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Nonlinear analysis in a Lorenz-like system

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ABSTRACT

In this paper we study the nonlinear dynamics of a Lorenz-like system. More precisely, we study the stability and bifurcations which occur in a new three parameter quadratic chaotic system. We also study the existence of singularly degenerate heteroclinic cycles for a suitable choice of the parameters. As a consequence we show the existence of chaotic attractors when these cycles disappear.

1. Introduction

In this paper we study the stability and bifurcations with the respective qualitative changes in the dynamics of the following system of nonlinear differential equations

$$\begin{cases} x' = a(y-x), \\ y' = a(bx - xz), \\ z' = -cz + xy, \end{cases} \tag{1}$$

where $(x, y, z) \in \mathbb{R}^3$ are the state variables and $(a, b, c) \in \mathbb{R}^3$ are real parameters. Despite the simplicity, system (1) has a rich dynamical behavior, ranging from stable equilibrium points to periodic and even chaotic oscillations, depending on the parameter values.

Landmark for the study of chaotic behavior is the work of Lorenz [1]. Dating to 1963, it can be taken as the starting point for the theory of *chaos* with the introduction of the so called *Lorenz system* (see also [2])

$$x' = \sigma(y-x), \qquad y' = rx - y - xz, \qquad z' = -bz + xy. \tag{2}$$

Since Lorenz presented its chaotic system several other systems with this type of behavior have been studied, such as the Chen system [3]

$$x' = a(y-x), \qquad y' = (c-a)x + cy - xz, \qquad z' = -bz + xy,$$

the Liu system [4]

$$x' = a(y-x), \qquad y' = bx - kxz, \qquad z' = -cz + hx^2,$$

the Rössler system [5]

$$x' = -y - z, \qquad y' = x + ay, \qquad z' = b + z(x-c),$$

the Rikitake system [6]

$$x' = -\mu x + yz, \qquad y' = -\mu y + (z-a)x, \qquad z' = 1 - xy,$$

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the Lü system [7]

$$x' = a(y - x), \qquad y' = cy - xz, \qquad z' = -bz + xy,$$

the Genesio system [8]

$$x' = y, \qquad y' = z, \qquad z' = ax + by + cz + x^2,$$

among several others.

All the above systems are in fact families of quadratic systems in $\mathbb{R}^3$ depending on parameters. Such a system is here called *Lorenz-like system*. Of course, there are other Lorenz-like systems not listed above as well as chaotic non-quadratic systems in $\mathbb{R}^3$ such as the Chua system [9], for example. Families of quadratic systems also appear as classical epidemiological models such as the SIR epidemic models [10] and pest control [11].

The natural question about the Lorenz-like systems is: have the Lorenz-like systems listed above and the Lorenz system different dynamical behaviors? The answer is yes. For example, the Rössler, the Rikitake and the Genesio systems have at most two isolated equilibria while the Lorenz system have three such equilibria; the Lü and Liu systems have degenerate Hopf bifurcations at the nontrivial equilibria (see [12] and [13] respectively) while the Hopf bifurcations in the Lorenz system are generic.

Quadratic systems in $\mathbb{R}^3$ are some of the simplest systems after linear ones. In this set an interesting problem is the study of the number of limit cycles. In $\mathbb{R}^2$ this number is finite [14,15]. For quadratic systems in $\mathbb{R}^3$ the scenario is very different. Recently Ferragut, Llibre and Pantazi [16] provide an example of quadratic vector field in $\mathbb{R}^3$ with infinitely many limit cycles.

System (1) was analyzed in [17] from the point of view of its chaotic behavior as well as its circuit implementation. The study carried out in the present paper may contribute to understand analytically the local stability and bifurcations. We also present a conjecture about the existence of infinitely many singularly degenerate heteroclinic cycles in system (1). These cycles were firstly studied by Kokubu and Roussarie in [18] in the Lorenz system. The importance of this type of cycle is that chaotic attractors can bifurcate from it.

This paper is organized as follows. In Section 2 we present the linear analysis of equilibria of system (1) and the Conjecture about the existence of infinitely many singularly degenerate heteroclinic cycles. A brief review of the methods used to study codimension one Hopf bifurcations are presented in Section 3. In general the Lyapunov coefficients are very difficult to obtain. These methods are used in Section 3 to prove the main results of this paper. Some numerical investigations that suggest the existence of chaotic attractors bifurcating from the singularly degenerate heteroclinic cycles are present in Section 4. In Section 5 we make some concluding remarks.

