trading are punctuated by periods of extremely active trading before, during, and after major *news announcements*. This variation can be modeled using event uncertainty (Easley and O'Hara (1992)). Taking the basic model as the starting point, a *random step* is placed at the *beginning* of the day: whether an information event has occurred.

Figure 2 below depicts the new event tree:

Event Uncertainty

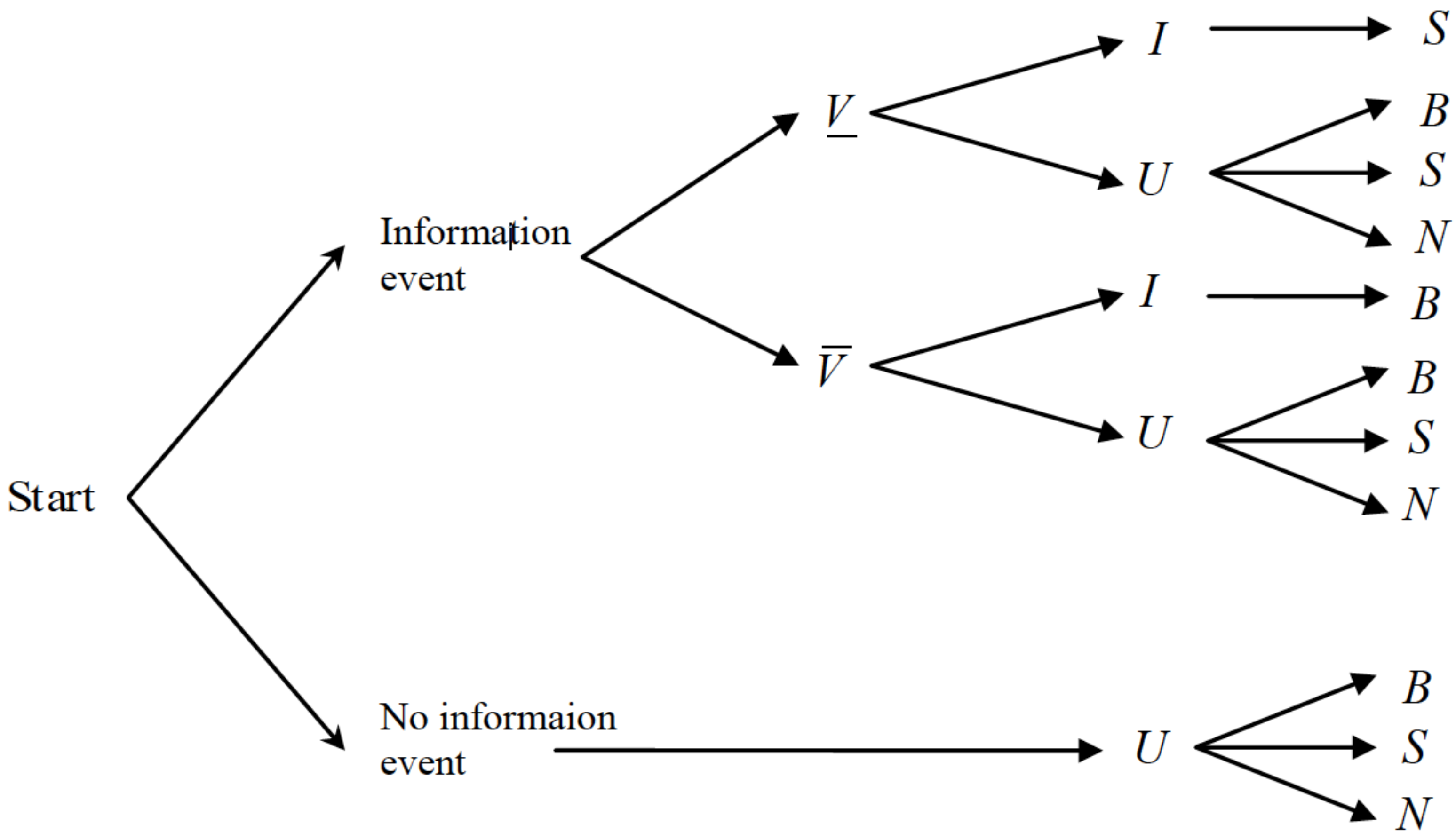


Figure 2: Event Tree at the Start of the Trading Day

Event Uncertainty

Besides the random occurrence of information events, the tree also features an additional type of trader action: *no trade*. For "no trade" to convey useful information, it must be observable. In the empirical implementation discussed in the next chapter, "no trade" will be reinterpreted as something more akin to "slow trade." In the present context, though, it is most straightforward to envision a market where agents physically approach the dealer and then decide whether to trade.

The *updating* of dealers beliefs is fairly intuitive. A customer buy might arise from an informed buyer, and so increases the conditional probability of $\overline{V}$; a customer sale increases the perceived likelihood of $\underline{V}$. A no-trade event *decreases* the dealer's conditional belief that an *information event* has occurred. Because information asymmetries can only arise subsequent to an information event, the dealer's perceived cost of informed trading drops after a no-trade event, and the spread narrows.

An informed trader *always* trades (in the direction of her knowledge). An *uninformed* trader *might not* trade. The no-trade probabilities for uninformed agents are the same whether an information event has occurred or not, but the proportion of uninformed traders in the customer mix is higher in the absence of an information event. To the dealer, therefore, *absence of trade* suggests a *decreased* likelihood of an *information event*.

Price Impact

Price impact refers to the effect of an incoming order on *subsequent* prices. In general this process is complex and subtle. It involves signal extraction mediated through the forces of economic competition.

Signal extraction entails learning about an unknown (in this case, value) from conditioning information consisting of indirect or noisy observations (the trades). Put another way, the dealer's key *inference* is a prediction of the closing revelation of true value based on the *order flow*.

