A Fast Robust Optimization Methodology Based on Polynomial Chaos and Evolutionary Algorithm for Inverse Problems

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This paper explores the potential of polynomial chaos in robust designs of inverse problems. A fast numerical methodology based on combinations of polynomial chaos expansion and evolutionary algorithm is reported in this study. With the proposed methodology, polynomial chaos expansion is used as a stochastic response surface model for efficient computations of the expectancy metric of the objective function. Additional enhancements, such as the introduction of a new methodology for expected fitness assignment and probability feasibility model, a novel driving mechanism to bias the next iterations to search for both global and robust optimal solutions, are introduced. Numerical results on two case studies are reported to illustrate the feasibility and merits of the present work.

***Index Terms*—Evolutionary algorithm, polynomial chaos expansion, robust design, robust optimization.**

I. Introduction

In practical engineering designs, imprecision and uncertainty are often inevitable and unavoidable. For example, it is usually very difficult to produce a product according to exact design specifications. Operating conditions will also vary with time. Hence, if the optimal solution is very sensitive to small variations of the designed parameters, it is possible that slight deviations in the optimized variables could result in either significant performance degradation or even infeasible solutions. The preferred design is thus not the global optimal solution in terms of objective function, it should be the one with good performance in terms of objective function and robustness against slight perturbations. Consequently, it is important to develop new robust optimal models and techniques in engineering design studies to address inevitable and unavoidable uncertainties [1]–[4]. So far, in robust optimal design studies, the problem is modeled as either an optimal problem using robust performance as the sole objective, or a multi-objective one employing both the objective function and robust performance as the objectives to give more freedom to the decision makers when formulating the trade-off between quality and robustness [1]. In this paper, the robust design problem is modeled as a single objective optimal one using robust performance as the sole objective.

Robustness means some degree of insensitivity to small perturbations in either design or environmental variables. It is obviously essential to define and incorporate the concept of robustness in robust designs. In this regard, the mean and standard deviation are commonly used as the gauge to assess the robustness of objective functions in literature. This means that a number of points in the neighborhood of a given solution are sampled and their function values are used to approximate the robust metric as there is no close-form solution for such evaluation in real-world problems. The computational burden for a robust optimizer is thus significantly higher than that for its global counterpart. With respect to the sampling mechanism, the Monte-Carlo simulation and its variants are generally used [5], [6]. Although simple to implement, the Monte-Carlo type approach is computational intensive since a large amount of sampling points are required. This situation is further exacerbated by applications of high fidelity numerical methods in inverse problems. To address such deficiency, a fast robust optimizer framed on evolutionary and polynomial chaos is proposed.

II. A Fast Robust Methodology

A. Robust Design Under Uncertainty

Uncertainties in typical engineering design problems can be classified into four categories [1]. This study only looks at the second and fourth types of uncertainties in inverse problems, i.e., the *production tolerances and actuator imprecision*, and the *feasibility uncertainties*. For an inverse problem involving such uncertainties, it is mathematically modeled as

$$\begin{array}{ll} \text{Minimize} & f(x,\delta) \\ \text{Subject to} & g_i(x,\delta)\le 0 \quad (i=1,2,\ldots,k) \end{array} \tag{1}$$

where, x is the vector of the design (decision) parameters (variables), δ is the vector of uncertainty variables.

To measure the robustness of a given solution, the mean or expectancy metric and the worst case scenario are generally used, respectively, for the objective and constraint functions as

$$f_{\exp}(x)=\int f(x,\delta)p(\delta)d\delta=\sum_{i=1}^{N_s} f(x_i)\Big/N_s \tag{2}$$

$$(\mathrm{g}_i)_{\mathrm{worst}}(x)=\sup_{\delta}\mathrm{g}_i(x,\delta) \tag{3}$$

where, $p(\delta)$ is the probability function of the uncertainties or disturbances, N_s is the number of sampling points generated in a small neighborhood of the specific point x.

