CHARACTERIZATION OF STABILITY REGIONS OF NONLINEAR DISCRETE DYNAMICAL SYSTEMS

Hsiao-Dong Chiang*, Jaewook Lee†, Fabiolo Moraes Amaral‡, Luís Fernando Costa Alberto§

*School of Electrical and Computer Engineering
Cornell University
Ithaca, NY, USA

†Department of Industrial and Management Engineering
Pohang University of Science and Technology
Pohang, South Korea

‡Departamento de Matemática
Federal Institute of Education, Science and Technology of Bahia
Eunápolis, BA, Brasil

§Escola de Engenharia de São Carlos
Universidade de São Paulo
São Carlos, SP, Brasil

Emails: chiang@ece.cornell.edu, jaewookl@postech.ac.kr, fabiolo@ifba.edu.br, lfcalberto@usp.br

Abstract— In this paper, a comprehensive theory for both the stability boundary and the stability regions (or domain of attraction) of nonlinear discrete dynamical systems is developed. Several topological and dynamical characterizations of the stability boundaries for a large class of nonlinear discrete dynamic systems are developed. For this class of nonlinear systems, the stability boundary is shown to equal the union of the stable manifolds of all the fixed points on the stability boundary. Several necessary and sufficient conditions for a given fixed point to lie on the stability boundary are derived. Numerical examples are presented to illustrate the theoretical developments presented in this paper.

Keywords— Stability Region, Domain of Attraction, Attraction Area, Discrete Systems.

Resumo— Neste artigo, a teoria de região de estabilidade de sistemas dinâmicos não lineares discretos é desenvolvida. Várias caracterizações da fronteira da região de estabilidade para uma ampla classe de sistemas dinâmicos não lineares discretos são desenvolvidas. Para esta classe, demonstra-se que a fronteira da região de estabilidade é igual à união das variedades estáveis de todos os pontos fixos na fronteira da região de estabilidade. Diversas condições necessárias e suficientes para um ponto fixo pertencer à fronteira da região de estabilidade são desenvolvidas. Exemplos numéricos ilustram os desenvolvimentos teóricos apresentados no artigo.

Keywords— Região de estabilidade, domínio de atração, área de atração, sistemas discretos

1 Introduction

Nonlinear discrete dynamical systems have been used to model a variety of practical nonlinear systems and the concept of stability region is essential in many of these applications. For instance, the dynamics of power systems with LTCs are modelled by difference equations (Vournas and Sakelladaridis, 2006), iterated-map neural networks are other example of discrete-time dynamical systems (Marcus and Westervelt, 1989), discrete-time dynamical systems are also present in studies of dynamics of ecosystems (Wang et al., 2011) and economic models (Yu et al., 2010). Discrete dynamical models also appear in the stability analysis of sample-data systems, which typically appear in systems that are controlled by a computer (Laila et al., 2011), or in systems that cannot be continuously measured (see (Elaiw and Xia, 2009) for an application of discrete-time approximate models in the analysis of HIV dynamics and treatment schedules).

There has been significant work on the analysis of stability and asymptotic behavior of nonlinear discrete-time dynamical systems, see (LaSalle, 1986; LaSalle, 1977; LaSalle, 1976; Rodrigues et al., 2011; Alberto et al., 2007)for invariance principles for discrete-time systems and (Blanchini, 1999) for a comprehensive survey on the theory of positively invariant sets in the analysis and control of discrete-time nonlinear dynamical systems. Recent advances on the stability theory of these systems can be found in (Coutinho and Souza, 2010; Mira et al., 1994; Grizzle and Kang, 2001; Lukyanova, 2009; Lee and Jiang, 2006; Kellet and Teel, 2004). In spite of the enormous amount of work done in the analysis of asymptotic behavior of solutions of discrete-time nonlinear dynamical systems, the problem of estimating stability region of these systems is still an open problem.

Significant progress was made in the devel-

ISBN: 978-85-8001-069-5

opment of theory and characterization of stability regions of nonlinear continuous dynamical systems (Chiang et al., 1988; Zaborszky et al., 1988; Amaral and Alberto, 2011), nevertheless, a few analytical results on the characterization of stability regions of nonlinear discrete dynamical systems exist.

In this paper, a comprehensive theory for both the stability boundary and the stability regions of a class of nonlinear discrete dynamical systems will be developed. In particular, topological properties of the stability region and stability boundary will be derived and a complete characterization of the stability boundary of a class of discrete dynamical systems will be developed.

One key application of the characterization of the stability boundary is that it leads to the development of practical methods to estimate stability regions of large-scale nonlinear dynamical systems.

