Advantages of an Ellipse when Modeling Leisure Utility *

Richard W. Evans[†] Kerk L. Phillips[‡]

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Abstract

This paper characterizes a specification for the utility of leisure that is based on the general equation for an ellipse. We show that this functional form has multiple benefits. The elliptical utility function provides Inada conditions at both the upper-bound and lower-bound constraints on labor supply, which is not the case for the two most common alternative functions. The presence of these two Inada conditions in the elliptical utility of leisure specification speeds up the computation by a factor between three and six times. We further show that the elliptical utility of leisure function is a close approximation to the constant relative risk aversion (CRRA) and constant Frisch elasticity (CFE) functions in terms of marginal utilities, microeconomic outcomes in a life cycle model, and macroeconomic outcomes in a simple real business cycle (RBC) model.

keywords: ellipse, labor, leisure, utility, Inada conditions, occasionally binding constraints, interior solutions.

JEL classification: C02, C60, C63, J20

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†Brigham Young University, Department of Economics, 167 FOB, Provo, Utah 84602, (801) 422-8303, revans@byu.edu.

‡Brigham Young University, Department of Economics, 166 FOB, Provo, Utah 84602, kerk_phillips@byu.edu.

1 Introduction

Constrained optimization problems are the hallmark of many economic models. It is well known that these models become difficult to compute when they have occasionally binding inequality constraints.[1] This computational difficulty is multiplied when the model has high dimensions of heterogeneity among the agents with occasionally binding constraints. Judd et al. (2003) document this issue in the context of non-adaptive grid methods. One of the key household decisions that includes important upper-bound and lower-bound occasionally binding constraints is the labor supply decision.

In this paper, we characterize a specification for the utility of leisure that is based on the general equation for an ellipse. We show that this functional form has multiple benefits. Computationally, our proposed elliptical utility function provides Inada conditions at both the upper-bound and lower-bound constraints on labor supply.[2] The two main functional forms for the utility of leisure—namely, constant relative risk aversion (CRRA) and constant Frisch elasticity (CFE)—each only have a single Inada condition on either the upper bound or lower bound of labor supply. We find that the presence of these two Inada conditions in the elliptical utility of leisure specification speeds up the computation by a factor between three and six times.

Further, we use a number of metrics to show that the elliptical utility of leisure specification is a close approximation to the common CRRA and CFE specifications. We show that the fitted marginal utilities, the microeconomic household outcomes in a life cycle model, and the macroeconomic outcomes in a simple real business cycle (RBC) model are all very similar.

Lastly, the elliptical utility function has the empirically attractive property that the Frisch elasticity of labor supply is decreasing as labor supply increases. This is also true of CRRA utility of leisure. However, the elliptical utility Frisch elasticity approaches a finite level as labor supply goes to zero, whereas the Frisch elasticity approaches infinity as labor goes to zero for the CRRA specification.

It is true that corner solutions of labor supply are important occurrences empirically in household decisions. However, even in macroeconomic models with large degrees of het-

[1] See Guerrieri and Iacoviello (2015), Brumm and Grill (2014), Judd et al. (2003), and Christiano and Fisher (2000).

[2] See Inada (1963). An Inada condition has come to be known as a condition that guarantees a strict interior solution that satisfies any inequality constraints.

erogeneity, we rarely see bins of individuals fine enough that their empirical labor supply averages are either zero or the maximum.[3] This suggests that, at least in macroeconomic models, the use of the elliptical utility of leisure function that bounds solutions away from the corners in labor supply does not cost the modeler much realism. Further, even in microeconomic models, the lack of corner solutions may be worth the speedup in computation time.

Section 2 presents the general household constrained optimization with a focus on the labor supply decision. We then present the two main specifications for the utility of leisure (CRRA and CFE) followed by the specification of our proposed elliptical utility functional form. Section 3 compares the economic outcomes from these three specifications in terms of marginal utilities of leisure, Frisch elasticities of labor supply, microeconomic outcomes from a life cycle model, and macroeconomic outcomes from a simple RBC model.

2 Utility of Leisure Alternatives

We first present the general formulation of a static partial-equilibrium constrained utility maximization problem in which an individual chooses consumption c and leisure ℓ subject to a budget constraint and subject to leisure being between zero and an upper bound time endowment of one unit. We show that both of the two most common additively separable functional forms for the utility of leisure only provide Inada conditions for one of the two conditions for an interior solution in leisure ℓ.[4] We then show how our elliptical utility of leisure (disutility of labor) function satisfies both conditions for an interior solution in leisure ℓ.

2.1 General constrained maximization problem

A generalized static partial-equilibrium version of an individual's constrained utility maximization problem choosing consumption c and leisure ℓ often assumes an additively sepa-

[3] DeBacker et al. (2015) use this elliptical utility of leisure function in a large-scale overlapping generations model. They find that the empirical distribution of labor is never zero on average for any bind of income percentiles or age groups between age 21 and 85.

[4] Another commonly used functional form is constant elasticity of substitution (CES) between consumption and leisure and is not additively separable in consumption and leisure. See İmrohoroğlu and Kitao (2012), King and Rebelo (1999), and Kydland (1995).

rable period utility function of the following form,

$$U\big(c,\ell\big) \equiv u(c) + \chi v(\ell) \tag{2.1}$$

where $c > 0$ is consumption, ℓ is leisure, $\chi > 0$ is a scale parameter on the utility of leisure, and the functional forms for the utility of consumption and the utility of leisure satisfy $u', v' \geq 0$ and $u'', v'' \leq 0$ for all c and ℓ. The constrained maximization problem with a budget constraint and a constraint that leisure be between zero and an upper-bound time endowment of one is the following,

$$\max_{c,\ell} \;\; u(c) + \chi v(\ell) \quad \text{s.t.} \quad c \leq w(1-\ell) + y \quad \text{and} \quad 0 \leq \ell \leq 1 \tag{2.2}$$

where $w > 0$ is the exogenous real wage and $y > 0$ is exogenous net non-labor income. The inequality constraint on leisure ℓ implies that individuals are endowed with a unit of time. This constrained maximization problem has three possible solutions with regard to leisure.

The Lagrangian for the constrained maximization problem in (2.2) has three multipliers for the three inequality constraints, with the Karush-Kuhn-Tucker (KKT) conditions holding.

$$\mathcal{L} = u(c) + \chi v(\ell) + \lambda_1\Big(w[1-\ell] + y - c\Big) + \lambda_2\big(1-\ell\big) + \lambda_3\ell \tag{2.3}$$

The standard interior solution for leisure holds when the marginal utility of leisure satisfies the following two conditions.

$$\begin{aligned} \ell \in (0,1),\; \lambda_2, \lambda_3 = 0 \quad \text{when} \quad \chi v'(0) > wu'(y+w) \quad \text{and} \quad \chi v'(1) < wu'(y+w) \\ \text{therefore} \quad \chi v'(\ell) = wu'\big(y + w[1-\ell]\big) \end{aligned} \tag{2.4}$$

The corner solution of $\ell = 0$ (and, therefore $\lambda_3 > 0$) occurs when the marginal utility of zero leisure is less than the marginal utility of consumption with zero leisure.

$$\ell = 0,\; \lambda_3 > 0 \quad \text{when} \quad \chi v'(0) < wu'(y+w) \quad \iff \quad \chi v'(0) = wu'(y+w) - \lambda_3 \tag{2.5}$$

Without the inequality constraint $\ell \geq 0$, the optimization problem would solve for $\ell < 0$ in this case. The corner solution of $\ell = 1$ (and, therefore $\lambda_2 > 0$) occurs when the marginal utility of leisure equal to 1 is less than the marginal utility of consumption with leisure equal to 1.

$$\ell = 1,\ \lambda_2 > 0 \quad \text{when} \quad \chi v'(1) > wu'(y) \quad \iff \quad \chi v'(1) = wu'(y) + \lambda_2 \tag{2.6}$$

Without the inequality constraint $\ell \leq 1$, the optimization problem would solve for $\ell > 1$ in this case.

Functional forms for utility functions $u(\cdot)$ and $v(\cdot)$ that bound them in such a way as to guarantee interior solutions are said to satisfy Inada conditions.[5] It is such an Inada condition on $u(\cdot)$ that usually guarantees an interior solution on consumption such that $c > 0$ and $\lambda_1 = 0$.

