Econ 3389 Big Data: Homework 0

Due Monday, Jan 25th (Beginning of class)

1 Calculus

Derivatives Compute the derivatives of the following functions *with respect to x*. Describe the points at which the derivative does not exist, if any exist in the domain of the function.

> Examples:
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> - $f(x) = \log(x)$ for $x > 0$. Answer: $f'(x) = \frac{1}{x}$ everywhere on $x > 0$.
> - $f(x, y) = \frac{2y}{x}$. Answer:$\frac{\partial f(x,y)}{\partial x} = -\frac{2y}{x^2}$ except at $x = 0$.

1. $f(x) = \frac{x}{1+x}$
2. $f(x) = \sin(x) + i\cos(x)$ at $x = \pi$ Note: $i = \sqrt{-1}$
3. $f(x) = |x|$
4. $f(x, y) = y^x$ for $y \geq 0$.

Critical points Find the critical points of the following functions over $\mathbb{R}$. Use the second derivative to prove whether the critical point is a minimum or a maximum.

> Example: $f(x) = -x^2 + 3$
> Answer: $f'(0) = 0$ and $f''(0) = -2 < 0$, hence $x = 0$ is a maximum. There are no minima.

1. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$
2. $f(x) = \sqrt{25 - x^2}$
3. $f(x) = e^{-x^2/2}$

2 Linear Algebra

Matrix algebra Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$. Compute:

1. $A + B$
2. A^{-1}
3. B^T Note: B^T is the transpose of B. Alternatively written B'.
4. AB

Matrix Properties Prove or find a counterexample.

1. Given *any* square matrices A, B of the same dimension, it is always true that $AB = BA$.
2. If A is *any* invertible matrix, then A^T is also invertible.
3. If $AB = I$, then $BA = I$, where $I = \begin{bmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 1 \end{bmatrix}$, the identity matrix.

3 Probability

Moments Let X be the outcome of one dice roll.

1. What is the probability mass function of this random variable?
2. What is the mean outcome of one dice roll?
3. What is the variance of one dice roll?
4. What is the mean of the sum of two rolls?
5. What is the mean of the sum of one thousand rolls?

Properties of Expectation and Variance Let X, Y be two random variables with finite variance, and define $W = Y - E[Y|X]$.

1. Compute $E[W]$ and $E[W|X]$
2. Compute $Var[W]$for the case $W|X \sim \mathcal{N}(0, X^2)$ and $X \sim \mathcal{N}(0, 1)$

Note: The notation $Z \sim N(\mu, \sigma^2)$ means "the random variable Z is normally distributed with mean μ and variance σ^2."

4 Statistical inference

Testing We have one thousand values drawn from $\mathcal{N}(\mu, 1)$, that is, from a normal distribution with unknown mean μ and known variance 1. We are interested in testing whether this distribution has mean $\mu = 0$.

1. Suggest an unbiased and consistent estimator $\widehat{\mu}$ for μ. How is your estimator distributed?
2. Formalize your test in terms of a null and an alternative hypothesis.
3. Given a size α (i.e., a confidence level $1 - \alpha$), explain under what circumstances you would reject your null hypothesis.
4. Now suppose that, instead of being normally distributed, the values are drawn from some other distribution, say the Binomial distribution. Would your answers above change? (Hint: Use the Central Limit Theorem)