This prediction per se does not fully account for the revision of the bid and ask, however. For the prediction to be fully reflected in the quotes, there must be other dealers, who also observe the trade, and who will compete away any attempt by any other dealer to extract a profit by setting an inferior bid or ask. To take an extreme example, a monopolist dealer could possibly set $B = \underline{V} + \varepsilon$ and $A = \underline{V} + \varepsilon$ for some trivial $\varepsilon > 0$, irrespective of prior orders. The order flow and (therefore) the dealer's beliefs about V would evolve exactly as in the competitive case, but with no price changes.

In this light, orders do not "impact" prices. It is more accurate to say the orders *forecast* prices. The distinction is important, but empirical resolution is difficult. "Functional" causality and forecast enhancement are generally indistinguishable. The usual (Granger- Sims) test for causality is implemented as a test of forecasting ability (Hamilton (1994) p. 302).

Price Impact

Focusing on the signal extraction process leads to interesting implications. Most important, market dynamics reflect the **beliefs** of market participants, not necessarily the **reality**. In particular, μ is the dealers' belief about the likelihood of an informed trader. If we subsequently determine that on a particular day a corporate insider had been illegally trading in the market, we are in a position to assert that the actual likelihood of an informed trade on that day was higher than what the dealers believed. It is the latter beliefs, however, that determine the price reaction. Absent any changes in these beliefs, we should not expect an empirical analysis to detect any elevation in the price impact.

A trade causes a revision in the expectation of *V* for all agents in the market, with the important exception of the agent who actually traded. This customer has no better information than he did before the trade. This is clearly the case for an informed trader, who possesses perfect knowledge. But it is also true of an uninformed trader. The uninformed trader who completes transaction *k* knows that the probability of informed trader on that trade is not μ but zero. For this trader, then, $\delta_k = \delta_{k-1}$. Whether informed or uninformed, the agent's ultimate effect of his trades is succinctly summarized in the aphorism, "The stock doesn't know that you own it" (Goodman 1967).

Price Impact

Alternatively the expectation of terminal value conditional on one's own contemplated trades does not depend on whether the trades are actually completed. For example, if the incoming order is a purchase by an unidentified agent, everyone in the market will revise upward their expectation of the terminal security value, *except* for the agent who actually sent in the order.

In a sense, the trading process creates superior information for uninformed traders. From the perspective of an uninformed agent who has just made the *k*th trade, the dealers' subsequent quote revisions are erroneous. Once the trade is completed, any uninformed trader possesses information superior to the dealers.

Does this create a manipulation opportunity? It is difficult to generally characterize market manipulations, but one strategy that might be offered up in example involves buying, moving the price upward in the process, and selling at the higher price. Traders in the basic model have only one chance to buy or sell, so there is no possibility of any such round-trip. But suppose that this limitation is removed, and we let an uninformed buyer immediately return to the market. Can he reverse his trade at a profit? (As an exercise, verify that the answer is "no.")

Price Impact

It is clear, however, that a sequence of uninformed trades in the same direction will move the market's conditional assessment of δ. This may create a profit opportunity for an agent who *knows* that the trades were *uninformed*, even though the agent is ignorant of the fundamental value information. Suppose that the first two trades are uninformed sells. The ask quote preceding the third trade is set to reflect the updated δ:

$$\delta_3(Sell_1, Sell_2, Buy_3) = \delta(1+\mu)/[1+(2\delta-1)\mu] > \delta_0 \qquad \text{(EQ 10)}$$

It therefore follows that $EV - Ask_3 > 0$, where $EV = \delta_0 \underline{V} + (1+\delta_0)\overline{V}$, the unconditional expectation. Knowing that all trades to this point (including his own) have been uninformed (ignorant of the true *V*), *EV* is also the buyer's conditional expectation. In buying at the ask, he is *paying less* than the security is *worth*.

What sort of agent might have the information required to behave in this fashion? Orders typically reach a market through *brokers*. Brokers typically know something about their customers, and so may be in a good position to judge if someone is uninformed. Trading ahead of your customer is generally *prohibited*, but the strategy described only requires trading *after* the customer.

Order Flow and
Strategic Trade Models

Order Flow and Strategic Trade Model

In part II of this lecture we will first review the relationship between order flow and trade prices, by examining the impact of the order arrival process on transaction prices.

We will also introduce an example of a strategic trade model, where a single informed trader can enter the market multiple times, and will behave strategically in an effort to reduce adverse price concessions resulting from large trades.

Order Flow and the Probability of Informed Trading

Though the richest implications of the sequential trade models involve the joint dynamics of trades and prices, inference can also be based solely on the *order arrival process*.

To illustrate the general features of the problem, we examine the statistical properties of the numbers of buys and sells from the basic sequential trade model.

We next turn to a more flexible model that allows for event uncertainty (Easley, Kiefer, and O'Hara (1997) and Easley, Hvidkjaer, and O'Hara (2002)).

The Distribution of Buys and Sells

For the model described in the previous lecture the probabilities of a buy order conditional on low and high outcomes are $\Pr(Buy \mid \underline{V}) = (1-\mu)/2$ and $\Pr(Buy \mid \overline{V}) = (1+\mu)/2$. These probabilities are *constant* over successive trades. Also, conditional on V, *order directions* are *serially independent*: The probability of a buy does not depend on the direction of the last or any prior orders. If we observe n trades, the conditional distributions of b buys, $\Pr(b \mid n, \underline{V})$ and $\Pr(b \mid n, \overline{V})$ are binomial:

$$\Pr(b \mid n, V) = \binom{n}{b} p^b (1-p)^{n-b} \quad \text{where} \quad p = \begin{cases} (1-\mu)/2 & \text{if } V = \underline{V} \\ (1+\mu)/2 & \text{if } V = \overline{V} \end{cases} \qquad \text{(EQ 1)}$$

The *number of buys* conditional only on n, then, is a *mixture of binomials*:

$$\Pr(b \mid n) = \delta \Pr(b \mid n, \underline{V}) + (1-\delta)\Pr(b \mid n, \overline{V}) \qquad \text{(EQ 2)}$$

With *no* informed trading, the distribution is *approximately normal*. As it increases, the component distributions become more *distinct*. With μ=0.1, the distribution is broader but still unimodal. With μ=0.5, the distribution is bimodal.

The Distribution of Buys and Sells

Figure 1 below depicts these distributions.