As mentioned in Section 1, optimization problems with occasionally binding constraints are notorious for being computationally difficult. The difficulty for optimization is that the inequality conditions introduce kinks in the objective function and one-sided support for the multipliers in the KKT representation of the Lagrangian formulation of the problem. That computational difficulty multiplies exponentially when an optimization problem has many occasionally binding constraints as is the case with heterogeneous agent models.

2.2 Constant Relative Risk Aversion (CRRA)

Many studies use a constant relative risk aversion (CRRA) functional form for the utility of leisure.[6] This functional form also implies a constant intertemporal elasticity of substitution (IES) for labor, where the IES is the inverse of the coefficient of relative risk aversion. Any study that uses the natural logarithm of leisure is a nested case of CRRA utility of leisure. This functional form is characterized as follows,

$$v(\ell) \equiv \frac{\ell^{1-\eta} - 1}{1-\eta}, \quad \eta \geq 0 \tag{2.7}$$

where η is the coefficient of relative risk aversion.[7]

This CRRA utility of leisure function provides an Inada condition for the lower bound of leisure (the upper bound of labor supply n) because the utility of zero leisure $\ell = 0$ produces infinite marginal utility. In this case, (2.5) cannot occur because $\chi v'(0) > wu'(y + w)$ for

[5] See Inada (1963).

[6] See Hansen (1985), Cahuc and Zylberberg (2004, p. 22), and Rogerson and Wallenius (2013).

[7] The CRRA function nests log utility $\ln(\ell)$ for $\eta = 1$. More specifically, $\lim_{\eta\to 1} \frac{\ell^{1-\eta}-1}{1-\eta} = \ln(\ell)$ by L'Hospital's rule.

all w and y. This Inada condition removes the need for the λ_3 constraint in the Lagrangian (2.3) because $\lambda_3 = 0$ and $\ell > 0$ always.

On the other hand, the upper-bound condition for leisure (lower bound of labor supply n) is not always satisfied with the CRRA functional form (2.7). There is no Inada condition for the constraint that $\ell \leq 1$. There are values for χ, w, y, and the functional form $u(\cdot)$ such that $\chi > wu'(y)$ and the optimal leisure is greater than one in the unconstrained problem, thereby violating the inequality constraint that $\ell \leq 1$. This is especially true in general equilibrium problems in which w and y are endogenous. This lack of an Inada condition for the upper bound of leisure requires the user of a CRRA utility of leisure function to solve a more difficult constrained optimization problem.

2.3 Constant Frisch Elasticity (CFE)

The other most popular utility of leisure functional form is the constant Frisch elasticity (CFE) utility of leisure (disutility of labor) function. This functional form is used more often in studies that focus on the outcomes in the labor market.[8] The functional form is the following,

$$v(\ell) \equiv -\frac{(1-\ell)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}} = -\frac{n^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}, \quad \theta > 0 \tag{2.8}$$

where n is the individual supply of labor and θ is the Frisch elasticity of labor supply. We assume that individuals can use their time endowment for either leisure ℓ or labor n such that $n = 1 - \ell$.

The CFE utility of leisure function provides an Inada condition for the upper bound of leisure (the lower bound of labor supply n) because the utility of maximum leisure $\ell = 1$ produces zero marginal utility. In this case, (2.6) cannot occur because $0 < wu'(y)$ for all w and y. This Inada condition removes the need for the λ_2 constraint in the Lagrangian (2.3) because $\lambda_2 = 0$ and $\ell < 1$ always.

On the other hand, the lower-bound condition for leisure (upper bound of labor supply n) is not always satisfied with the CFE functional form (2.8). There is no Inada condition for the constraint that $\ell \geq 1$. There exist values for χ, w, y, and the functional form $u(\cdot)$ such that $\chi < wu'(y+w)$ and the optimal leisure is less than zero in the unconstrained

[8] See Peterman (2016), Dyrda et al. (2012), and Cho and Cooley (1994). Note that when models are log-linearized, the CFE functional form can be made to be isomorphic to the CRRA functional form. However, this is certainly not the case in their original respective nonlinear forms.

problem, thereby violating the inequality constraint that $\ell \geq 0$ characterized in (2.6). This lack of an Inada condition for the lower bound of leisure requires the user of a CFE utility of leisure function to solve a more difficult constrained optimization problem.

2.4 Elliptical utility of leisure

The general functional form for an ellipse in x and y space is the following,

$$\left(\frac{x-h}{a}\right)^{\mu} + \left(\frac{y-k}{b}\right)^{\mu} = 1, \quad a,b > 0 \quad \text{and} \quad \mu > 1 \tag{2.9}$$

where the centroid of the ellipse is at coordinates $(x_0, y_0) = (h, k)$, the horizontal radius is $a > 0$, the vertical radius is $b > 0$, and the curvature is controlled by $\mu > 1$.[9] Figure 1 shows an ellipse with the parameterization $[h, k, a, b, \mu] = [1, -1, 1, 2, 2]$. The upper-right quadrant of the ellipse is highlighted because we focus on this portion of the function.

In (2.9), let x be labor supply n or $1-\ell$ and let y be the utility of leisure $v(\ell)$.

$$\left(\frac{n-h}{a}\right)^{\mu} + \left(\frac{v(\ell)-k}{b}\right)^{\mu} = 1 \tag{2.10}$$

We only care about values of labor supply between 0 and 1, and the condition $v'(\ell) > 0$ for all ℓ implies that $\frac{\partial v(\ell)}{\partial n} < 0$ for all n. For this reason, we restrict ourselves to the upper-right quadrant of an ellipse of the form (2.10), where the x-coordinate of the centroid is zero $h = 0$, the radius in the x-dimension is one $a = 1$, and we can normalize the centroid in the y-dimension to zero $k = 0$ because it will not appear in any marginal utility that affects behavior.

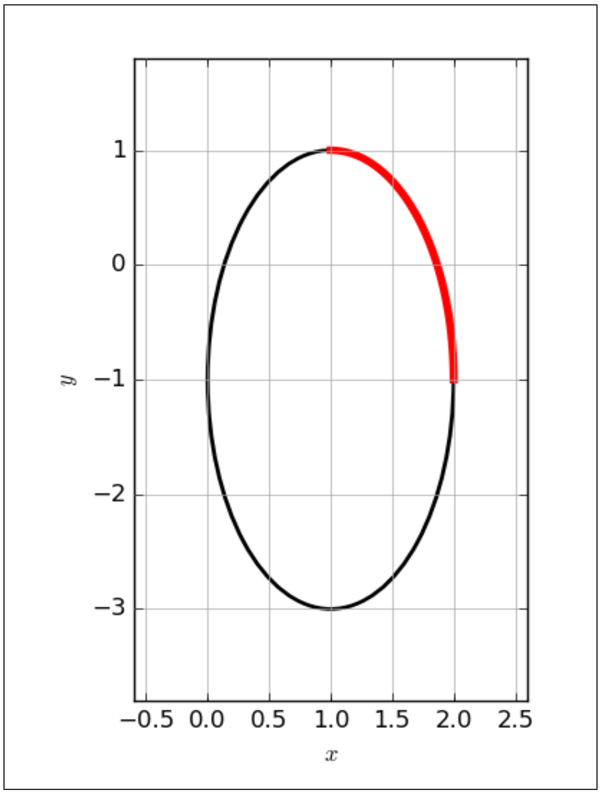
$$n^{\mu} + \left(\frac{v(\ell)}{b}\right)^{\mu} = 1 \tag{2.11}$$

We can then solve (2.11) for $v(\ell)$ to arrive at our elliptical utility functional form for the utility of leisure.

$$v(\ell) = b\left(1 - n^{\mu}\right)^{\frac{1}{\mu}} \quad \text{or} \quad v(\ell) = b\Big(1 - [1-\ell]^{\mu}\Big)^{\frac{1}{\mu}} \tag{2.12}$$

[9] In the equation for an ellipse, the parameter μ is usually restricted to equal two. However, in our case, where we are focusing on the upper-right quadrant of the ellipse, we can allow $\mu > 1$ more generally. Increasing this parameter simply increases the curvature of the upper-right quadrant of the ellipse. If we were concerned about the other three quadrants of the ellipse, we would need absolute value operators in the respective numerators of both terms on the left-hand-side of (2.9).