B. Polynomial Chaos and Its Application

To address the deficiency of computational inefficiency in using Monte-Carlo type approaches to determine the expectancy metric of (2), the polynomial chaos is proposed as a stochastic

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response of the objective function. Polynomial chaos is virtually a polynomial representation of a Gaussian random process originally proposed in [7], and it is further generalized to cover other random processes in [8]. For sampling points with a size of M, the error of polynomial chaos approximation for a stochastic process decreases with e^M; whereas those of Monte-Carlo and its variant Latin Hypercube sampling are inversely proportional to $\sqrt{M}$ and M, respectively. In other words, polynomial chaos is an effective and efficient means for solving problems involving random variables and processes [9].

For the objective function $f(x,\delta)$, its polynomial chaos approximation, in a finite order of P, is [9]

$$f^P(x,\delta)=\sum_{i=1}^{I}\hat{f}_i(x)\phi_i(\delta), \quad I=\binom{N+P}{N} \tag{4}$$

where $\phi_i(\delta)$ is a set of N-variate normalized orthogonal polynomials of a total degree up to P.

An exclusive characteristic of polynomial chaos is that the polynomial bases are orthogonal in an inner product sense with respect to a real positive measure, ϕ, which is the probability function of the uncertainties of $p(\delta)$. This key feature renders the algorithm so designed to have the capability to develop fast and efficient numerical procedures. Furthermore,

$$E\left[\phi_i(\delta)\phi_j(\delta)\right]=\int \phi_i(\delta)\phi_j(\delta)p(\delta)d\delta=\delta_{mn} \tag{5}$$

where δ_{mn} is the Kronecker delta function.

The expansion coefficients can be determined from

$$\hat{f}_i(x)=E\left[f(x,\delta)\phi_i(\delta)\right]=\int f(x,\delta)\phi_i(\delta)p(\delta)d\delta. \tag{6}$$

It is clear that the polynomial chaos can be regarded as a stochastic response of the objective function. Moreover, it is evident, from the exclusive feature of polynomial chaos, that

$$f_{\exp}(x)\approx f_{\exp}^P(x)=\int\left(\sum_{i=1}^{I}\hat{f}_i(x)\phi_i(\delta)\right)p(\delta)d\delta=\hat{f}_1(x). \tag{7}$$

For the present application, it suffices for one to compute only the coefficient of the first term of the polynomial chaos expansions.

C. Handling Constraints Using Feasibility Probability

Under the worst case scenario of (3), the robust solution of a constraint function g_{i} is the maximal value of g_{i} in a user-defined neighborhood of the design point x. Such solution philosophy is however rather conservative. This is because the objective and constraint functions are generally conflicting and incompatible, and the usage of the worst case solution philosophy may produce robust solutions with very poor objective function performance to render the resulting design becoming useless in extreme cases [1]. In this regard, it would be better to consider robustness measures based on probability models. For this purpose, a statistical approach, the chance constraints programming [10], is used to address the inequality constraints probabilistically. Mathematically, the statistical model for the i^{th} constraint function g_{i} is given as

$$\text{Probability}\left(\mathrm{g}_i(x,\delta)\le 0\right)\ge P_0^i \tag{8}$$

where P_0^i is the confidence probability.

To determine the probability of (8), the Latin Hyper-cube sampling technique [6] is used.

D. Performance Criterion Biasing Subsequent Searches

In existing robust oriented algorithms, the robust performance, rather than the original objective function, is used as the only driving force in selecting/updating the candidate solutions in iterative procedures. However, it is well known that only local optimal solutions and boundary solutions have the potential to be a robust one. In this regard, the original objective function should be selected as the driving force. However, if the objective function is solely used as the driving force, the global "robust" optimal solution may not be found since the information of the robust parameters is not being used to guide the searches. In this regard, the robust performance parameter and the objective function will be alternatively selected as the driving forces of a robust optimizer to equip it with the ability to find both the global and the "globally" robust optimal solutions. To achieve this goal, a new mechanism and procedure to transform different performance criteria as the driving forces is proposed. More concretely, the iterative procedures will start, from phase one, to use the objective function as the driving force to guide the searching process. These iterative procedures will jump to phase two when the number of consecutive searches without improvements in the so far searched best objective function exceeds a predefined value. In phase two, the expected fitness function will be used as the biasing force to evaluate and select the candidate solutions for the next iterative cycles, and the searching procedures will terminate when the number of consecutive searches without improvement in the so far searched best expected fitness function exceeds a predefined value. The two phases will be alternated until a "global" stop criterion is satisfied.