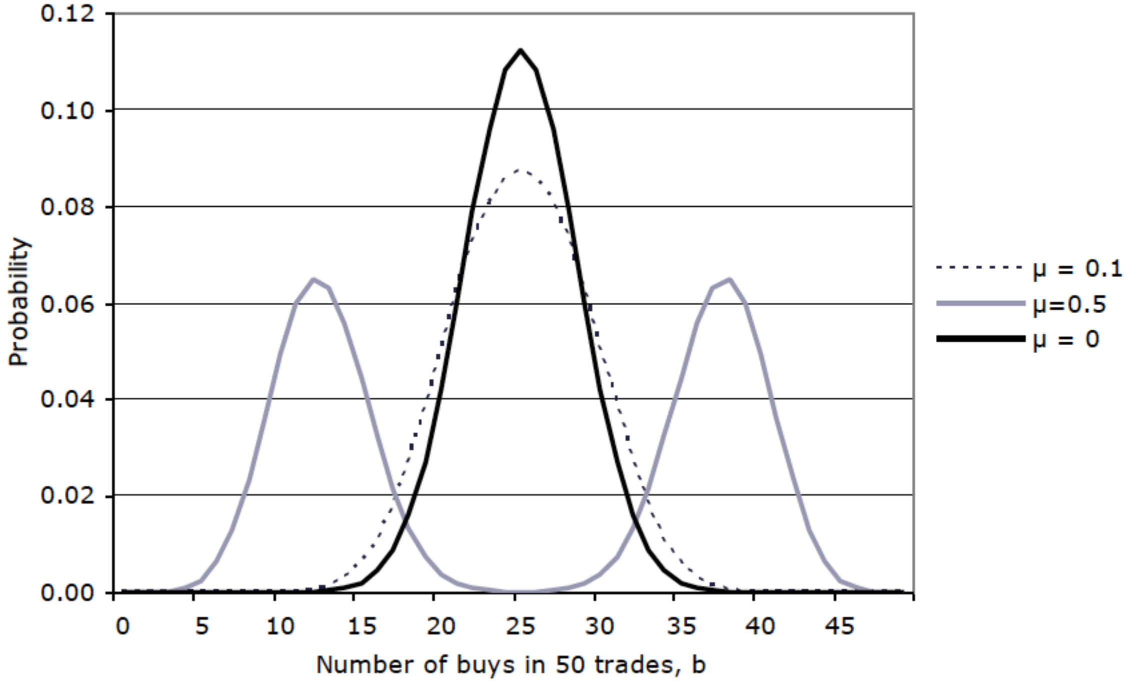


Figure 1: Distribution of the Number of Buy Orders

This suggests the properties of the data that will identify an *estimate* of μ. With *high* μ, all days will tend to have *one-sided* order flow, a preponderance of buys or sells, depending on the outcome of the value draw. Although this model could be estimated by maximum likelihood, actual applications are based on a modified version, described below.

Event Uncertainty and Poisson Arrivals

This model is a variation of the sequential trade model *with* event uncertainty. The principal difference is that agents are *not* sequentially drawn in discrete time but *arrive randomly* in continuous time. These events are modeled as a *Poisson arrival process*.

Definition (Poisson Process): a random variable n whose probability distribution with parameter θ has the form $\Pr(n) = e^{-\theta}\theta^n / n!$ for $n = 0, 1, \ldots$ The mean and variance of this distribution are $En = Var(n) = \theta$.

Specifically, suppose that the traders arrive randomly in time such that the *probability* of an arrival in a time interval of length Δt is $\lambda \Delta t$ where λ is the *arrival intensity per unit time*, and the probability of two traders arriving in the same interval goes to zero in the limit as $\Delta t \to 0$. Then,

- The *number of trades* in any finite interval of length Δt is *Poisson* with parameter $\theta = \lambda \Delta t$.

- The *duration* τ between two *successive* arrivals is exponentially distributed: $f(\tau) = \tau e^{-\lambda \tau}$.

- If two types of traders arrive *independently* with intensities λ_1 and λ_2, then the arrival intensity for undifferentiated traders, that is, traders of any type, is $\lambda_1 + \lambda_2$.

Definition (Poisson Process)

The situation is shown in Figure 2 below:

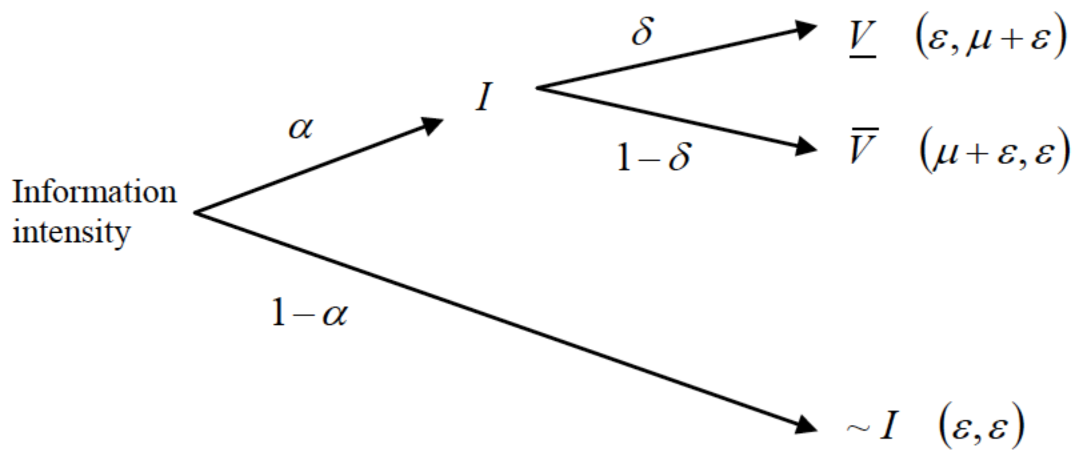


Figure 2: Sequential Trade Model with Event Uncertainty

Definition (Poisson Process)

The arrival intensities for informed and uninformed traders are μ and ε. The effect of these parameters on the arrival intensities of buyers and sellers depends on the information state of the market. In all states, *uninformed* buyers and sellers arrive with intensity ε. If there is an information event that results in a *low* value realization, for example, *informed* sellers appear, the total arrival intensity of sellers is $\varepsilon + \mu$.