2 Discrete Dynamical Systems

Consider the following class of autonomous nonlinear discrete dynamical systems:

$$x_{k+1} = f(x_k) \tag{1}$$

where $k \in \mathbb{Z}, x_k \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued map. The solution of (1) starting from $x_o \in \mathbb{R}^n$ at $k = 0$, denoted by $\phi(\cdot, x_o) : \mathbb{Z} \to \mathbb{R}^n$, is called an orbit (or trajectory) of (1). The solution of the discrete dynamical system (1) is an infinite sequence x_k that can be obtained by successive applications of the map f, i.e. $x_k = \phi(k, x_o) = f^k(x_o)$. The existence and uniqueness of solutions of system (1) for times greater than zero ($k > 0$) is not an issue for discrete systems. If map f is a well defined function on $\mathbb{R}^n$, then the solution starting at x_o is defined in $\mathbb{Z}_+$ and can be easily obtained by successive application of the map. On the other hand, solutions may not exist for negative times ($k < 0$) and when they exist, the solution may not be unique. However, if function f is invertible, then solutions of (1) exist and are defined in $\mathbb{Z}$.

A point x^* is a periodic point of period p if $f^p(x^*) = x^*$ and $f^k(x^*) \neq x^*$ for every k satisfying $0 < k < p$. If x^* has period one, i.e. $f(x^*) = x^*$, then x^* is called a fixed point of system (1). For a fixed point x^*, it is stable, if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that $\| x_0 - x^* \| < \delta \Longrightarrow \| x_k - x^* \| < \epsilon, \forall k \in \mathbb{Z}^+$ and it is asymptotically stable, if it is stable and δ can be chosen such that $\| x_0 - x^* \| < \delta \Longrightarrow \lim_{k \longrightarrow \infty} x_k = x^*$. A fixed point x^* is called unstable, if it is not stable.

When function f is continuous and differentiable, we say a fixed point of (1) is hyperbolic if the Jacobian matrix at x^*, denoted by $Df(x^*)$, has no eigenvalues with modulus one.

A set M is positively invariant with respect to the discrete system (1) if $f(M) \subset M$, which implies that every orbit x_k starting in M remains in M for all $k \geq 0$. A set M is invariant if $f(M) = M$ and negatively invariant if $f^{-1}(M) \subset M$.

A point p is said to be in the ω-limit set (or α-limit set) of x_o if for any given $\epsilon > 0$ and $N > 0$ (or $N < 0$), there exists a $k > N$ (or $k < N$) such that $\|x_k - p\| < \epsilon$. This is equivalent to the condition that there is a sequence $k_i \in \mathbb{Z}$ with $k_i \to \infty$ (or $k_i \to -\infty$) as $i \to \infty$ such that $p = lim_{i \to \infty} x_{k_i}$.

One favorable property about the ω-limit set $\omega(x)$ is that it is a closed and positively invariant set. If function f is invertible, then the ω-limit set is closed and invariant. If f is continuous and the orbit $\{x_k\}$ is also bounded for $k > 0$, then the ω-limit set is nonempty, compact, invariant and invariantly connected. We note that a closed and invariant set M is said to be invariantly connected if it is not the union of two non-empty disjoint closed invariant sets (LaSalle, 1976). Moreover,

$$x_k = f^k(x) \to \omega(x) \text{ as } k \to \infty$$

The hyperbolic fixed point x^* is called a *type-k fixed point* if the Jacobian $Df(x^*)$ has exactly k eigenvalues with a modulus greater than one. The function f is a diffeomorphism if it is differentiable and invertible and its inverse f^{-1} is differentiable. If f is a $\mathcal{C}^r$-diffeomorphism, with $r \geq 1$, and x^* is a hyperbolic fixed point of (1), then the tangent space at x^*, $T_{x^*}(\mathbb{R}^n)$, can be uniquely decomposed as a direct sum of two subspaces denoted by E^s and E^u, which are invariant with respect to the linear operator $Df(x*)$:

$$E^s = span \ \{e_1, ..., e_s\},$$

$$E^u = span \ \{e_{s+1}, ..., e_{s+u}\},$$

where $\{e_1, ..., e_s\}$ and $\{e_{s+1}, ..., e_{s+u}\}$ are the generalized eigenvectors of $Df(x^*)$ respectively associated with the eigenvalues of $Df(x^*)$ that have a modulus less and greater than one. Subspaces E^s and E^u are respectively called stable and unstable subspaces.

There are local manifolds $W^s_{loc}(x^*)$, and $W^u_{loc}(x^*)$ of class C^r, invariant with respect to (1) (Abraham and Robbin, 1967; Hirsch et al., 1970), that are tangent to E^s and E^u at x^*, respectively. $W^s_{loc}(x^*)$ and $W^u_{loc}(x^*)$ are respectively called local stable and unstable manifolds. These local stable and unstable manifolds are unique. Every orbit x_k starting in $W^s_{loc}(x^*)$ approaches x^* as $k \to \infty$, while every orbit starting in $W^u_{loc}(x^*)$ approaches x^* as $k \to -\infty$.