Figure 1: **Ellipse with** $[h, k, a, b, \mu] = [1, -1, 1, 2, 2]$



Further, because $v(\ell)$ is multiplied by χ in the period utility function (2.1), we can normalize $b = 1$ and let χ from the utility function be the level parameter.

$$v(\ell) = (1 - n^\mu)^{\frac{1}{\mu}} \quad \text{or} \quad v(\ell) = \Big(1 - [1 - \ell]^\mu\Big)^{\frac{1}{\mu}} \tag{2.13}$$

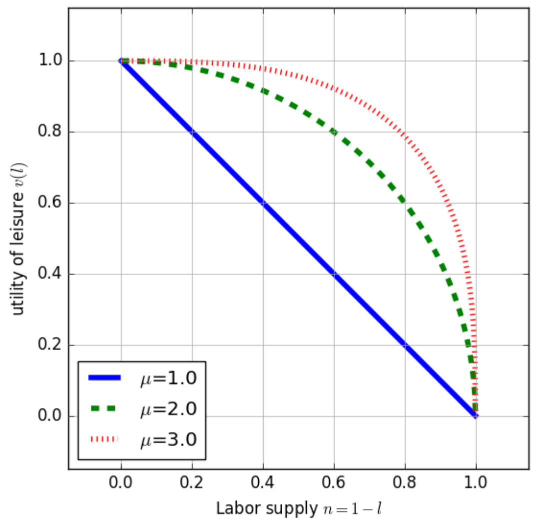
This functional form satisfies both $v'(\ell) > 0$ and $v''(\ell) < 0$ for all $\ell \in (0, 1)$.

Figure 2 shows the upper-right quadrant of the elliptical utility of leisure $v(\ell)$ from (2.13) on the y-axis and labor supply n on the x-axis for three different values of μ.[10] Analogous to η in the CRRA utility of leisure function (2.7) and θ in the CFE disutility of labor function (2.8), the parameter μ in the elliptical utility of leisure function (2.13) simply increases the curvature of the function.

Most importantly for this paper, the elliptical utility of leisure function $v(\ell)$ in (2.13) includes Inada conditions for both the upper-bound and lower-bound constraints on leisure.

[10] If we had plotted these functions in terms of leisure ℓ on the x-axis, they would have been the upper-left quadrant of an ellipse with centroid $(h, k) = (1, 0)$ and radii $a, b = 1$.

Figure 2: Upper-right quadrant of ellipses with $[h, k, a, b] = [0, 0, 1, 1]$ for $\mu = [1, 2, 3]$



The elliptical utility function provides an Inada condition for the lower bound of leisure (the upper bound of labor supply n) because the utility of minimum leisure $\ell = 0$ produces infinite marginal utility (look at the slope of the right-hand-side of the graph of the function in Figure 2). In this case, (2.5) cannot occur because $\infty > wu'(y+w)$ for all w and y. This Inada condition removes the need for the λ_3 constraint in the Lagrangian (2.3) because $\lambda_3 = 0$ and $\ell > 0$ always.

The elliptical utility function (2.13) also provides an Inada condition for the upper bound of leisure (the lower bound of labor supply n) because the utility of maximum leisure $\ell = 1$ produces zero marginal utility (look at the slope of the left-hand-side of the graph of the function in Figure 2). In this case, (2.6) cannot occur because $0 < wu'(y)$ for all w and y. This Inada condition removes the need for the λ_2 constraint in the Lagrangian (2.3) because $\lambda_2 = 0$ and $\ell < 1$ always. With Inada conditions on both the upper and lower bounds of leisure, the elliptical utility function removes the need to solve the constrained optimization problem. This allows the modeler to compute the much easier unconstrained maximization problem.

One last comparison to note about the elliptical utility of leisure relative to CRRA and CFE functions are their respective Frisch elasticities of labor supply as functions of labor supply. Let ε be the Frisch elasticity of labor supply for a particular utility function. The

Frisch elasticity is defined as the wage elasticity of labor supply holding the marginal utility of wealth constant.[11] The Frisch elasticities of labor supply for each of the three functional forms are given below.[12]

$$\text{(CRRA)} \quad \varepsilon = \frac{1}{\eta}\left(\frac{1-n}{n}\right) \tag{2.14}$$

$$\text{(CFE)} \quad \varepsilon = \theta \tag{2.15}$$

$$\text{(Ellipse)} \quad \varepsilon = \frac{1-n^{\mu}}{\mu-1} \tag{2.16}$$

For both CRRA (2.14) and elliptical utility (2.16), the Frisch elasticity of labor supply is decreasing with labor supply for all n. All three elasticities are positive $\varepsilon \geq 0$ for all allowed values of their respective inputs. CRRA and elliptical utility elasticities approach zero as n goes to 1 $\lim_{n\to 1}\varepsilon = 0$, although the CRRA elasticity goes to zero faster. On the other hand, the limit of the CRRA elasticity as n goes to zero is infinity $\lim_{n\to 0}\varepsilon_{crra} = \infty$, whereas the limit of the elliptical utility elasticity goes to a finite positive scalar as n goes to its lower bound $\lim_{n\to 0}\varepsilon_{ellip} = \frac{1}{\mu-1}$. In Section 3.1, Figure 5 compares the Frisch elasticities as a function of labor supply for estimated versions of elliptical utility, CRRA, and CFE functional forms.

3 Evaluation of elliptical approximation

In this section, we fit the elliptical utility of leisure function (2.13) to a standard calibration of the CRRA utility of leisure function (2.7) and to the CFE disutility of labor function (2.8). Notice in the three functional forms that, in each case, $v(\ell)$ has only one parameter (η, θ, or μ), which controls the curvature of the utility of leisure or the disutility of labor. And in the period utility specification (2.1), the utility of leisure $v(\ell)$ is multiplied by a positive scalar parameter χ that controls the level of the function. The level parameter χ and the scale parameter μ in the elliptical utility function (2.13) are completely analogous to the other two-parameter functional forms and can be calibrated in the same way.

Our exercise in this section is not to say which of the three utility of leisure functions

[11] In the context of the general model described in (2.3), the Frisch elasticity is $\frac{\partial n^*}{\partial w}\frac{w}{n^*}$ where n^* is characterized by the interior solution (2.4) in which the marginal utility of wealth $\lambda_1 > 0$ is held constant (compensated elasticity) and the other two multipliers equal zero $\lambda_2, \lambda_3 = 0$.

[12] See Appendix A-1 for a derivation of the Frisch elasticities.

is best, but is rather to show that the elliptical utility function (2.13) is a close approximation to both canonical forms and that it provides the added benefit of greatly simplifying computation with its two Inada conditions on the labor supply constraints.

3.1 Fitting marginal utility of leisure and Frisch elasticity

As our baseline, we use a simple calibration for the CFE disutility of labor function (2.8). Peterman (2016) finds that the microeconomic literature estimates a Frisch elasticity in the range $\varepsilon \in [0.0, 0.5]$ and the macroeconomic literature estimates a Frisch elasticity in the range $\varepsilon \in [2.0, 4.0]$. We choose a value at the upper end of the microeconomic estimates for the Frisch elasticity in the CFE function of $\theta = 0.5$. And without loss of generality for fit purposes, we calibrate the level parameter in the CFE function to $\chi_{cfe} = 1$.

We choose parameters to fit functions by focusing on matching associated marginal utilities $v'(\ell)$ across the support of leisure ℓ. We choose this criterion because the marginal utilities are what influences behavior in the models. The respective marginal utilities of the three functions are the following.

$$\text{(CRRA)} \quad v'(\ell) = \ell^{-\eta}, \tag{3.1}$$

$$\text{(CFE)} \quad v'(\ell) = (1-\ell)^{\frac{1}{\theta}}, \tag{3.2}$$

$$\text{(Ellipse)} \quad v'(\ell) = (1-\ell)^{\mu-1}\Big(1-[1-\ell]^{\mu}\Big)^{\frac{1-\mu}{\mu}}, \tag{3.3}$$

Recall that each of these functions is preceded by the level parameter χ from the period utility function specification (2.1).