On any day the unconditional numbers of buys and sells are jointly distributed as a Poisson mixture:

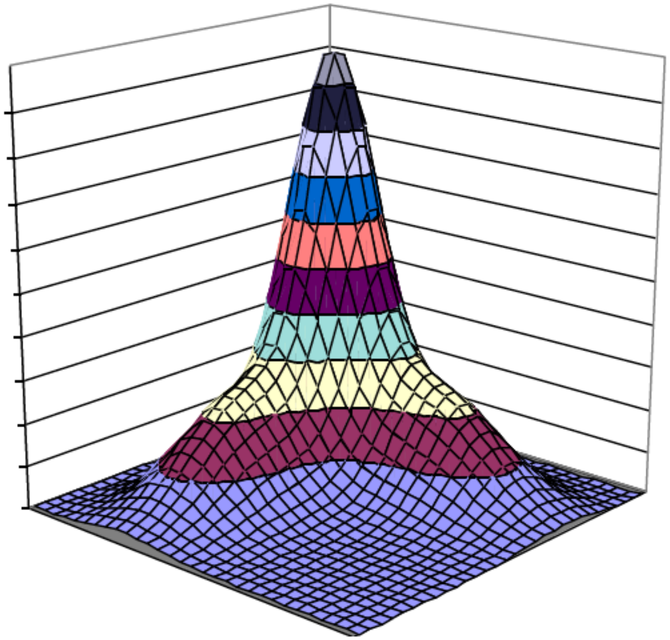
(EQ 3)

$$\Pr(b,s) = (1-\alpha)\Pr(b;\varepsilon)\Pr(s;\varepsilon) + \alpha[\delta \Pr(b;\varepsilon)\Pr(s;\mu+\varepsilon) + (1-\delta)\Pr(b;\mu+\varepsilon)\Pr(s;\varepsilon)]$$

where $\Pr(n;\lambda)$ denotes the probability of n arrivals when λ is the intensity parameter.

Figure 6.3 depicts this distribution for parameter values $\alpha = 0.4$, $\mu = \varepsilon = 10$, (per day). The distribution is shown both as a contour map and in three dimensions. In both depictions it is clear that the dominant features are *two lobes*, corresponding to the days when the order flow tends to be *one-sided*.

Definition (Poisson Process)



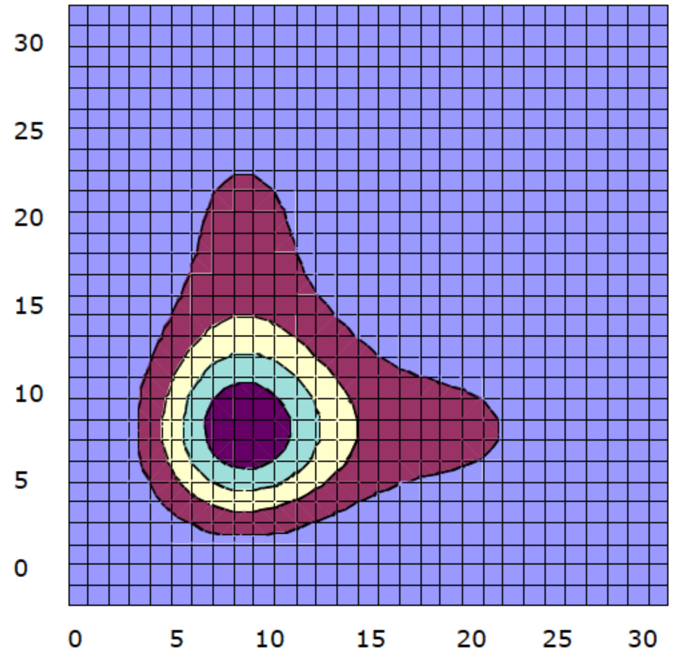


Figure 3: Joint Probability Density of the Number of Buy and Sell Orders

All parameters in this model may be estimated by maximum likelihood. Most interest, however, centers on a transformation of the parameters that corresponds to the probability of informed trading (*PIN*). This may be developed as follows.

Probability of Informed Trading (PIN)

Probability of Informed Trading (PIN) - The expected *total* order arrival intensity is $2\varepsilon + \alpha\mu$, consisting of *uninformed buyers*, *uninformed sellers*, and (with probability α) *informed traders*. *PIN* is the *unconditional probability* that a *randomly chosen* trader on a randomly chosen day is *informed*. Thus,

$$PIN = \alpha\mu / (\alpha\mu + 2\varepsilon) \qquad \text{(EQ 4)}$$

In the basic model, the probability of an informed trader is μ. This can be obtained as a special case of the present model.1 When μ increases in the basic model, daily order flows became increasingly *one-sided*.

There interplay between α and μ in this model is interesting. These parameters enter into *PIN* only as the product α μ, and so can offset each other. We can obtain similar probability distributions for buys and sells whether informed traders are many (high μ) and information events are infrequent (low α) or informed traders are few (low μ) and information events are frequent (high α). This is important for estimation. Order one-sidedness in a sample may yield a relatively precise estimate for *PIN*.

Discussion

Although the expanded model is more realistic than the simple version, it remains highly stylized.

- Information events that can occur at most *once per day*, events that always occur at the *beginning of the day*, two possible value realizations, etc., seem to push this model far away from anything that might be found in reality. As a result, it is probably a mistake to take the model's estimates too literally.
- The obvious difference between the Roll and the sequential trade models is that the former focuses on price and the latter on orders.
- There is also, however, a profound difference in how the time series are viewed. The Roll model exists in a covariance stationary framework, where information arrives uniformly and all relevant probability distributions are time-invariant.
- The setup in the sequential trade models, though, is neither stationary nor ergodic.
  - Stationarity is *violated*, for example, because even before the day begins, we know that the distribution of the spread immediately prior to the tenth trade *differs* from that prior to the ninth trade.
  - Ergodicity is violated because successive probabilities of buys, for example, always depend on the outcome of the *initial* value draw.
- The dynamics of the sequential trade model *resemble* an *adjustment process* after a shock. This might be most accurate, for example, in modeling trading leading up to a significant prescheduled corporate *news announcement*.
- The covariance stationary framework of the Roll model, on the other hand, would conform more closely to an average trading day in which diverse information arrives continually during the day.