2. Linear analysis of system (1)

In this section we study some of the generalities and linear stability of system (1). In a vectorial notation which will be useful in the calculations, system (1) can be written as $\mathbf{x}' = f(\mathbf{x}, \boldsymbol{\zeta})$, where

$$f(\mathbf{x}, \boldsymbol{\zeta}) = (a(y - x), a(bx - xz), -cz + xy), \tag{3}$$

$\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and $\boldsymbol{\zeta} = (a, b, c) \in \mathbb{R}^3$.

2.1. The case $a = 0$

For the case $a = 0$ system (1) has the form

$$x' = 0, \qquad y' = 0, \qquad z' = xy - cz.$$

Consider two cases: $c = 0$ and $c \neq 0$.

Case $c = 0$: If $c = 0$ then system (1) has two planes of equilibria: the planes $\{x = 0\}$ and $\{y = 0\}$. For an initial condition (x_0, y_0, z_0) with $x_0 y_0 \neq 0$ the corresponding solution $\phi(t)$ of system (1) is on the straight line (x_0, y_0, z), $z \in \mathbb{R}$, and

$$\lim_{t\to\infty} z(t) = \infty \ (-\infty, \text{respec.}), \qquad \lim_{t\to-\infty} z(t) = -\infty \ (\infty, \text{respec.}),$$

if $x_0 y_0 > 0$ ($x_0 y_0 < 0$, respectively). See Fig. 1.

Case $c \neq 0$: If $c \neq 0$ then system (1) has a surface of equilibria: the surface $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : z = xy/c\}$. For an initial condition $(x_0, y_0, z_0) \notin \mathcal{S}$ the corresponding solution $\phi(t)$ of system (1) is on the straight line (x_0, y_0, z), $z \in \mathbb{R}$, and

$$\lim_{t\to\infty} \phi(t) = (x_0, y_0, x_0 y_0/c) \in \mathcal{S}, \quad \text{if } c > 0,$$

$$\lim_{t\to-\infty} \phi(t) = (x_0, y_0, x_0 y_0/c) \in \mathcal{S}, \quad \text{if } c < 0.$$

See Fig. 2(a) for $c > 0$ and (b) for $c < 0$.

2.2. The case $a \neq 0$

From now on we consider the set of parameters

$$\mathcal{W} = \{(a, b, c) \in \mathbb{R}^3 : a \neq 0\}.$$

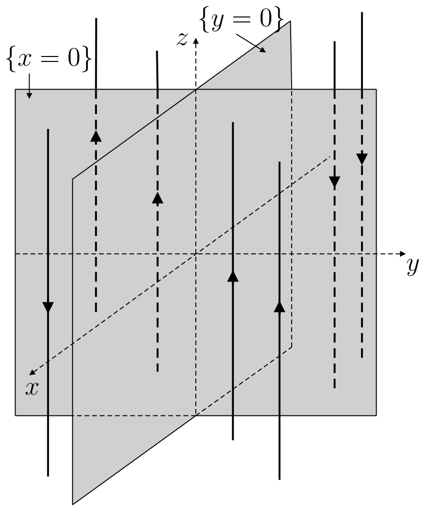

Fig. 1. Planes of equilibria $\{x = 0\}$, $\{y = 0\}$ and the flow of system (1) for parameters $a = 0$ and $c = 0$.

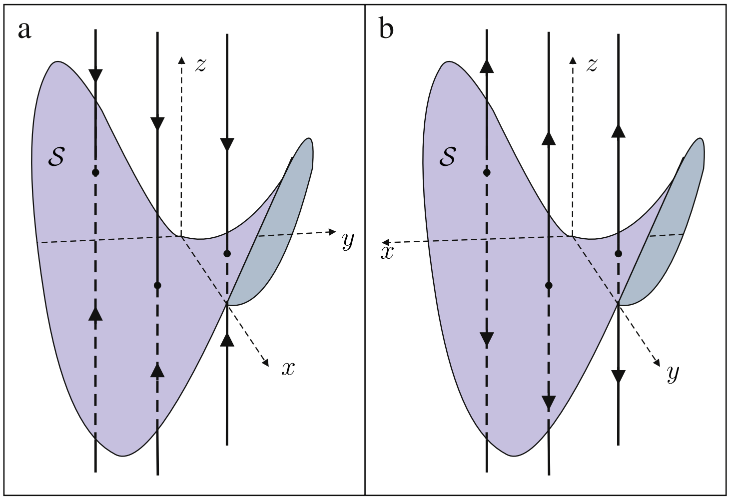

Fig. 2. Surface of equilibria $\mathcal{S}$ and the flow of system (1) for parameters $a = 0$ and $c > 0$ (a), $c < 0$ (b).