Obviously, both the predefined number and the "global" stop criterion affect the algorithm performance. If the predefined number is too large or the "global" stop criterion is too strict, the algorithm will slowly converge to a final solution. Conversely, the algorithm will prematurely terminate. Based on rules of thumb, the predefined value is set to 20. The global stop criterion is that the algorithm will be stopped once the number for consecutive phase changes without improvements on so far best robust and global optimal solutions reaches 2.

E. A Particle Swarm Optimization Based Robust Algorithm

It should be pointed out that the previous robust methodologies and approaches can be readily integrated into any evolutionary algorithm. Nevertheless, the particle swarm optimization algorithm (PSO) [11] is used in this paper.

As explained previously, the robust optimal solutions of an optimal design problem are either those of the local/global optima of the objective function or those distributed on the boundaries of the feasible parameter space. It is unnecessary to compute the robust performances of all intermediate solutions. In other words, there should be some "intelligence" and "memory" for an ideal robust optimizer to judge the status of an intermediate solution so as to assign robust performances only to those promising solutions in order to reduce unnecessary computational burdens. However, it is not easy to check if an intermediate solution is a local/global optimal one during the optimization process. To incorporate the previous ideas into the design

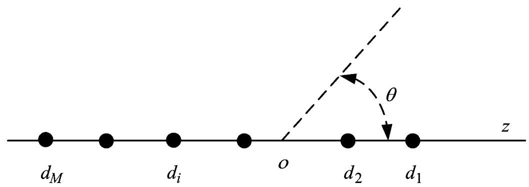

Fig. 1. The configuration of a M-element array placed on the z-axis.

of a robust oriented PSO algorithm while excluding the difficulty of evaluating the states of intermediate solutions, and to further enhance the computational efficiency of the optimizer, the assignment procedure for the expected fitness and feasibility probability of the proposed algorithm is activated only to every newly searched personal best solution and boundary solutions of the constraint functions.

III. Numerical Validations

A. Case Study One

The first case study is to seek the robust solution of a high frequency inverse problem. The problem is to reformulate a general design problem of antenna arrays [12], as shown in Fig. 1, in a robust sense, to produce a beam pattern of the desired shape. The desired pattern is defined as

$$\begin{aligned} F_d(\cos\theta) &= \begin{cases} \text{cosecant}(\cos\theta) & (0.1 \le \cos\theta \le 0.5) \\ 0 & (\text{elsewhere}) \end{cases} \\ \text{MSLL}_d &= -35\ \text{dB}\ (-1 \le \cos\theta \le -0.3) \\ \text{MSLL}_d &= -25\ \text{dB}\ (-0.3 < \cos\theta \le 0) \\ \text{MSLL}_d &= -40\ \text{dB}\ (0.6 \le \cos\theta \le 1) \end{aligned} \tag{9}$$

where MSLL is the maximum sidelobe level.

To produce a radiation pattern which is as close as possible to this desired pattern, one will optimize a completely non-uniform antenna array with the minimal number of elements with respect to the following objective function:

$$\begin{aligned} &\min f \\ &= \sqrt{\sum_{i=1}^{N}\left[F_d^{\text{norm}}(\theta_i) - F_{\text{designed}}^{\text{norm}}(\theta_i)\right]^2 \Big/ \sum_{i=1}^{N}[F_d^{\text{norm}}(\theta_i)]^2} \\ &\text{s.t} \begin{cases} MSLL_{\text{designed}} \le -35\ \text{dB}(-1 \le \cos\theta \le -0.3) \\ MSLL_{\text{designed}} \le -25\ \text{dB}(-0.3 < \cos\theta \le 0) \\ MSLL_{\text{designed}} \le -40\ \text{dB}(0.6 \le \cos\theta \le 1) \\ |d_i - d_{i-1}| \ge 0.4\lambda \end{cases} \end{aligned} \tag{10}$$

where, $F_d^{\text{norm}}(\theta_i)$ is the value of the normalized desired radiation pattern at the sampling point θ_i, $F_{\text{designed}}^{\text{norm}}(\theta_i)$ is the value of the normalized radiation pattern produced by a designed array of M elements, N is the number of total sampling or discretizing points.