Strategic Trade Models

In the sequential trade framework, there are many informed agents, but each can trade *only once* and only if he or she is “drawn” as the arriving trader. Furthermore, if order size is a choice variable, the informed agent will always trade the *largest* quantity. The model (Kyle (1985)) discussed here, differs in both respects.

- Here, there is a *single informed trader* who behaves *strategically*. She sets her trade size taking into account the *adverse* price concession associated with *larger* quantities.
- Furthermore, She can (in the multiple-period) version of the model, return to the market, *spreading out* her trades over time.

Strategic Trade Models, cont'd.

The practice of distributing orders over time to **minimize trade impact** is perhaps one of the most common strategies used in practice.

- With decimalization and increased fragmentation of trading activity, market participants have fewer opportunities to easily trade large quantities.
- In the present environment, therefore, *order-splitting* strategies are widely used by all sorts of traders (uninformed as well as informed).
- Although the Kyle model allows for strategic trade, whereas the sequential trade models do not, it is more stylized in some other respects. There is *no* bid and ask, for example; all trades *clear* at an informationally *efficient price*.

The Single-Period Model

The terminal security value is $v \sim N(p_0, \Sigma_0)$. There is *one informed trader* who knows v and enters a *demand* x. Liquidity ("noise") traders submit a *net order flow* $u \sim N(0, \sigma_u^2)$ u), *independent* of v. The *market maker* (MM) *observes* the total demand $y = x + u$ and then *sets* a price, p. All of the trades are *cleared* at p. If there is an *imbalance* between buyers and sellers, the MM *makes up* the difference.

Nobody knows the market clearing price when the submit their orders. Because the liquidity trader order flow is exogenous, there are really only two players we need to concentrate on: the informed trader and the market maker.

- The informed trader wants to trade aggressively, for example, buying a large quantity if her information is *positive*.
- The MM knows that if he sells into a large net customer buy, he is likely to be on the wrong side of the trade. He protects himself by setting a price that is *increasing* in the net order flow.
- This acts as a brake on the informed trader's desires: If she wishes to buy *a lot*, she'll have to pay a *high price*.

The solution is a formal expression of this trade-off.

Informed Trader's Problem

Informed Trader's Problem - We first consider the informed trader's problem (given a conjectured MM price function) and then show that the conjectured price function is *consistent* with the informed trader's optimal strategy. The informed trader *conjectures* that the MM uses a *linear* price adjustment rule $p=\lambda y+\mu$ where y is the *total order flow*: $y=u+x$. λ is an *inverse* measure of *liquidity*. The informed trader's profits are

$$\pi=(v-p)x \qquad \text{(EQ 5)}$$

Substituting in for the price conjecture and y yields

$$\pi=x[v-\lambda(u+x)-\mu] \qquad \text{(EQ 6)}$$

The *expected profits* are

$$E\pi=x(v-\lambda x-\mu) \qquad \text{(EQ 7)}$$

In the sequential trade models, an informed trader *always* makes money. This is *not* true here. For example, if the informed trader is buying ($x>0$), it is possible that a large surge of uninformed buying ($u>>0$) drives the $\lambda(u+x)+\mu$ above v. The informed trader chooses x to *maximize* $E\pi$, yielding $x=(v-\mu)/2\lambda$. The second-order condition is $\lambda>0$.

Market Maker Problem

Market Maker Problem - The Market maker conjectures that the informed trader's demand is linear in v: $x = \alpha + \beta v$. Knowing the optimization process that the informed trader followed, the MM can solve for α and β : $(v - \mu)/2\lambda = \alpha + \beta v$ for all v. This implies

$$\alpha = -\mu/2\lambda \text{ and } \beta = 1/2\lambda \qquad \text{(EQ 8)}$$

The inverse relation between β and λ is particularly important. As liquidity *drops* (i.e., as λ rises) the *informed* agent trades *less*. Now the MM must figure out $E[v|y]$. This computation relies on the following result, which are also used later in other contexts.

Bivariate normal projection

Bivariate normal projection - Suppose that X and Y are *bivariate normal* random variables with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and covariance σ_{XY}. The conditional expectation of Y given X is

$$E[Y \mid X = x] = \mu_Y + (\sigma_{XY} / \sigma_X^2)(x - \mu_X) \qquad \text{(EQ 9)}$$

Because this is linear in X, conditional expectation is equivalent to projection. The variance of the projection error is

$$Var[Y \mid X = x] = \sigma_Y^2 - \sigma_{XY}^2 / \sigma_X^2 \qquad \text{(EQ 10)}$$

Note that this does not depend on x.

Here, given the definition of the order flow variable and the MM's conjecture about the informed traders behavior, $y = u + \alpha + \beta v$, we have $Ey = \alpha + \beta Ev = \alpha + \beta p_0$, $Var(y) = \sigma_u^2 + \beta^2 \Sigma_0$, and $Cov(y, v) = \beta \Sigma_0$.