The local unstable manifold can be extended via dynamics of system (1) to form the (global) unstable manifold:

$$W^u(x^*) = \bigcup_{k \geq 0} \phi(k, W^u_{loc}(x^*)) = \bigcup_{k \geq 0} f^k \left(W^u_{loc}(x^*)\right).$$

ISBN: 978-85-8001-069-5

The (global) stable manifold can also be obtained via backward iterations of system (1):

$$W^s(x^*) = \bigcup_{k\leq 0} \phi(k, W^s_{loc}(x^*)) = \bigcup_{k\leq 0} f^k\left(W^s_{loc}(x^*)\right)$$

The manifolds $W^s(x^*)$ and $W^u(x^*)$ are of class C^r. If x^* is a type-k fixed point, then the dimension of $W^u(x^*)$ is k and the dimension of $W^s(x^*)$ is $n-k$.

The idea of transversality is basic in the study of nonlinear dynamic systems. The stable and unstable manifolds of hyperbolic fixed points are said to satisfy the transversality condition if either (i) they do not intersect at all, or (ii) at every intersection point $x \in W^u(x^*)\bigcap W^s(y^*)$, the tangent space of $W^u(x^*)$ and $W^s(y^*)$ spans $\mathbb{R}^n$ at x, i.e,

$$T_x(W^u(x^*)) \oplus T_x(W^s(y^*)) = T_x(\mathbb{R}^n)$$

One of the most important features of a hyperbolic fixed point x^* is that its stable and unstable manifolds intersect transversally at x^*(Shub, 1987). This transversal intersection is important because it persists under small perturbation of the map (Palis, 1969; Peixoto, 1967).

3 Topological Properties of Stability Regions

Typically, asymptotically stable fixed points of physical systems are not globally asymptotically stable. In this case, knowledge of the stablitity region of an asymptotically stable fixed point x^s of (1), defined below, is of great importance.

$$A(x^s) := \{x \in \mathbb{R}^n : lim_{k\to\infty} f^k(x) = x^s\}.$$

The topological boundary of the stability region $A(x^s)$ is called the stability boundary of x^s and will be denoted by $\partial A(x^s)$.

In the following, we derive several topological characterizations of stability regions.

Theorem 1 (Topological Characterization 1) *The stability region $A(x^s)$ of an asymptotically stable fixed point x^s of (1) is positively invariant and negatively invariant. The stability boundary $\partial A(x^s)$ is a closed set.*

Theorem 1 provides topological information regarding the stability region and stability boundary without imposing any condition on the map f. As we impose conditions on the map f, refined results about the topological properties of these sets can be obtained. The next proposition assumes that the map f is a continuous function.

Theorem 2 (Topological Characterization 2) *Let x^s be an asymptotically stable fixed point of (1) and suppose that f is a continuous function. The stability region $A(x^s)$ is an open, positively and negatively invariant set. The stability boundary $\partial A(x^s)$ is a closed positively invariant set formed by forward orbits. Moreover the stability boundary $\partial A(x^s)$ is of dimension less than n and if $A(x^s)$ is not dense in R^n, then $\partial A(x^s)$ is of dimension $n-1$.*

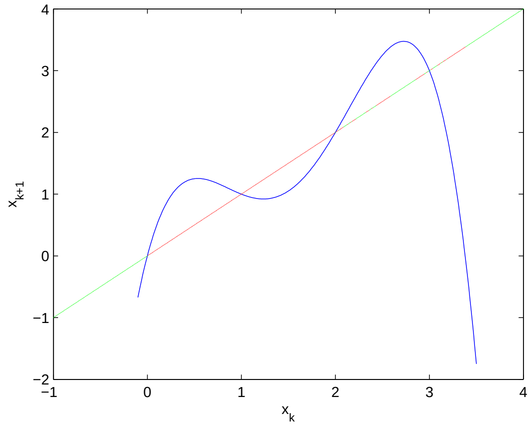

Figure 1: The stability region of system (2) is indicated in red in the diagonal. The stability region is not a path connected set.

The previous theorem indicates that the existence of at least two asymptotically stable fixed points is sufficient to guarantee that the dimension of the stability boundary of each fixed point is $n-1$.

In the next theorem, the map f is assumed to be surjective. The stability region of surjective maps is shown to be invariant.

Theorem 3 (Topological Characterization 3) *Let x^s be an asymptotically stable fixed point of system (1) and suppose that f is surjective, then the stability region $A(x^s)$ is an invariant set.*

Another topological property of the stability region of fixed points of discrete systems is that it may not be path connected, as it is shown in the following one-dimensional example.

Example 3.1 *The following discrete system:*

$$x_{k+1} = x_k - 0.8x_k(x_k-1)(x_k-2)(x_k-3) \quad (2)$$

possesses four fixed points, $x^1 = 0$, $x^2 = 1$, $x^3 = 2$ and $x^4 = 3$. The fixed point $x^2 = 1$ is an asymptotically stable fixed point. Its stability region is indicated in red in the graph of Figure 1. The stability region is quite complex even for this one-dimensional dynamical system. In particular, the stability region is not a connected set and it is not invariant.

However, if f is invertible and the inverse is a continuous function, then the stability region is path connected, as shown in Theorem 4.