We first fit the CRRA utility of labor function (2.7) to most closely match our baseline CFE function by choosing χ and η in the CRRA function so that the CRRA marginal utility (3.1) most closely matches the baseline CFE marginal utility (3.2) along the support of leisure ℓ. Figure 3 compares the marginal utility of the CRRA utility of leisure function with estimated level parameter $\chi_{crra} = 0.0810$ and curvature parameter $\eta = 1.4112$ to the baseline CFE marginal utility.

With our specifications for the CFE and CRRA utility of leisure functions, we now fit the elliptical utility of leisure function to the CFE and CRRA functions, respectively, by matching marginal utility functions. Table 1 shows the estimates and fit measures, and

Figure 3: Marginal utility of leisure for baseline CFE ($\chi = 1.0, \theta = 0.5$) and fitted CRRA ($\chi = 0.0810, \eta = 1.4112$)

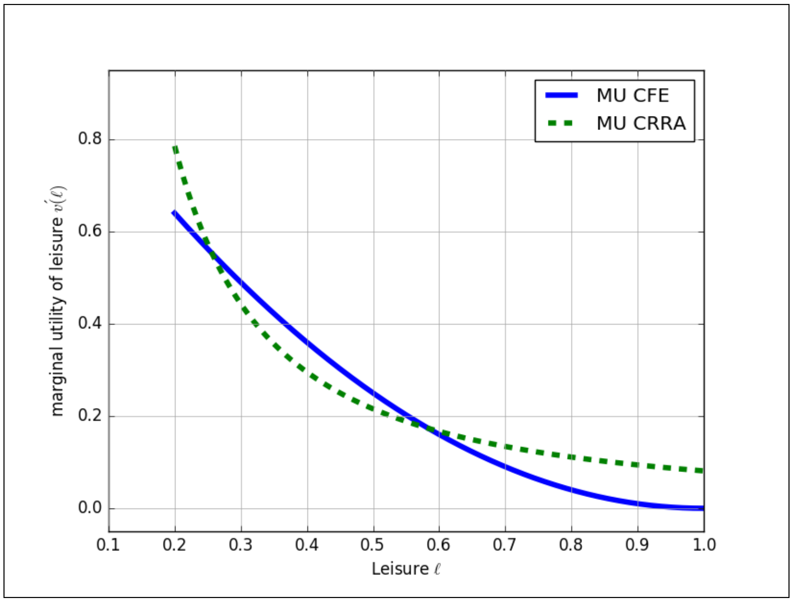


Table 1: Elliptical utility approximations to CRRA and CFE functions

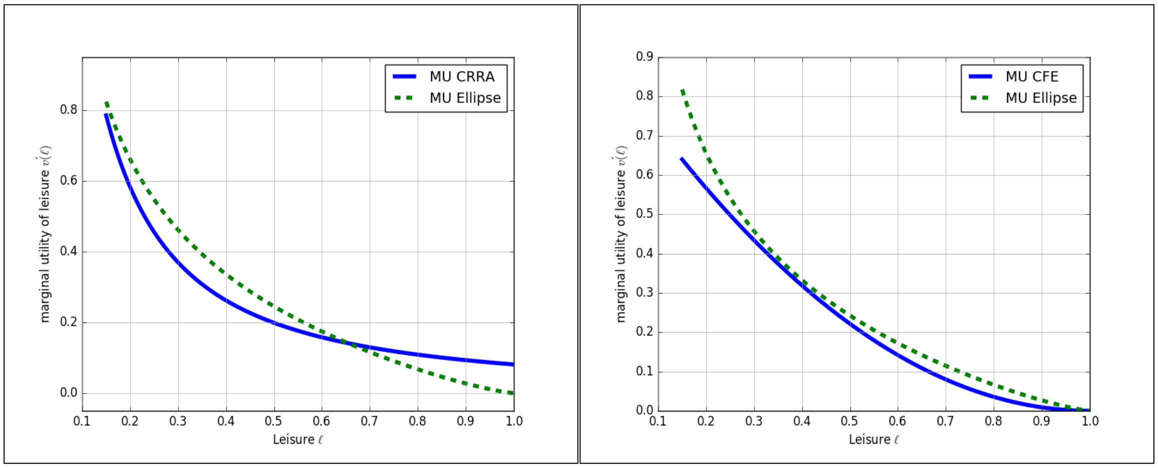
	Elliptic (2.13) to CRRA (2.7)		Elliptic (2.13) to CFE (2.8)	
	CRRA[c]	Elliptic	CFE	Elliptic
Level (χ)	0.0810	0.5259	1.0000	0.5223
Curvature (η, θ, μ)	1.4112	2.2863	0.5000	2.2926
Leisure support[a]	$(0.20, 0.90)$	$(0.20, 0.95)$	n.a.	$(0.15, 0.95)$
SSE marg. util.[b]	3.0177	1.6898	n.a.	0.6356

[a] The estimation of each function did not use the entire support of leisure because the marginal utility was sometimes infinity at $\ell = 0$. For this reason, we report the support of leisure used in each estimation.
[b] We report the sum of squared errors of the estimated fit of respective marginal utilities over 1,000 equally spaced points over the support of leisure. [c]These parameters, leisure support, and sum of squared errors are estimated to match the baseline CFE function.

Figure 4 plots to two respective sets of functions. Panel (a) shows that the elliptical utility function fits the CRRA marginal utility of leisure for $\ell < 0.2$, overestimates the CRRA marginal utility for $\ell \in (0.2, 0.6)$, and understimates the CRRA marginal utility for $\ell > 0.6$. Panel (b) of Figure 4 shows that the elliptical utility function fits the CFE marginal utility of leisure well for $\ell > 0.3$ but it overestimates the CFE marginal utility for $\ell < 0.3$. The fact that the elliptical utility function fits the CFE utility of leisure function better is also evidenced in the respective sum of squared error values in columns 2 and 4 of Table 1.

Figure 5 compares the implied Frisch elasticities of labor supply for each of the function

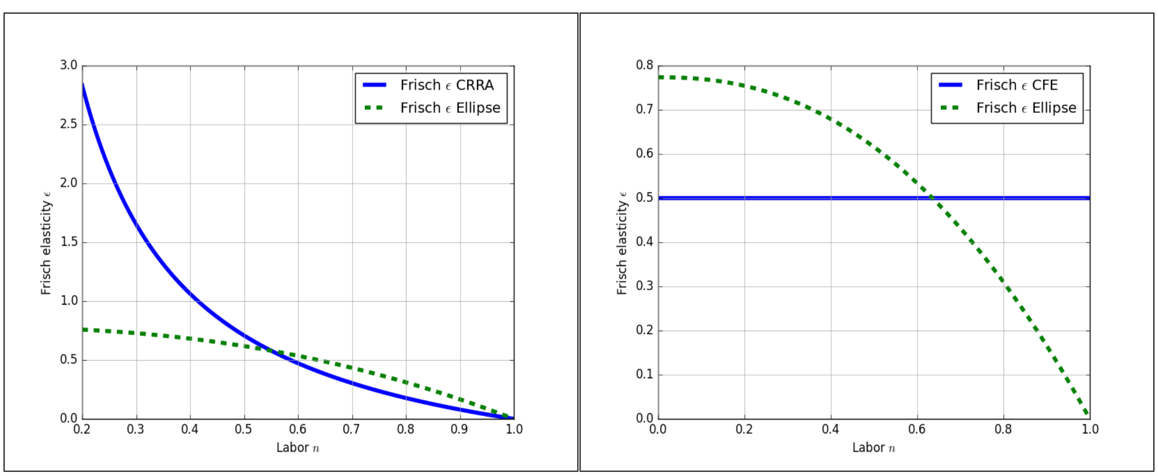
Figure 4: Fit of elliptical marginal utility to CRRA and CFE marginal utilities



(a) Elliptical to CRRA **(b)** Elliptical to CFE

specifications listed in Table 1. These elasticities are plotted on the x axis in terms of labor supply $n = 1 - \ell$ rather than leisure ℓ. Panel (a) is the most interesting comparison. It shows that, while both CRRA and elliptical utility specifications have decreasing Frisch elasticities as a function of labor supply, there is an important difference at labor supply equal to zero. The elliptical utility Frisch elasticity at $n = 0$ is finite, whereas the CRRA Frisch elasticity is infinity at $n = 0$. Having finite Frisch elasticity at zero labor supply might be a strength of the elliptic utility of leisure specification.