For an antenna array of Fig. 1, its array factor is given by

$$F_{\text{M}}(\theta) = \sum_{i=1}^{M} R_i e^{jkd_i\cos\theta} \tag{11}$$

where, R_{i} is the complex excitation coefficient of the i^{th} element located at $z = d_{\text{i}}$ along the linear array direction z, and $k = (2\pi/\lambda)$ is the spatial wavenumber.

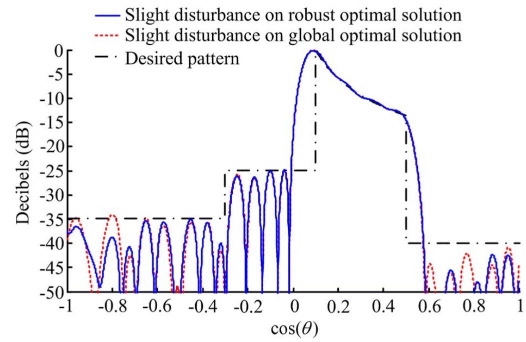

Fig. 2. Field pattern degradations of the global and robust solutions.

In numerical experiments, P_0^i of (8) is set to 0.9, the number of the antenna array is 26 and thus the total design variables are 78 [real R_{i}, imaginary R_{i} and d_{i} $(\text{i} = 1, \ldots 26)$]. Also, a Gaussian random process is assumed for the uncertainties in the design variables. To facilitate implementations of polynomial chaos, the distance (radial) variable in the feasible space is used as the only uncertain variable. For performance comparisons, PSO is integrated with, respectively, the proposed (Proposed) and the Latin Hypercube sampling technique (LH) to compute the robust performance as well as finding both the global and robust optimal solutions. After 58 779 iterations, the Proposed finds both robust and global optimal solutions in a single run. To give a reasonable comparison, the number of sampling points for Latin Hypercube sampling technique is determined by incrementally increasing the number of sampling points of the polynomial chaos until the errors for the expected fitness computation of 100 observing points reaches the same level as those of the proposed one. Under such condition, it is observed that it requires 79 665 iterations using LH to arrive at the same optimal solutions. The expected fitness values for the robust and global optimal solutions are, respectively, 1.831620 and 1.834763. The objective function values for the robust and global optimal solutions are, respectively, 1.805495 and 1.800775. Moreover, the global optimal solution searched by the proposed algorithm is validated using a general purpose PSO algorithm which requires 47 664 iterations. To study violation probabilities for the constrained conditions of the two optimal solutions with uncertainties, small variations of up to 0.2% on the decision variables following a Gaussian random process are randomly perturbed to each optimal solution 100 times to yield 200 new solutions. It is found that the violation probabilities for the global and robust optimal solutions are, respectively, 60% and 20%. Fig. 2 depicts intuitively the field pattern degradations when the two optimal solutions are disturbed by small tolerances.

From these numerical results, it is observed that:

1) Compared with the Latin Hypercube sampling technique, the application of the polynomial chaos expansion for a random process can efficiently reduce the sampling points without compromising on the precision of robust performance computations, resulting in a significant reduction in the total iteration numbers, from 79 665 to 58 779 as in this case study.
2) The robustness of the solutions of the proposed one is much stronger than that of the final solutions obtained using traditional global optimal techniques, the infeasibility probability when small perturbations are applied to the opti-

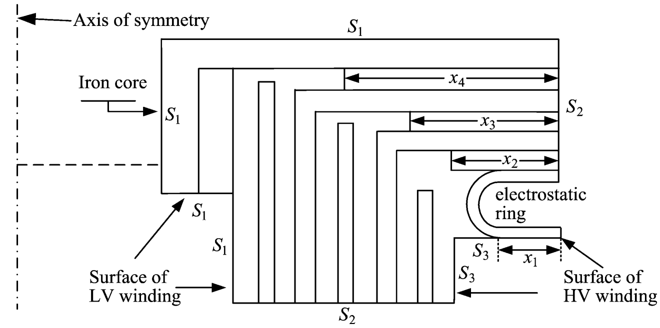

Fig. 3. Schematic diagram of the end region of the studied transformer.