ISBN: 978-85-8001-069-5

Theorem 4 (Topological Characterization 4) *Let x^s be an asymptotically stable fixed point of the discrete system (1). If function f is invertible and its inverse is a continuous function, then the stability region is path connected.*

If function f is a homeomorphism, i.e., f is continuous, invertible and its inverse is also continuous, then all properties of the stability region and stability boundary that have been shown in the theorems 1-4 are valid. The next theorem asserts these properties and shows that the stability boundary is also an invariant set.

Theorem 5 (Topological Characterization 5) *Let x^s be an asymptotically stable fixed point of the discrete system (1). If function f is a homeomorphism, then the stability region $A(x^s)$ is: (i) open, (ii) positively and negatively invariant, (iii) invariant and (iv) path connected; and the stability boundary $\partial A(x^s)$ is: (i) closed and (ii) invariant.*

4 Stability Region Characterizations

Our aim in this section is to present a comprehensive characterization of stability regions for the nonlinear discrete dynamical system (1). Our approach starts from a local characterization of the stability boundary and progresses towards a global characterization of the stability boundary.

We first derive a complete characterization for a fixed point to lie on the stability boundary, which is a key step in the characterization of the stability region $A(x_s)$. We do this in two steps. First we impose only one generic assumption, namely, that fixed points are hyperbolic, and derive conditions for a fixed point to be on the stability boundary in terms of both its stable and unstable manifolds. Additional conditions are then imposed on the discrete dynamical system and the results are further sharpened.

Theorem 6 (Fixed Points on the Stability Boundary) *Let $A(x^s)$ be the stability region of an asymptotically stable fixed point x^s of system (1). Let $\hat{x} \neq x^s$ be a hyperbolic fixed point and suppose that f is a diffeomorphism. Then,*

(i) $\hat{x} \in \partial A(x^s)$ *if and only if* $\{W^u(\hat{x}) - \hat{x}\} \bigcap \overline{A(x^s)} \neq \phi$

(ii) *If $\hat{x}$ is not a source, then $\hat{x} \in \partial A(x^s)$ if and only if $\{W^s(\hat{x}) - \hat{x}\} \bigcap \partial A(x^s) \neq \phi$.*

As a corollary to Theorem 6, if $\{W^u(\hat{x}) - \hat{x}\} \cap A \neq 0$, then $\hat{x}$ must be on the stability boundary. Since any orbit in $A(x^s)$ approaches x^s, we see that a sufficient condition for $\hat{x}$ to be on the stability boundary is the existence of an orbit in $W^u(\hat{x})$ which approaches x^s. The favorable property of this condition is that it can be checked numerically. From a practical point of view we would like to see when this condition is necessary. We will show that this condition becomes necessary under two additional assumptions, the transversality condition, which is generic for a nonlinear dynamical system (1), and the assumption that every orbit on the stability boundary approaches one of the fixed points. Hence, we consider the following assumptions:

(A1) All the fixed points on $\partial A(x^s)$ are hyperbolic;

(A2) The stable and unstable manifolds of fixed points on $\partial A(x^s)$ satisfy the transversality condition;

(A3) Every orbit on $\partial A(x^s)$ approaches one of the fixed points as $t \to \infty$.

Assumptions (A1) and (A2) are generic properties of diffeomorphisms, while assumption (A3) is not a generic property.

Now, we present the key theorem which characterizes a fixed point being on the stability boundary in terms of both its stable and unstable manifolds. From the practical point of view, this result is more useful in the numerical verification of fixed points on the stability boundary than Theorem 6.

Theorem 7 (Fixed Points on the Stability Boundary) *Let $A(x^s)$ be the stability region of an asymptotically stable fixed point x^s of system (1). Let $\hat{x}$ be a hyperbolic fixed point and suppose that f is a diffeomorphism and assumptions (A1)-(A3) are satisfied. Then, the following characterizations hold:*

(i) $\hat{x} \in \partial A(x^s)$ *if and only if* $W^u(\hat{x}) \cap A(x^s) \neq \emptyset$

(ii) *if $\hat{x}$ is not a source, then $\hat{x} \in \partial A(x^s)$ if and only if $W^s(\hat{x}) \subseteq \partial A(x^s)$*

Now we are in a position to develop a complete characterization of the stability boundary of a class of nonlinear discrete dynamical systems satisfying assumptions (A1)-(A3).

Theorem 8 (Stability Boundary Characterization) *Let x^s be an asymptotically stable fixed point of the discrete system (1) and suppose that f is a diffeomorphism and assumptions (A1)-(A3) are satisfied. Let $x_1, x_2, \ldots$ be the hyperbolic unstable fixed points on the stability boundary $\partial A(x^s)$ of x^s. Then*

$$\partial A(x^s) = \bigcup_i W^s(x_i)$$

Theorem 8 asserts that the stability boundary is composed of the union of the stable manifolds of all fixed points that lie on the stability boundary. Next examples illustrate these analytical results.