Figure 5: Comparison of Frisch elasticity of labor supply for estimated Elliptical, CRRA, and CFE utility of leisure functions



(a) Elliptical to CRRA **(b)** Elliptical to CFE

3.2 Comparison in life cycle model

In this section, we compare the effects of the different estimated utility of leisure functions from Table 1 on the household responses from a partial equilibrium life cycle model. We assume an individual lives for $S = 20$ periods, with age indexed by s. If we assume that an economically active lifetime is empirically 80 years, then each model period equals 4 years. We assume a constant net interest rate of $\bar{r} = 0.2155$ (or 21.55%). If each model period equals four years, then this interest rate is equivalent to an annual interest rate of 5 percent. We assume a constant average wage of $\bar{w} = 1$ and a discount factor of $\beta = 0.8227$ that corresponds to $1/(1+\bar{r})$. As in the rest of this paper, we assume that leisure is bounded above and below $\ell_s \in [0, 1]$ and that the one unit of time endowment is allocated between leisure and labor $\ell_s = 1 - n_s$. The constrained optimization problem for this agent over his lifetime is the following,

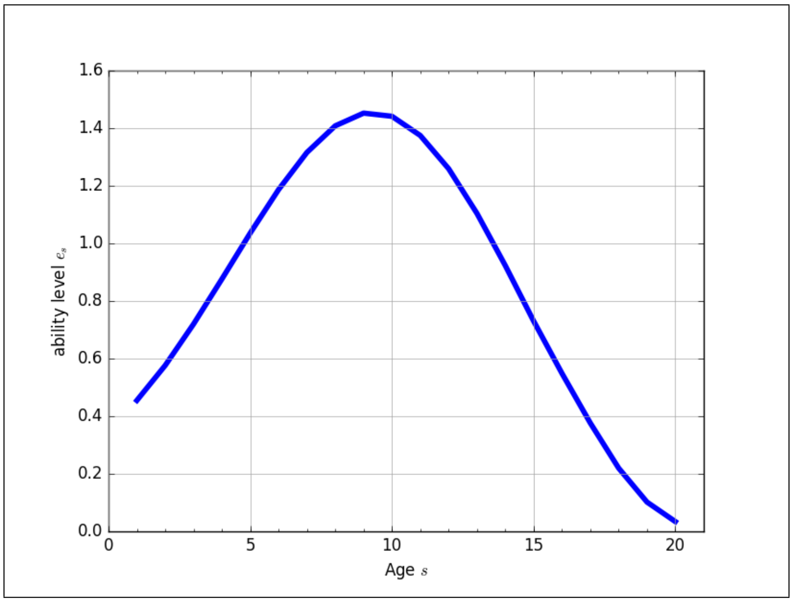
$$\begin{aligned} &\max_{c_s,n_s} \sum_{s=1}^{S} \beta^{s-1}\left[\frac{(c_s)^{1-\gamma}-1}{1-\gamma} + \chi v\big(1-n_s\big)\right] \\ &\text{s.t} \quad c_s + b_{s+1} \leq (1+\bar{r})b_s + \bar{w}e_s n_s \\ &\text{and} \quad 0 \leq n_s \leq 1 \end{aligned} \tag{3.4}$$

where b_{s+1} is the savings decision at age s and b_s is the amount of wealth that the agent enters the period with at age s.

The variable e_s in the last term of the budget constraint in (3.4) represents an ability multiplier on the average wage $\bar{w}$ that varies over the life cycle s. We take the values of this multiplier over the life cycle from DeBacker et al. (2015). They use IRS unadjusted gross income data that is not top coded and impute hourly wages from the Current Population Survey to get a measure of hourly wages over the life cycle. We then normalized this variable so that the average is equal to one so that the average wage over the life cycle is $\bar{w}$. Figure 6 shows this estimated life cycle earnings ability profile.

As described in Section 2.1, the utility of consumption function $u(c_s)$ has an Inada condition at $c_s = 0$ so the budget constraint in (3.4) binds with equality. Let $\lambda_{2,s}$ be the multiplier on the constraint that $n_s \geq 0$ and let $\lambda_{3,s}$ be the multiplier on the constraint that $n_s \leq 1$ as in the Lagrangian (2.3) in Section 2.1. Then the characterizing first order conditions for the individual described in the constrained maximization problem (3.4) are

Figure 6: Life cycle earnings profile e_s from DeBacker et al. (2015)



the following,

$$(c_s)^{-\gamma} = \beta(1+\bar{r})(c_{s+1})^{-\gamma} \quad \text{for} \quad 1 \leq s \leq S-1 \tag{3.5}$$

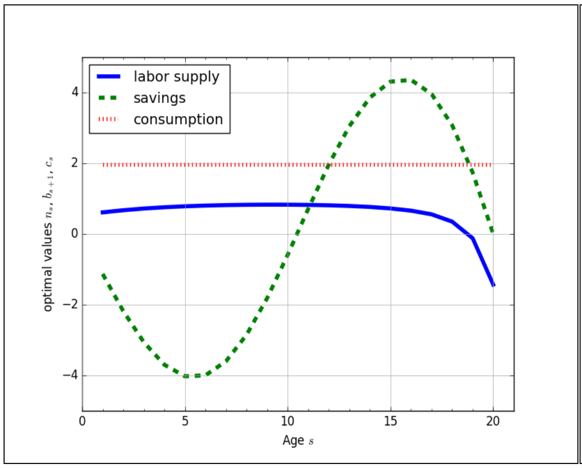
$$we_s(c_s)^{-\gamma} = \chi v'(1-n_s) - \lambda_{2,s} + \lambda_{3,s} \quad \text{for} \quad 1 \leq s \leq S \tag{3.6}$$

where the multipliers $\lambda_{2,s}$ and $\lambda_{3,s}$ satisfy the standard KKT complimentary slackness conditions.

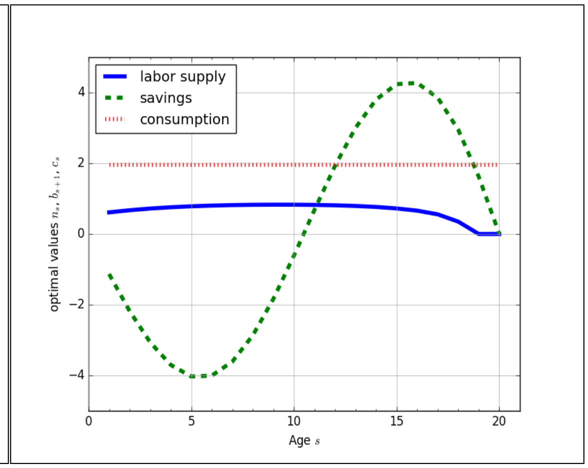
Figure 7 shows the optimal time paths over the life cycle for consumption c_s, labor supply n_s and savings b_{s+1} for the lifetime constrained maximization problem in (3.4) using the CRRA utility of leisure function $v(\ell)$ from (2.7) with values $\chi = 0.0810$ and $\eta = 1.4112$ from Table 1. Panel (c) shows the time paths of the endogenous variables for the fitted elliptical utility of leisure function (2.13) with values $\chi = 0.5259$ and $\mu = 2.2863$ from Table 1. Panel (d) compares the three time paths of labor supply n_s from panels (a), (b), and (c). We increased the average wage to $\bar{w} = 3$ in order to make the $n_s \geq 0$ constraint bind in old age.