TABLE I
PERFORMANCE COMPARISON ON EXAMPLE 2

Algorithm	Iterative No.	E_{max} (pu)	$(E_{max})_{\exp}$ (pu)
LH(global)	6524	0.90	0.870
LH(robust)		0.91	0.858
Proposed (global)	4391	0.90	0.870
Proposed (robust)		0.91	0.858

mized designs using the Proposed is decreased from 60% to 20% in this case study.

3) The Proposed can be used readily and efficiently to search, in a single run, both global and robust optimal solutions using any existing robust optimizer.

B. Case Study Two

The geometric optimal design of the end region of a power transformer is used as an application of the proposed methodology to study low frequency inverse problems. The aim is to reduce the maximum end region electric fields $(E_{\max})$ in order to avoid the transformer suffering from unnecessary flashovers, and thus the objective is formulated as

$$\begin{aligned} &\min && E_{\max}(x) \\ &\text{subject to} && x_{i,b} \le x_i \le x_{i,a} (i = 1, 2, \cdots, 4) \\ &&& x_2 + \Delta_{\min} \le x_3 \le x_4 - \Delta_{\min} \end{aligned} \tag{12}$$

where $E_{\max}$ is the maximum electric field in the end region; x_1 is the width of the electrostatic rings of the winding; $\Delta_{\min}$ is a predefined constant; x_{ib} and x_{ia} are, respectively, the lower and upper limits of x_{i}; x_i (i = 2, 3, 4) is the width of spacer i (Fig. 3).

The end fields of the transformer are simplified as a 2-D planar field, which is computed using finite element method and the boundary value problem is

$$\varepsilon \frac{\partial^2 \varphi}{\partial x^2} + \varepsilon \frac{\partial^2 \varphi}{\partial y^2} = 0, \quad \varphi|_{S_1=0}, \varphi|_{S_3=1}, \quad \partial\varphi/\partial n|_{S_2} = 0. \tag{13}$$

In the numerical experiment, an adaptive meshing technique based on Delaunay criterion is used to consider the variation of dimensional parameters. Also, the parameters and implemental conditions are the same as those of case study one except that a Gamma distribution random process is assumed for the uncertainty. For performance comparisons, the PSO is integrated with, respectively, the proposed (Proposed) and the Latin Hypercube sampling technique (LH) to compute the robust performance as well as finding both the global and robust optimal solutions of this ill-conditioned problem. Also, to give a reasonable comparison, the number of sampling points for the Latin Hypercube sampling technique is determined by incrementally increasing the number of sampling points of the polynomial chaos until the errors for the expected fitness computation of 50 observing points reaches the same level as those of the proposed one. Under such condition, typical numerical results reveal that the final solutions of the two approaches are nearly the same, as given in Table I; and the iterative numbers are 4391 and 6524, respectively for the Proposed and LH. The optimized design parameters of x_i (i = 1, 2, 3, 4) in units of millimeter are sequentially 40.0, 40.0, 85.0, 112.2; 39.8, 40.2, 84.8, 113.1; for the global and robust optimal solutions. Moreover, some post-processing experiments are conducted to evaluate the violation probability of the two optimal solutions when small variations of up to 0.9% on the design variables with a Gamma distribution are randomly introduced. It is found that the violation probabilities for the global and robust optimal solutions are, respectively, 70% and 30%. From these numerical results, the same conclusions as those in case study one are observed for a Gamma distribution random process.

IV. CONCLUSION

A fast robust optimizer framed on particle swarm optimization and polynomial chaos approximation is proposed. The numerical experiments have demonstrated that 1) the proposed methodology is robust for different random process; 2) the proposed methodology can efficiently find the robust optimal solutions without any compromise on the precision of the final optimal solution; 3) the exclusive characteristic of the proposed methodology is that it can find both global and robust optimal solutions in a single run.

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