ISBN: 978-85-8001-069-5

Example 4.1 *Consider the nonlinear discrete system:*

$$\begin{aligned} x_{k+1} &= dx_k + x_k^3 + ey_k^2 \\ y_{k+1} &= cy_k \end{aligned} \tag{3}$$

with $c = d = e = 0.5$. It is straightforward to show that $f(x_k, y_k)$ is a diffeomorphism. The fixed points of (3) are $(0, 0)$, which is an asymptotically stable fixed point, and $x_1 = (+\sqrt{1-d}, 0)$ and $x_2 = (-\sqrt{1-d}, 0)$, which are type-one fixed points. All fixed points are hyperbolic and, as a consequence, assumption (A1) is satisfied. Figure 2 displays the stability region $A(0, 0)$ of the fixed point $(0, 0)$. The unstable fixed points x_1 and x_2 lie on the stability boundary $\partial A(0, 0)$. Assumptions (A2) and (A3) are also satisfied and therefore the stability boundary is composed of the union of the stable manifolds $W^s(x_1)$ and $W^s(x_2)$, in accordance with the results of Theorem 8.$\square$

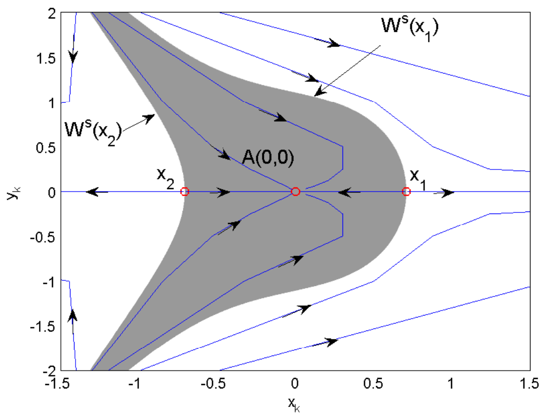

Figure 2: Stability region of the fixed point $(0, 0)$ of system (3). The stability boundary is composed of the union of the stable manifolds $W^s(x_1)$ and $W^s(x_2)$ of the type-one fixed points x_1 and x_2

.

Example 4.2 *Consider the following discrete version of the nonlinear pendulum equation:*

$$\begin{aligned} x_{k+1} &= x_k + hy_k \\ y_{k+1} &= (1 - dh)y_k - hc\sin(x_k) \end{aligned} \tag{4}$$

with $c, d > 0$ and $0 < h < \frac{1}{d}$. For sufficiently small h, the vector field is a diffeomorphism. All the fixed points are hyperbolic and assumptions (A1),(A2) and (A3) are satisfied. Hence, the characterization of the stability boundary of Theorem 8 is valid. Figure 3 displays the stability region and stability boundary of system (4) for $h = 0.1$, $k = 1$ and $d = 0.5$. There are two hyperbolic fixed points on the stability boundary of the asymptotically stable fixed point $(0, 0)$. The stability boundary is composed of the union of the stable manifolds of these two fixed points, confirming the results of Theorem 8.$\square$

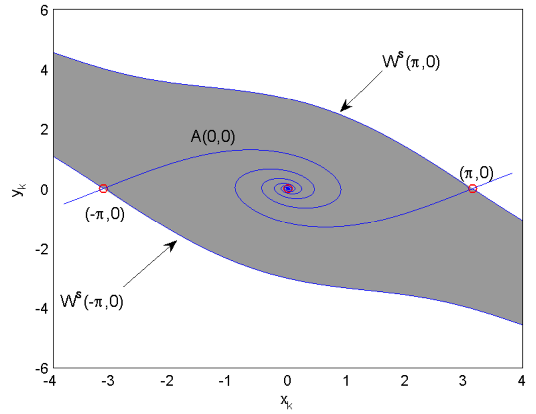

Figure 3: Stability region of system (4) for $k = 1$, $d = 0.5$ and $h = 0.1$.

Since the stability boundary $\partial A(x^s)$ is of dimension $n - 1$, the stable manifolds of non-type-one fixed points are thin sets on $\partial A(x^s)$. Exploring this property, next theorem offers a complete characterization of the boundary of $\bar{A}$ in terms of the stable manifolds of type-one fixed points that lie on the stability boundary.

Theorem 9 *Let x^s be an asymptotically stable fixed point of the discrete system (1) and suppose that f is a diffeomorphism and assumptions (A1)-(A3) are satisfied. Let x_1^1, $x_2^1, \ldots$ be the type-one hyperbolic unstable fixed points on the stability boundary $\partial\overline{A}(x^s)$ of x^s. Then*

$$\partial\overline{A}(x^s) = \bigcup_i \overline{W^s(x_i^1)}$$

The following theorem gives an interesting result on the structure of the fixed points on the stability boundary. Moreover, it presents a necessary condition for the existence of certain types of fixed points on a bounded stability boundary.