In the unconstrained CRRA problem, shown in Panel (a) of Figure 7, the individual chooses negative labor in the last two periods of life. Including the constraints in the optimization problem sets labor supply equal to zero in those last two periods, as shown

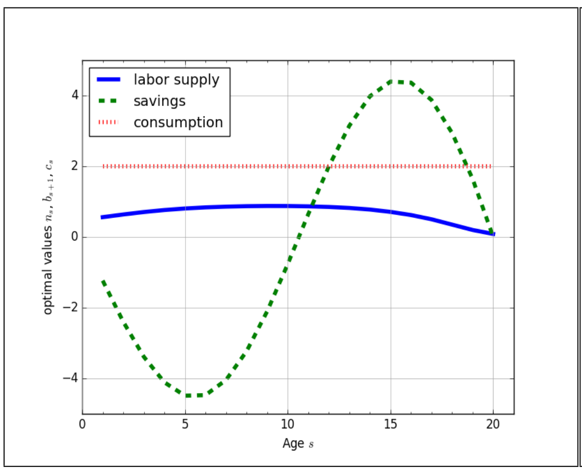
Figure 7: Life cycle solutions for CRRA versus Elliptical utility of leisure ($\bar{w} = 3.0$)



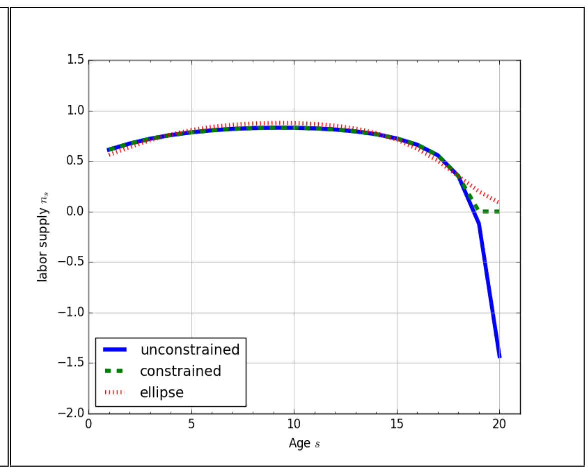
(a) CRRA unconstrained



(b) CRRA constrained



(c) Elliptical



(d) Compare labor supply n_s

in panel (b). The fitted elliptical utility of leisure function in panel (c) forces labor supply to approach zero in the last two periods of life, but never reach it. Panel (d) shows that the elliptical utility of leisure function fits the labor supply of the CRRA function very well over most of the life cycle, with only a small deviation from the constrained optimization problem in the last two periods.

Of further interest is the computation time required for each of the three specifications in Figure 7. The first column of Table 2 shows the computation times for panels (a), (b), and (c) from Figure 7. The unconstrained problem in panel (a) solves the fastest because it ignores the multipliers $\lambda_{2,s}$ and $\lambda_{3,s}$ and performs an unconstrained root finding algorithm on the vector of Euler errors. However, the constrained problem in panel (b) takes an order of magnitude longer because it must include the constraints. These constraints not only introduce new endogenous variables, but they also create kinks in the objective function due to the asymmetry of the complimentary slackness conditions. The elliptical utility specification in panel (c) takes only slightly longer to solve than the unconstrained CRRA specification. This is a major advantage of using elliptical utility of leisure with its two Inada conditions.

Table 2: Elliptical utility approximations to CRRA and CFE functions

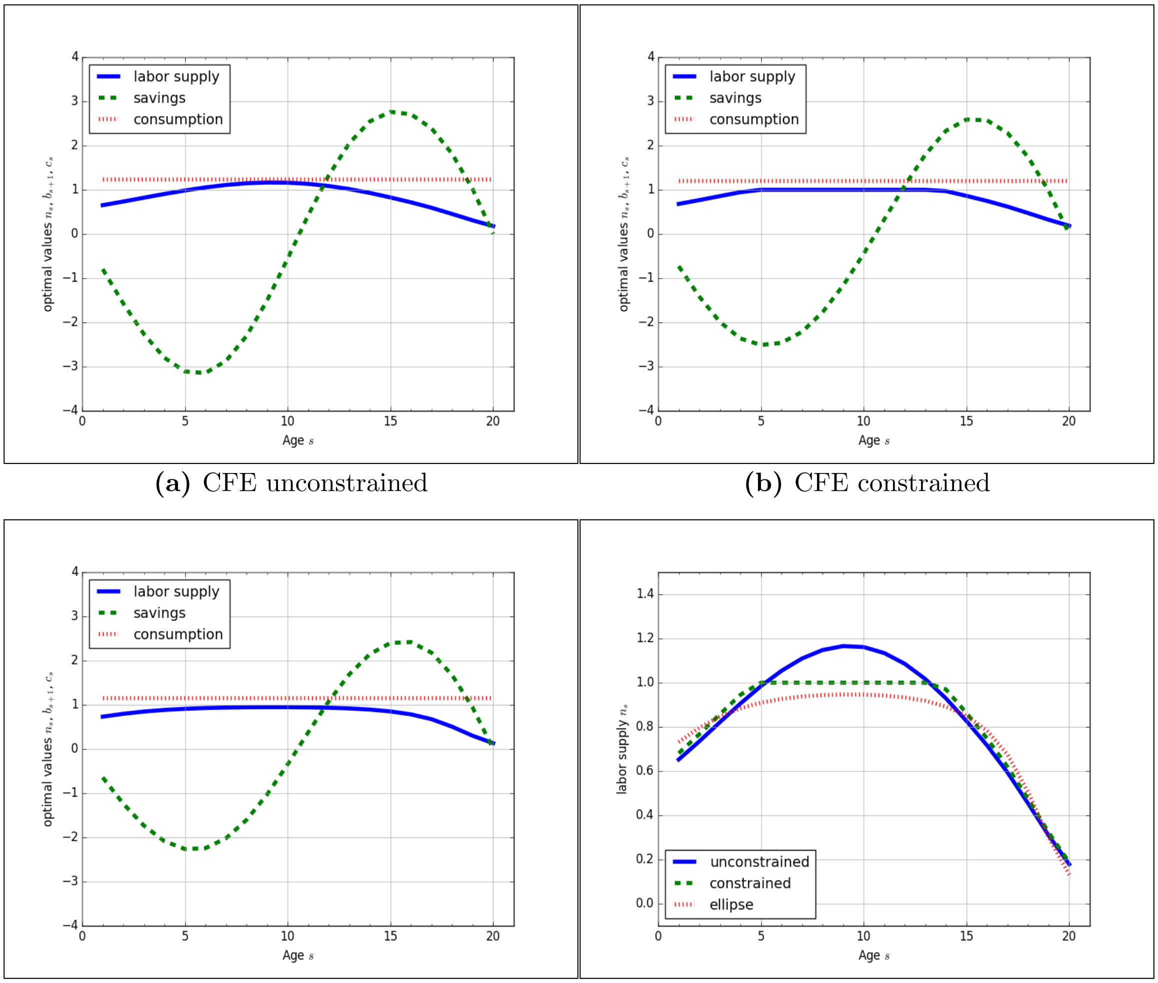
	Figure 7 CRRA vs. Ellipse (seconds)	Figure 8 CFE vs. Ellipse (seconds)
Unconstrained: panel (a)	0.0393	0.0364
Constrained: panel (b)	0.4136	0.1617
Ellipse: panel (c)	0.0509	0.0564

Figure 8 shows the optimal time paths over the life cycle for consumption c_s, labor supply n_s and savings b_{s+1} for the lifetime constrained maximization problem in (3.4) using the CFE utility of leisure function $v(\ell)$ from (2.8) with values $\chi = 1.0$ and $\eta = 0.5$ from Table 1. Panel (c) shows the time paths of the endogenous variables for the fitted elliptical utility of leisure function (2.13) with values $\chi = 0.5223$ and $\mu = 2.2926$ from Table 1. Panel (d) compares the three time paths of labor supply n_s from panels (a), (b), and (c). We increased the average wage to $\bar{w} = 3$ in order to make the $n_s \geq 0$ constraint bind in old age.

In the unconstrained CFE problem, shown in Panel (a) of Figure 8, the individual chooses labor supply greater than 1 in ages $6 \leq s \leq 13$. Including the constraints in the optimization problem sets labor supply equal to one in those middle eight periods, as shown in panel (b). The fitted elliptical utility of leisure function in panel (c) forces labor supply to approach one in those constrained periods, but never reach it. Panel (d) shows that the elliptical utility of leisure function fits the constrained labor supply of the CFE function fairly well.