Bivariate normal projection

Using these in the projection results gives

$$E[v \mid y] = p_0 + \frac{\beta(y - \alpha - \beta p_0)\Sigma_0}{\sigma_u^2 + \beta^2\Sigma_0} \text{ and}$$

$$Var[v \mid y] = \frac{\sigma_u^2\Sigma_0}{\sigma_u^2 + \beta^2\Sigma_0} \qquad \text{(EQ 11)}$$

This must equal $p=\lambda y + \mu$ for all values of y, so

$$\mu = \frac{-\alpha\beta\Sigma_0 + \sigma_u^2 p_0}{\sigma_u^2 + \beta^2\Sigma_0} \text{ and } \lambda = \frac{\beta\Sigma_0}{\sigma_u^2 + \beta^2\Sigma_0} \qquad \text{(EQ 12)}$$

Solving for the parameters yields:

$$\alpha = -p_0\sqrt{\frac{\sigma_u^2}{\Sigma_0}} \; ; \quad \mu = p_0 \; ; \quad \lambda = \frac{1}{2}\sqrt{\frac{\Sigma_0}{\sigma_u^2}} \text{ and } \beta = \sqrt{\frac{\sigma_u^2}{\Sigma_0}} \qquad \text{(EQ 13)}$$

Discussion

Both the liquidity parameter λ and the informed trader's order coefficient β depend only on the value uncertainty Σ_0 relative to the intensity of noise trading σ_u^2.
The informed trader's expected profit is:

$$E\pi = \frac{(v-p_0)^2}{2}\sqrt{\frac{\sigma_u^2}{\Sigma_0}} \qquad \text{(EQ 14)}$$

This is *increasing* in the *divergence* between the *true value* and the *unconditional mean*. It is also *increasing* in the *variance* of noise trading. We can think of the noise trading as providing *camouflage* for the informed trader.

This if of practical importance. All else equal, an agent trading on inside information will be able to make more money in a widely held and frequently traded stock (at least, prior to broad acquisition of the knowledge by the market).

How much of the private information is impounded in the price? Using the expression in EQ 11 gives $Var[v|p] = Var[v|y] = {}^{\Sigma_0}/_{2}$. That is, *half* of the insider's information gets into the price. This does *not* depend on the intensity of the trading noise.

The essential properties of the Kyle model that make it tractable arise from the multivariate normality (which gives linear conditional expectations) and a quadratic objective function (which has a linear first-order condition). The multivariate normality can accommodate a range of modifications.

Multiple Rounds of Trading

A practical issue in market design is the determination of when trading should occur. Some firms on the Paris Bourse, for example, trade in twice-per-day call auctions, others continuously within a trading session. What happens in the Kyle model as we increase the number of auctions, ultimately converging to continuous trading? We will consider the case of N auctions that are equally spaced over a unit interval of time. The time between auctions is $\Delta t = 1/N$. The auctions are indexed by n=1, . . .N. The noise order flow arriving at the nth auction is defined as u_n. This is distributed normally, $\Delta u_n \sim N(0, \sigma_u^2 \Delta t)$ where σ_u^2 has the units of variance per unit time. The use of the difference notation facilitates the passage to continuous time. The equilibrium has the following properties.

- The informed trader's demand is $\Delta x_n = \beta_n (v - p_{n-1}) \Delta t_n$, where β_n is trading intensity per unit time.
- The price change at the nth auction is $p_n = p_{n-1} + \lambda_n (\Delta x_n + \Delta u_n)$
- Market efficiency requires $p_n = E[v \mid y_n]$ where y_n is the cumulative order flow over time.

In *equilibrium* the informed trader "slices and dices" her order flow, distributing it over N auctions. This corresponds to the real-world practice of splitting the orders, which is used by traders of all stripes, not just informed ones.

Appendix: Poisson Process

Intuitive Construction of a Poisson Process

A Poisson process $N(t)$ is an increasing, integer-valued process (i.e. values 0, 1, 2….) with jump times $(\tau_1, \tau_2, \tau_3, \ldots)$ such that the *probability of jump* in the next small time interval Δt is *proportional* to the interval Δt:

$$\mathbf{P}[N(t+\Delta t) - N(t) = 1] = \lambda \Delta t \qquad \text{(A1)}$$

that the jumps by more than 1 *do not* occur, and that jumps in *disjoint* time intervals happen *independently* of each other. This means, conversely, that the probability of the process remaining constant is:

$$\mathbf{P}[N(t+\Delta t) - N(t) = 0] = 1 - \lambda \Delta t \qquad \text{(A2)}$$

Figure A1 is a depiction of one period evolution of the Poisson process in discrete time:

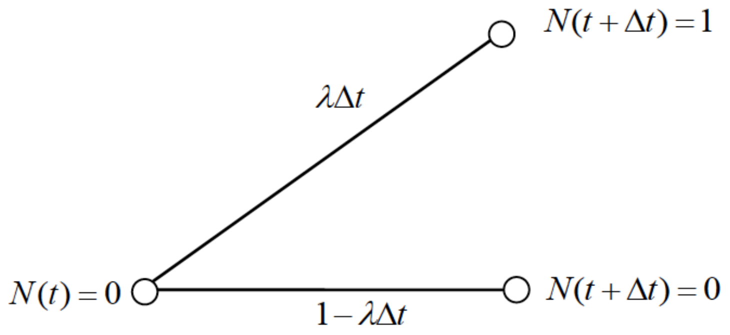


Figure A1: One-Period Evolution of a Poisson Process in Discrete Time

Appendix: Poisson Process

Over the interval $[t, t+2\Delta t]$ this probability is

$$\mathbf{P}[N(t+2\Delta t)-N(t)=0]$$

$$=\mathbf{P}[N(t+\Delta t)-N(t)=0].\mathbf{P}[N(t+2\Delta t)-N(t+\Delta t)=0] \qquad \text{(A3)}$$

$$=(1-\lambda\Delta t)^2$$

Now we can start to construct a Poisson process. We *subdivide* the interval $[t, T]$ into n subintervals of length $\Delta t=(T-t)/n$. In each subintervals the process $N(t)$ has a jump with probability $\lambda\Delta t$. We construct n independent *binomial experiments* each with a probability $\lambda\Delta t$ for a jump outcome. The probability of *no jump* at all in $[t, T]$ is in the limit of small Δt given by:

$$\mathbf{P}[N(T)-N(t)=0]=(1-\lambda\Delta t)^n=\left(1-\frac{1}{n}\lambda(T-t)\right)^n \to \exp(-\lambda(T-t)) \qquad \text{(A4)}$$