Theorem 10 *Let x^s be an asymptotically stable fixed point of the discrete system (1) and suppose that f is a diffeomorphism and assumptions (A1)-(A3) are satisfied. If the stability region $A(x^s)$ is not dense in $\mathbb{R}^n$ then $\partial A(x^s)$ must contain at least one type-1 fixed point. Moreover, if $A(x^s)$ is bounded, then $\partial A(x^s)$ must contain at least one type-n fixed point (i.e. a source).*

Theorem 10 offers a sufficient condition to check wether the stability region is unbounded. This condition is formally stated in the next corollary.

Corollary 11 *If the stability boundary $\partial A(x^s)$ of the asymptotically stable fixed point x^s of the discrete system (1) has no source and assumptions (A1)-(A3) are satisfied, then $A(x^s)$ is unbounded.*

ISBN: 978-85-8001-069-5

To illustrate Corollary 11, we note that the stability regions of both system (3) and system (4) are unbounded and there are no type-2 fixed points lying on the stability boundaries of both systems.

5 Characterization via Energy Functions

In this section, we focus on how to characterize, via an energy function, the stability region of high-dimension nonlinear discrete systems.

Before defining the concept of energy function, it is important to understand the concept of first difference, which plays the role of derivative for discrete systems. Consider the scalar function $V : \mathbb{R}^n \to \mathbb{R}$. The first difference of V relative to (1), or to the map f, at point $x \in \mathbb{R}^n$ is given by

$$\Delta V(x) = V(f(x)) - V(x) \tag{5}$$

If x_k is a solution of (1) for $k \geq 0$, then the first difference of V along the solution x_k is given by

$$\Delta V(x_k) = V(x_{k+1}) - V(x_k), \quad k \geq 0.$$

Definição 5.1 *A continuous function $V : R^n \to R$ is called an energy function for the discrete system (1) if it satisfies the following conditions*

(E1) $\Delta V(x) \leq 0$ *for all $x \in R^n$;*

(E2) $\Delta V(x_k) = 0$ *implies x_k is a fixed point.*

(E3) *If $V(x_k)$ is bounded for $k \in \mathbb{Z}_+$, then the orbit x_k is itself bounded for $k > 0$.*

Property (E1) indicates that the energy function is non-increasing along any orbit, but it alone does not imply that the energy function is strictly decreasing along any nontrivial orbit. There may exist a time k such that $\Delta V(x_k) = 0$. Properties (E1) and (E2) together imply that the energy function is strictly decreasing along any nontrivial system orbit. Property (E3) states that the energy function is a dynamic proper map along any system orbit but need not be a proper map for the entire state space. Recall that a proper map is a function $f : X \to Y$ such that for each compact set $D \in Y$, the set $f^{-1}(D)$ is compact in X. From the above definition of the energy function, it is obvious that an energy function may not be a Lyapunov function. We note that if function V is proper or radially unbounded, then assumption (E3) is satisfied.

Energy functions are useful for global analysis of system orbits and for estimating stability regions and quasi-stability regions, among others. We next present a global analysis of system orbits of nonlinear discrete dynamical systems that have energy functions.

Theorem 12 (Global Analysis) *If the nonlinear discrete dynamical system (1) admits an energy function, the map f is continuous with all fixed points being isolated, then every bounded trajectory x_k converges to a fixed point as $k \to \infty$.*

Theorem 12 asserts that nonlinear discrete systems admitting energy function do not present complex behavior, i.e. their limit sets are exclusively composed of fixed points, in particular, the state space of this class of nonlinear systems does not admit nontrivial periodic solutions, quasi-periodic solutions and chaos. It will be shown in the next theorem that orbits on the stability boundary of nonlinear discrete dynamical systems admitting an energy function are bounded for $k > 0$ although the stability boundary itself can be unbounded.

Theorem 13 (Boundedness of Orbits on the Stability Boundary) *Let x^s be an asymptotically stable fixed point of the discrete system (1) that admits an energy function and suppose that f is continuous. Then every trajectory x_k on the stability boundary $\partial A(x^s)$ is bounded for $k > 0$.*

Combining the two previous theorems, a sufficient condition for the satisfaction of assumption (A3) is obtained.

Corollary 14 (Sufficient Condition for (A3)) *Let x^s be an asymptotically stable fixed point of the discrete system (1) that admits an energy function, suppose that f is continuous and all fixed points are isolated. Then assumption (A3) is satisfied, i.e. every trajectory on the $\partial A(x^s)$ converges to a fixed point.*

We are now in a position to present a complete characterization of the stability boundary for a class of discrete nonlinear dynamical systems that admit energy functions.

Theorem 15 (Stability Boundary Characterization 1) *Let x^s be an asymptotically stable fixed point of the nonlinear discrete system (1) that admits an energy function. Suppose that f is a continuos map satisfying assumption (A1) and let $x_1, x_2, \ldots$ be the unstable fixed points on the stability boundary $\partial A(x^s)$. Then, the stability boundary is contained in the union of the stable sets of all the fixed points on the stability boundary:*

$$\partial A(x^s) \subseteq \bigcup_i W^s(x_i)$$

where $W^s(x_i)$ is the stable set of the unstable fixed point x_i.