Table 2 shows the same pattern of computation times in this CFE case as it does in the CRRA case. The second column of Table 2 shows the computation times for panels (a), (b), and (c) from Figure 8. The unconstrained problem in panel (a) again solves the fastest because it ignores the multipliers $\lambda_{2,s}$ and $\lambda_{3,s}$ and performs an unconstrained root

Figure 8: Life cycle solutions for CFE versus Elliptical utility of leisure ($\bar{w} = 1.5$)



(a) CFE unconstrained **(b)** CFE constrained

(c) Elliptical **(d)** Compare labor supply n_s

finding algorithm on the vector of Euler errors. Although the constrained problem in panel (b) takes longer, it is only about six times longer in the CFE case rather than more than 10 times longer in the CRRA case. And the elliptical utility specification in panel (c) takes only slightly longer to solve than the unconstrained CRRA specification.

In the life cycle model specified in (3.4), the elliptical utility of leisure function produces behavior that closely matches that of the CRRA and CFE functions. In addition, the elliptical utility of leisure function provides solutions that respect the constraints on leisure without a significant slowdown in computation time. Both of these benefits become extremely valuable in models with large degrees of heterogeneous agents.

3.3 Comparison in simple RBC Model

In this section, we compare the effects of the different estimated utility of leisure functions from Table 1 on the macroeconomic outcomes in a simple real business cycle (RBC) model with an endogenous labor decision.[13]

We assume that the representative household is infinitely lived and chooses consumption c_t, savings k_{t+1}, and labor supply n_t every period to maximize the following dynamic program,

$$\begin{aligned} V(k_t, z_t) = \max_{k_{t+1}, n_t} \; & \tfrac{1}{1-\gamma} c_t^{1-\gamma} + \chi v(1-n_t) + \beta E_{z_{t+1}|z_t}\Big[V(k_{t+1}, z_{t+1})\Big] \quad \forall t \\ \text{s.t} \quad & c_t + k_{t+1} \leq (1 + r_t - \delta)k_t + w_t n_t \\ \text{and} \quad & 0 \leq n_t \leq 1 \end{aligned} \tag{3.7}$$

where δ is the depreciation rate, z_t is the current period persistent value of labor augmenting technological productivity, and k_t is the holdings of capital in period t. If we assume a Cogg-Douglas production function with stochastic persistent labor augmenting technological productivity, the equilibrium in this model is characterized by the following seven equations.

$$z_t = \rho z_{t-1} + \varepsilon_t;\ \varepsilon_t \sim N(0, \sigma^2) \tag{3.8}$$

$$y_t = A k_t^{\alpha} (e^{z_t} n_t)^{1-\alpha} \tag{3.9}$$

$$r_t = \alpha \frac{y_t}{k_t} \tag{3.10}$$

$$w_t = (1-\alpha)\frac{y_t}{n_t} \tag{3.11}$$

$$c_t = (1 + r_t - \delta)k_t + w_t n_t - k_{t+1} \tag{3.12}$$

$$(c_t)^{-\gamma} = \beta E\Big[(1 + r_{t+1} - \delta)(c_{t-1})^{-\gamma}\Big] \tag{3.13}$$

$$w_t (c_t)^{-\gamma} = \chi v'(1-n_t) \tag{3.14}$$

Equation (3.8) is the exogenous law of motion for labor augmenting technological change z_t, where $|\rho| < 1$ is the persistence of z_t. (3.9) is the Cobb-Douglas production function of the representative firm. Equations (3.10) and (3.10) are the firm's respective first order

[13] Examples of dynamic stochastic general equilibrium models that use the additively separable utility of consumption and leisure specifications described in this paper are Greenwood et al. (1988), King et al. (1988), and Jaimovich and Rebelo (2009).

conditions for capital and labor. The budget constraint is (3.12). And equations (3.13) and (3.14) are the two Euler equations associated with the representative household's problem in (3.7).

We solve and simulate this simple RBC model using the software package Dynare and use a quadratic approximation of the policy function which maps k_t, z_t and ε_t into k_{t+1}. We use four versions of utility of leisure function $v(1-n_t)$ corresponding to the CRRA, CFE, and two fitted elliptical specifications in Table 1. For our simulations we use the following parameter values which are appropriate for a quarterly model: $\alpha = 0.33$, $\beta = 0.995$, $\gamma = 2.2$, $\delta = 0.02$, $\rho = 0.9$, and $\sigma = 0.02$. Table 3 reports various behavioral moments and percent differences between elliptical utility and either the CRRA or CFE specification. We compare steady-state values, standard deviations, correlations, and autocorrelations of macroeconomic variables from the model.

The top panel of Table 3 reports the percent deviation of the steady-state moments in the RBC model for either the CRRA or CFE specifications from the elliptical utility steady-state moments. In both cases, the elliptical utility moments are almost all less than one percent away from the CRRA or CFE moments. Only the log of investment in the CRRA case (-1.61% difference) and the logs of consumption and investment in the CFE case (1.30% and -2.49% differences, respectively). We interpret this as being a very good fit in terms of steady-state values. Note in the last line of this panel that the CRRA utility specification, which is fitted to the baseline CFE specification, has steady-state moments in the RBC model that fit the CFE moments very closely.

In the second panel of Table 3, we compare the standard deviations of macroeconomic variables across specifications. The elliptical utility differences in standard deviation moments are all less than 3.6% away from their counterparts with the glaring exception of labor supply. The elliptical utility labor supply has a 43% higher standard deviation than the CRRA labor supply and a 29% lower standard deviation than the CFE labor supply. However, the last line of the second panel shows that the elliptical utility standard deviations are closer to both CRRA and CFE than are the CRRA standard deviations to CFE standard deviations. This same pattern holds for the correlations with the log of y_t in the third panel. The autocorrelation moments in the fourth panel are all very close to each other.

Table 3: Moment comparisons from simple RBC model

	$\ln(y_t)$	$\ln(c_t)$	$\ln(i_t)$	$\ln(k_t)$	$\ln(w_t)$	r_t	z_t	n_t
Steady-state values								
CRRA	0.9733	0.6673	-0.3636	3.5536	0.8704	0.0250	0.0000	0.7426
Ellipse	0.9796	0.6736	-0.3578	3.5599	0.8704	0.0250	0.0000	0.7473
% difference	**0.64%**	**0.94%**	**-1.61%**	**0.18%**	**0.00%**	**0.00%**	**n.a.**[a]	**0.63%**
CFE	0.9727	0.6668	-0.3650	3.5531	0.8705	0.0250	0.0000	0.7422
Ellipse	0.9814	0.6755	-0.3560	3.5618	0.8704	0.0250	0.0000	0.7487
% difference	**0.90%**	**1.30%**	**-2.49%**	**0.24%**	**0.00%**	**0.00%**	**n.a.**[a]	**0.88%**
% diff. (CRRA - CFE)	**0.06%**	**0.09%**	**-0.38%**	**0.02%**	**0.00%**	**0.00%**	**n.a.**[a]	**0.06%**
Standard deviations								
CRRA	0.0393	0.0190	0.1113	0.0430	0.0384	0.0009	0.0459	0.0038
Ellipse	0.0400	0.0186	0.1153	0.0432	0.0384	0.0009	0.0459	0.0054
% difference	**1.77%**	**-2.10%**	**3.60%**	**0.45%**	**0.07%**	**3.31%**	**0.00%**	**43.24%**
CFE	0.0406	0.0183	0.1187	0.0434	0.0385	0.0010	0.0459	0.0067
Ellipse	0.0399	0.0186	0.1152	0.0432	0.0384	0.0009	0.0459	0.0054
% difference	**-1.67%**	**1.41%**	**-2.96%**	**-0.58%**	**-0.06%**	**-2.72%**	**0.00%**	**-19.28%**
% diff. (CRRA - CFE)	**-3.35%**	**3.55%**	**-6.27%**	**-1.02%**	**-0.13%**	**-5.78%**	**0.00%**	**-43.42%**
Correlations with $\ln(y_t)$								
CRRA	1.0000	0.7974	0.9579	0.6534	0.9916	0.3524	0.9668	0.2306
Ellipse	1.0000	0.7855	0.9605	0.6373	0.9836	0.3776	0.9754	0.2965
% difference	**0.00%**	**-1.49%**	**0.27%**	**-2.47%**	**-0.81%**	**7.15%**	**0.88%**	**28.59%**
CFE	1.0000	0.7764	0.9624	0.6250	0.9754	0.3966	0.9810	0.3443
Ellipse	1.0000	0.7857	0.9604	0.6376	0.9838	0.3771	0.9752	0.2953
% difference	**0.00%**	**1.21%**	**-0.21%**	**2.01%**	**0.85%**	**-4.91%**	**-0.59%**	**-14.23%**
% diff. (CRRA - CFE)	**0.00%**	**2.71%**	**-0.47%**	**4.54%**	**1.66%**	**-11.14%**	**-1.45%**	**-33.02%**
Autocorrelations								
CRRA	0.9275	0.9924	0.8971	0.9988	0.9429	0.9091	0.9000	0.9231
Ellipse	0.9239	0.9921	0.8947	0.9988	0.9457	0.9074	0.9000	0.9167
% difference	**-0.38%**	**-0.03%**	**-0.27%**	**-0.01%**	**0.30%**	**-0.19%**	**0.00%**	**-0.69%**
CFE	0.9212	0.9919	0.8928	0.9987	0.9481	0.9060	0.9000	0.9121
Ellipse	0.9240	0.9921	0.8947	0.9988	0.9457	0.9074	0.9000	0.9169
% difference	**0.30%**	**0.02%**	**0.21%**	**0.01%**	**-0.25%**	**0.16%**	**0.00%**	**0.53%**
% diff. (CRRA - CFE)	**0.68%**	**0.05%**	**0.48%**	**0.02%**	**-0.55%**	**0.34%**	**0.00%**	**1.21%**