Appendix: Poisson Process

Next we can look at the probability of *exactly one* jump in $[t, T]$. There are n possibilities of having exactly one jump, given the total probability of

$$\mathbf{P}[N(T)-N(t)=1]=n\lambda\Delta t(1-\lambda\Delta t)^{n-1}$$

$$=\frac{\lambda(T-t)}{1-\frac{1}{n}\lambda(T-t)}\left(1-\frac{1}{n}\lambda(T-t)\right)^{n}\to\lambda(T-t)\exp\left(-\lambda(T-t)\right) \tag{A5}$$

Similarly one can reach the limit probabilities for *two jumps*:

$$\mathbf{P}[N(T)-N(t)=2]=\frac{1}{2}\lambda^2(T-t)^2\exp\left(-\lambda(T-t)\right) \tag{A6}$$

and for *n jumps*:

$$\mathbf{P}[N(T)-N(t)=n]=\frac{1}{n!}\lambda^n(T-t)^n\exp\left(-\lambda(T-t)\right) \tag{A7}$$

Appendix: Poisson Process

Figure A2 depicts multi-period evolution of the Poisson process in discrete time:

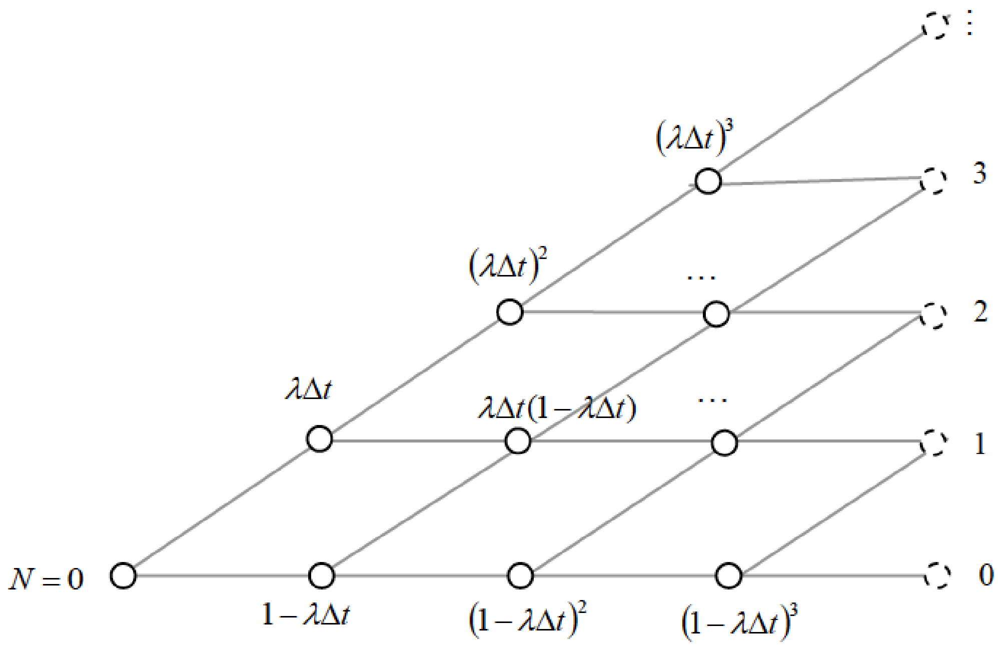


Figure A2: Multi-Period Evolution of the Poisson Process in Discrete Time

Appendix: Poisson Process

We now have derived the equation that is usually used to formally define a Poisson process.

Definition (Poisson Process): A Poisson processes with (constant) intensity $\lambda > 0$ is a non-decreasing, integer-valued process with initial value $N(0) = 0$ whose increments are independent and satisfy for all $0 \le t < T$

$$\mathbf{P}[N(T) - N(t) = n] = \frac{1}{n!}(T-t)^n \lambda^n \exp[-\lambda(T-t)] \qquad \text{(A8)}$$

The *discrete time* approximation of a Poisson process *resembles* the binomial approximation of a Brownian motion. In both constructions we take a number of binomially distributed random variables with only *two possible* values and add them up to get the full process.

- In the Poisson case we take the individual jumps in the time interval Δt.
- For Brownian motion we take the "up" and "down" movements at individual nodes.

Appendix: Poisson Process

Figure A3 below shows the comparison of (one-period) discrete time approximation of a Poisson process to that of a Brownian motion:

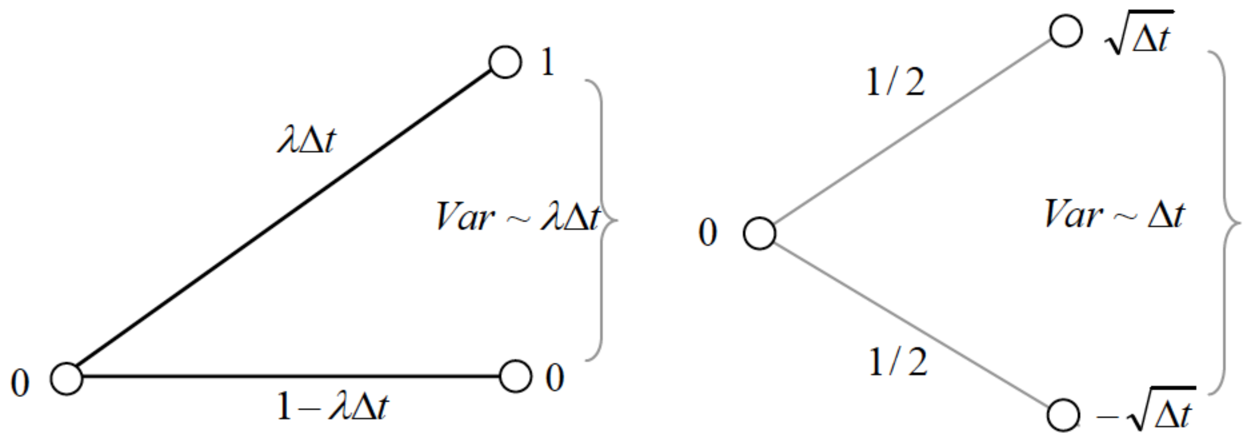


Figure A3: Discrete Time Approximations of Poisson Process vs. Brownian Motion

Appendix: Poisson Process

The difference lies in what we do in the *limit*. For the Brownian motion we decrease the jump size (proportional to $1/\sqrt{n}$) and keep the *probabilities* constant. For Poisson process, on the other hand, we keep the *jump size* constant (at 1) and decrease the probability of jump (proportional to $1/n$). Both limiting procedures ensure that the *variance* of the final value of the process remains finite (and proportional to time step), but the resulting behavior of the process is completely different in two cases.