The characterization of the stability boundary of discrete systems that admit energy functions, asserted in Theorem 15, only requires continuity of the vector field. A sharper characterization of the stability boundary of systems having energy

ISBN: 978-85-8001-069-5

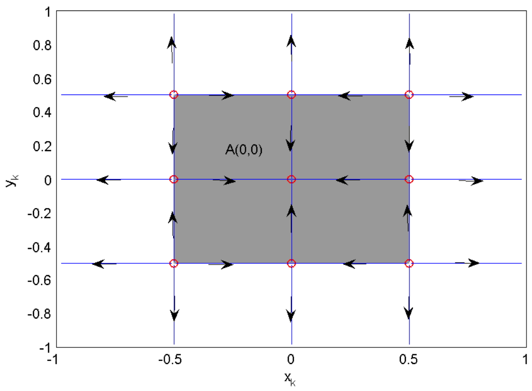

Figure 4: Stability region of system (6). Its stability boundary is composed of the union of the stable manifolds of the 8 unstable fixed points lying on the boundary.

functions can be obtained by assuming that all fixed points are hyperbolic and f is a diffeomorphism. We gain more structure on the stable set that in this case is a manifold at the expense of more conditions on the vector field f.

Theorem 16 (Stability Boundary Characterization 2) *Let x^s be an asymptotically stable fixed point of the nonlinear discrete system (1) that admits an energy function. Suppose that f is a diffeomorphism satisfying assumption (A1). Let x_1, $x_2, \ldots$ be the hyperbolic unstable fixed points on the stability boundary $\partial A(x^s)$ of x^s. Then*

$$\partial A(x^s) \subseteq \bigcup_i W^s(x_i)$$

Moreover, if assumption (A2) is satisfied, then

$$\partial A(x^s) = \bigcup_i W^s(x_i)$$

where $W^s(x_i)$ is the stable manifold of the hyperbolic unstable equilibrium point x_i.

Example 5.1 *Consider the following two-dimensional nonlinear discrete system:*

$$\begin{array}{rcl} x_{k+1} & = & x_k^3 + \frac{3}{4}x_k \\ y_{k+1} & = & y_k^3 + \frac{3}{4}y_k \end{array} \qquad (6)$$

The vector field is a diffeomorphism and the system possesses 9 hyperbolic fixed points, they are: $(0,0)$, *an asymptotically stable fixed point, and* $(0,\pm\frac{1}{2})$, $(\pm\frac{1}{2},0)$, $(\pm\frac{1}{2},\pm\frac{1}{2})$, *which are unstable fixed points. Consider the following candidate to energy function:*

$$V(x,y) = \begin{cases} |x|+|y| & \text{if } |x| \le \frac{1}{2} \text{ and } |y| \le \frac{1}{2} \\ 1-|x|+|y| & \text{if } |x| > \frac{1}{2} \text{ and } |y| \le \frac{1}{2} \\ 1+|x|-|y| & \text{if } |x| \le \frac{1}{2} \text{ and } |y| > \frac{1}{2} \\ 2-|x|-|y| & \text{if } |x| > \frac{1}{2} \text{ and } |y| > \frac{1}{2} \end{cases}$$

It is straightforward to show that $\Delta V(x,y) \le 0$ and $\Delta V(x,y) = 0$ if and only if (x,y) is a fixed point. Hence, assumptions (E1) and (E2) are satisfied. Assumption (E3) is also satisfied because $V(x,y)$ is a proper function. We concluded that V is an energy function for system (6) and all the conditions of Theorem 16 are satisfied. Thus the stability boundary $\partial A(0,0)$ is composed of the union of the stable manifolds of every unstable fixed point that lies on the stability boundary. Figure 4 illustrates the stability region and stability boundary of system (6). The stability boundary contains 8 unstable fixed points and is composed of the union of their corresponding stable manifolds.$\square$

6 Conclusions

A comprehensive theory of stability regions of general nonlinear autonomous discrete dynamical systems has been developed in this paper. Several topological properties of stability boundary and characterization of limit sets lying on the stability boundary for general nonlinear discrete dynamical systems have been derived. A complete characterization of stability boundaries for a fairly large class of nonlinear discrete dynamic systems has been obtained. For this class of nonlinear discrete systems, it was shown that the stability boundary of an asymptotically stable fixed point consists of the stable manifolds of all the fixed points on the stability boundary. Several necessary and sufficient conditions were derived to determine whether a given fixed point lies on the stability boundary. Numerical examples were given to illustrate the theoretical results developed in this paper.

Agradecimentos

This work was partially supported by the Consortium for Electric Reliability Technology Solutions. Funding was provided by the Office of Electricity Delivery and Energy Reliability, Transmission Reliability Program of the U.S.Department of Energy through the National Energy Technology Laboratory Cooperative Agreement No. DE-FC26-09NT43321. This work was also partially supported by FAPESP (Agreements No. 2010/00574-9 and 2011/06938-5).