[a] The percent differences here are very large because the levels are so close to zero that small changes create large percent differences.

4 Conclusion

This paper characterizes how to use a generalized function for an ellipse as the functional form for the utility of leisure in the economic model of household decision making with additively separable period utility of leisure. The primary motivation for considering using this elliptical functional form is its inclusion of two Inada conditions for both the upper-bound and lower-bound constraints on labor supply. These two properties are not simultaneously possessed by either of the main functional forms currently used for the utility of leisure—constant relative risk aversion (CRRA) utility of leisure and constant Frisch elasticity (CFE) utility of leisure.

In the context of a life cycle model, we show that the elliptical utility of leisure function successfully bounds optimal labor supply away from the corner solutions. We also show that the computation time of the elliptical utility specification is nearly as fast as the unconstrained solution for both CRRA and CFE. The computation time for CRRA and CFE, accounting for both the upper and lower bound constraints on labor supply, takes six times longer than elliptical utility for CRRA and three times longer than elliptical utility for CFE. This speedup is a significant contribution for models with large degrees of heterogeneity among agents.

We further show that the resulting economic outcomes of the elliptical utility of leisure function closely approximate the economics outcomes of the CRRA and CFE functions. In a comparison of moments from a simple real business cycle model, the elliptical utility specification has very similar results to that of the CRRA and CFE specifications.

With decreasing Frisch elasticity of labor supply as labor increases, computational speedup, Inada conditions on both labor constraints, and the ability to closely match the outcomes of CRRA and CFE specifications, the elliptic utility of leisure function is a valid alternative for period utility specifications. With its computational efficiency, this functional form should also make feasible the analysis and simulation using global solution methods of larger-scale DSGE models and heterogeneous agent models with occasionally binding constraints.

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APPENDIX

A-1 Derivation of Frisch Elasticities

The Frisch elasticity of labor supply is defined as the wage elasticity of labor supply holding the marginal utility of wealth constant. For this reason, the Frisch elasticity is a compensated elasticity that captures the pure substitution effect on labor supply from a change in the wage. Let ε represent the Frisch elasticity,

$$\varepsilon \equiv \frac{dn^*}{dw}\frac{w}{n}\bigg|_{\bar{\lambda}_1} \tag{A.1.1}$$

where n^* is the optimal labor supply.

The Lagrangian for this constrained maximization problem is given in (2.3). For the purpose of computing the Frisch elasticity, we assume an interior solution for leisure (labor) such that $\lambda_2, \lambda_3 = 0$.

$$\mathcal{L} = u(c) + \chi v(\ell) + \lambda_1\Big(w[1-\ell] + y - c\Big) \tag{A.1.2}$$

Because we assume that $u'(0) = \infty$ and $u'(c) > 0$ for all $c > 0$, we know that the solution to this problem is characterized by $\lambda_1 > 0$ and $c = y + w(1-\ell) = y + wn$. The variable λ_1 is the multiplier on the budget constraint and represents the marginal utility of an extra unit of wealth. The two first order conditions that characterize optimal c^* and n^* for the problem in (A.1.2), excluding the KKT condition for λ_1, are the following.

$$u'(c) = \lambda_1 \tag{A.1.3}$$

$$\frac{\chi v'(\ell)}{w} = \lambda_1 \tag{A.1.4}$$

Because the Frisch elasticity of labor assumes that λ_1 is constant, we can ignore (A.1.3) and derive this elasticity by totally differentiating (A.1.4), solving for $\frac{dn^*}{dw}$, and then multiplying by w/n.

$$\begin{aligned} &\frac{\chi v'(\ell)}{w} = \lambda_1 \quad \Rightarrow \quad -\frac{\chi}{w}v''(\ell)dn - \frac{\chi}{w^2}v'(\ell)dw = 0 \\ &\Rightarrow \quad wv''(\ell)dn = -v'(\ell)dw \quad \Rightarrow \quad \frac{dn}{dw} = -\frac{v'(\ell)}{wv''(\ell)} \end{aligned} \tag{A.1.5}$$

Multiplying (A.1.5) by w/n gives the Frisch elasticity, holding λ_1 constant.

$$\varepsilon = -\frac{v'(\ell)}{nv''(\ell)} \tag{A.1.6}$$

The following are the expressions for the respective Frisch elasticities of labor substitution for the CRRA, CFE, and elliptical utility of leisure functions from (2.7), (2.8), and (2.13).

$$\text{(CRRA)} \quad v(\ell) = \frac{\ell^{1-\eta}}{1-\eta}: \qquad \varepsilon = \frac{1}{\eta}\left(\frac{1-n}{n}\right) \tag{2.14}$$

$$\text{(CFE)} \quad v(\ell) = -\frac{(1-\ell)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}: \qquad \varepsilon = \theta \tag{2.15}$$

$$\text{(Ellipse)} \quad v(\ell) = \Big(1 - [1-\ell]^{\mu}\Big)^{\frac{1}{\mu}}: \quad \varepsilon = \frac{1-n^{\mu}}{\mu - 1} \tag{2.16}$$

For both CRRA (2.14) and elliptical utility (2.16), the Frisch elasticity of labor supply is decreasing with labor supply for all n. All three elasticities are positive $\varepsilon \geq 0$ for all allowed values of their respective inputs. CRRA and elliptical utility elasticities approach zero as n goes to 1 $\lim_{n\to 1}\varepsilon = 0$, although the CRRA elasticity goes to zero faster. On the other hand, the limit of the CRRA elasticity as n goes to zero is infinity $\lim_{n\to 0}\varepsilon_{crra} = \infty$, whereas the limit of the elliptical utility elasticity goes to a finite positive scalar as n goes to its lower bound $\lim_{n\to 0}\varepsilon_{ellip} = \frac{1}{\mu-1}$.

We provide here the expressions for the marginal utility of leisure $v'(\ell)$ and the derivative of the marginal utility of leisure $v''(\ell)$ as required by (A.1.6) for each of the three functional forms used to calculate (2.14), (2.15), and (2.16).

$$\text{(CRRA)} \quad v'(\ell) = \ell^{-\eta}, \tag{3.1}$$

$$v''(\ell) = -\eta\ell^{\eta-1} \tag{A.1.7}$$

$$\text{(CFE)} \quad v'(\ell) = (1-\ell)^{\frac{1}{\theta}}, \tag{3.2}$$

$$v''(\ell) = -\frac{1}{\theta}(1-\ell)^{\frac{1}{\theta}-1} \tag{A.1.8}$$

$$\text{(Ellipse)} \quad v'(\ell) = (1-\ell)^{\mu-1}\Big(1-[1-\ell]^{\mu}\Big)^{\frac{1-\mu}{\mu}}, \tag{3.3}$$

$$v''(\ell) = (1-\mu)(1-\ell)^{\mu-2}\Big(1-[1-\ell]^{\mu}\Big)^{\frac{1-\mu}{\mu}}\left[1+\frac{(1-\ell)^{\mu}}{1-(1-\ell)^{\mu}}\right] \tag{A.1.9}$$