Figures A4 and A5 below compare the continuous time behavior of a Poisson process versus Brownian motion:

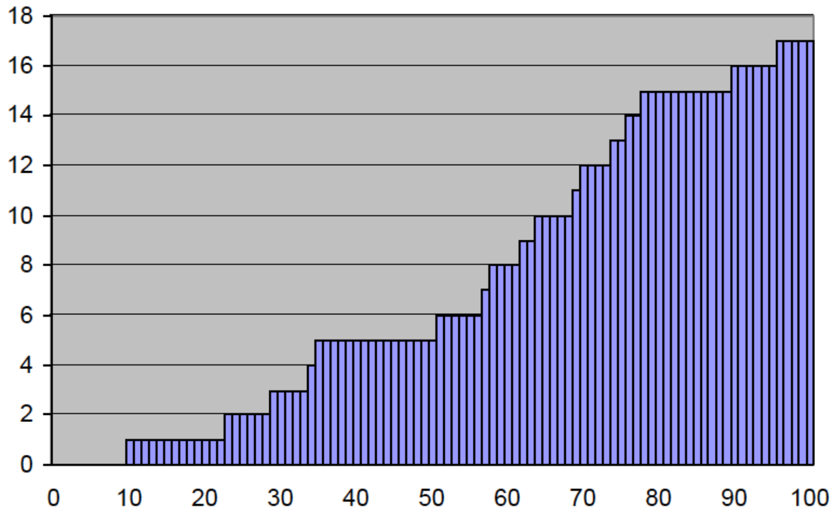


Figure A4: Typical Path of a Homogeneous Poisson Process

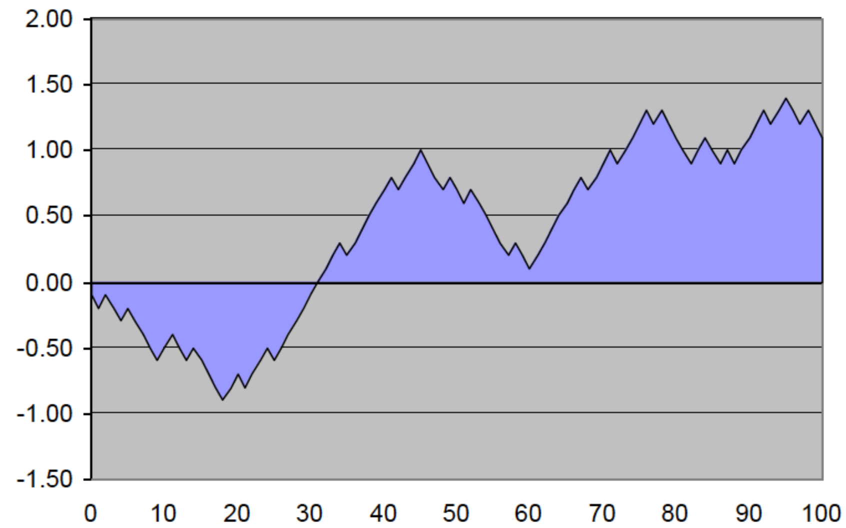


Figure A5: Typical Path of a Standard Brownian Motion

Properties of the Poisson Process

Poisson processes are usually used to model either rare events (e.g. insurance mathematics) or discretely countable events (e.g. radioactive decay). Both properties also apply to *default*, which are *rare* and *discrete* events, where one models the default of a firm as the *first* jump of a Poisson process.

Here are some additional properties of the Poisson process:

1. The Poisson process has *no memory*. The probability of n jumps in $[t, t+s]$ is *independent* of $N(t)$ and its *history* before t. In particular, a jump is *no* more likely just because the last jump occurred a long time ago.

2. The *inter-interval* times (i.e. times between two consecutive events) of a Poisson process ($\tau_{n+1} - \tau_n$) are *exponentially distributed* with density

$$P[\tau_{n+1} - \tau_n \in dt] = \lambda \, dt \exp(-\lambda t) \qquad \text{(A9)}$$

   This is the *density* of the time of the next jump of N.

3. *Two or more jumps* at exactly the same time have *zero* probability.

Properties of the Poisson Process

4. For financial modeling we need to know how to handle Poisson-type processes in stochastic differential equations. From the construction we have

$$\mathbf{E}[dN] = \lambda dt \tag{A10}$$

The *predictable compensator* of $N(t)$ is λt, i.e. the following "compensated" process is a *martingale*:

$$M(t) = N(t) - \lambda t \tag{A11}$$

5. The *covariations* of a Poisson process (with itself) follow directly from the construction:

$$dN.dN = dN, \quad E[dN^2] = \lambda dt \tag{A12}$$

Properties of the Poisson Process

6. The covariations with any *Brownian motion* W are identically zero:

$$[dN, dW] = 0, \quad d < N, W >= 0 \tag{A13}$$

To see this recall that on any time interval $[0, T]$ we have $dN = 0$ *except* for a *finite* number of points in time $T_1, T_2, \ldots, T_K$ (almost surely). This implies that

$$\int_0^T dN(t)dW(t) = \sum_{k=1}^{K} dW(T_k) \sim k\sqrt{dt} \to 0 \tag{A14}$$

due to the fact that the increments of the Brownian motion are infinitesimally small and go to zero like $\sqrt{dt}$.

7. Similarly, for any *differentiable functions* f and g at all times we have:

$$d\big(f(N(t))g(W(t))\big) = f(N(t))dg(W(t)) + g(W(t))df(N(t)) \tag{A15}$$

and all *martingales* generated by $N(t)$ are *uncorrelated* with all martingales generated by any Brownian motion (as will be shown later on in detail).