References

Abraham, R. and Robbin, J. (1967). *Transversal Mappings and Flows*, Benjamin, New York.

Alberto, L. F. C., Calliero, T. R., Martins, A. and Bretas, N. G. (2007). An invariance principle for nonlinear discrete autonomous dynamical systems, *IEEE Transactions on Automatic Control* **52**(4): 692–697.

Amaral, F. M. and Alberto, L. F. C. (2011). Stability region bifurcations of nonlinear autonomous dynamical systems: Type-zero saddle-node bifurcations, *International Journal of Robust and Nonlinear Control* **21**(6): 591–612.

Blanchini, F. (1999). Set invariance in control, *Automatica* **35**: 1747–1767.

Chiang, H. D., Hirsch, M. W. and Wu, F. F. (1988). Stability regions of nonlinear autonomous dynamical systems, *IEEE Transactions on Automatic Control* **33**(1): 16–27.

Coutinho, D. F. and Souza, C. E. (2010). Robust domain of attraction estimates for a class of uncertain discrete-time nonlinear systems, *8th IFAC Symposium on Nonlinear Control Systems* .

Elaiw, A. M. and Xia, X. (2009). Hiv dynamics: Analysis and robust multirate mpc-based treatment schedules, *Journal of Math. Anal. Appl.* **359**: 285–301.

Grizzle, J. W. and Kang, J.-M. (2001). Discrete-time control design with positive semi-definite lyapunov functions, *System & Control Letters* **43**: 287–292.

Hirsch, M. W., Pugh, C. C. and Shub, M. (1970). Invariant manifolds, *Bulletin of the American Mathematical Society* **76**(5): 1015–1019.

Kellet, C. M. and Teel, A. R. (2004). Smooth-lyapunov functions and robustness of stability for difference inclusions, *System & Control Letters* **52**: 395–405.

Laila, D. S., Lovera, M. and Astolfi, A. (2011). A discrete-time observer design for spacecraft attitude determination using an orthogonality-preserving algorithm, *Automatica* **47**: 975–980.

LaSalle, J. P. (1976). *Stability of Dynamical Systems*, Ed. Philadelphia, PA: SIAM.

LaSalle, J. P. (1977). Stability theory for difference equations. studies in ordinary differencial equation, *Studies in ordinary differencial equations, Studies in Math., Math. Assoc. of America, Washington, D.C (Reviewer G.R. Sell)* **14**: 1–31.

LaSalle, J. P. (1986). *The stability and Control of Discrete Processes*, Springer-Verlag New York Inc.

Lee, T. C. and Jiang, Z. P. (2006). On uniform global asymptotic stability of nonlinear discrete-time systems with applications, *IEEE Trans. on Automatic Control* **51**(10): 1644–1660.

Lukyanova, T. A. (2009). Stability and boundedness conditions for the motions of discrete-time mechanical systems, *International Applied Mechanics* **45**(8): 917–921.

Marcus, C. M. and Westervelt, R. M. (1989). Dynamics of iterated-map neural networks, *Physics Review A* **40**(1): 501–504.

Mira, C., Fournier-Prunaret, D., Gardini, L., Kawakami, H. and Cathala, J. C. (1994). Basin bifurcations of two-dimensional noninvertible maps: Fractalization of basins, *International Journal of Bifurcation and Chaos* **4**(2): 343–381.

Palis, J. (1969). On morse-smale dynamical systems, *Topology* **8**(4): 385–404.

Peixoto, M. M. (1967). On an approximation theorem of kupka and smale, *Journal of Differential Equations* **3**(2): 214–227.

Rodrigues, H. M., Wu, J. H. and Gabriel, L. R. A. (2011). Uniform dissipativeness, robust synchronization and location of the attractor of parametrized nonautonomous discrete systems, *International Journal of Bifurcation and Chaos* **21**(2): 513–526.

Shub, M. (1987). *Global Stability of Dynamical Systems*, Springer-Verlag, New York.

Vournas, C. D. and Sakelladaridis, N. (2006). Region of attraction in a power system with discrete ltcs, *IEEE Trans. on Circuits and Systems I: Regular Papers* **53**(7): 1610–1618.

Wang, W.-X., Zhang, Y.-B. and zhong Liu, C. (2011). Analysis of a discrete-time predator-prey system with allee effect, *Ecological Complexity* **8**(1): 81 – 85.
URL: *1*

Yu, T., Yu, J., Li, H. and Lin, H. (2010). Complex dynamics of industrial transferring in a credit-constrained economy, *Physics Procedia* **3**(5): 1677 – 1685. The International Conference on Complexity and Interdisciplinary Sciences. The 3rd China-Europe Summer School on Complexity Sciences.
URL: *1*

Zaborszky, J., Huang, G., Zheng, B. and Leung, T. C. (1988). On the phase portrait of a class of large nonlinear dynamic systems such as the power system, *IEEE Trans. on Automatic Control* **33**(1): 4–15.