2.2.1. The case $c = 0$

We have the following proposition.

Proposition 2.1. *For $c = 0$, $a \neq 0$ and $b \in \mathbb{R}$ system* (1) *has the straight line of equilibria $\mathcal{L} = \{(0, 0, z) : z \in \mathbb{R}\}$. Denote such an equilibrium by $E_z = (0, 0, z)$ and let $b_0 \in \mathbb{R}$. The following statements hold.*

(a) *If $z < b_0$ then the equilibria E_z are saddles normally hyperbolic to the line $\mathcal{L}$, that is, the linearized system at E_z has two real eigenvalues with opposite signs and the corresponding one-dimensional stable and unstable manifolds are normal to $\mathcal{L}$.*
(b) *If $0 < z - b_0 \leq 1/4$ then the equilibria E_z are (stable or unstable) node normally hyperbolic to the line $\mathcal{L}$, that is, the linearized system at E_z has two real eigenvalues with the signs opposite to the sign of the parameter a and the corresponding two-dimensional stable or unstable manifolds are normal to $\mathcal{L}$.*
(c) *If $z - b_0 > 1/4$ then the equilibria E_z are (stable or unstable) foci normally hyperbolic to the line $\mathcal{L}$, that is, the linearized system at E_z has two complex conjugate eigenvalues with the signs of the real part opposite to the sign of the parameter a and the corresponding two-dimensional stable or unstable manifolds are normal to $\mathcal{L}$.*

Proof. For $c = 0$, $a \neq 0$ and $b \in \mathbb{R}$ system (1) becomes

$$x' = a(y - x), \qquad y' = ax(b - z), \qquad z' = xy,$$

whose equilibria are $E_z = (0, 0, z)$. The characteristic polynomial of the Jacobian matrix of the above system at E_z is given by

$$p(\lambda) = \lambda\left(\lambda^2 + a\lambda - a^2(b - z)\right).$$

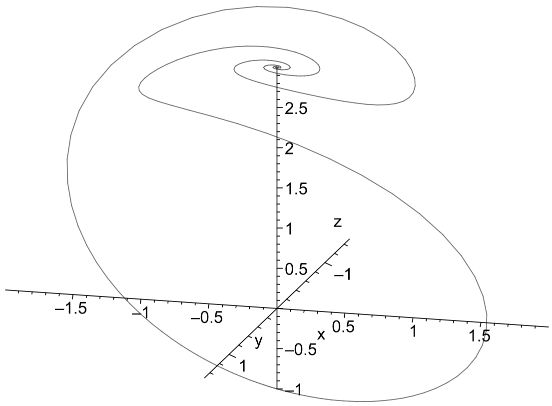

Fig. 3. Two orbits of system (1) for parameters $a = 1$, $b = 1$ and $c = 0$. Time of integration: [0, 100]. Initial conditions: (0.0005, 0.001, −1) and (−0.0005, −0.001, −1). Stepsize 0.1. The figure suggests the existence of two singularly degenerate heteroclinic cycles of system (1). Each of these cycles is formed by one of the one-dimensional unstable manifolds of the normally hyperbolic saddle $E_{z_0} = (0, 0, -1)$, which connects E_{z_0} with the normally hyperbolic focus $E_{z_1} = (0, 0, z_1)$, $z_1 > 5/4$, when $t \to \infty$ and the segment of equilibria between E_{z_0} and E_{z_1}.

Therefore the eigenvalues of the Jacobian matrix at E_z are

$$\lambda_1 = 0, \quad \lambda_{2,3} = -\frac{a}{2}\left[1 \pm \sqrt{1 + 4(b - z)}\right],$$

with corresponding eigenvectors given by

$$V_1 = (0, 0, 1), \qquad V_{2,3} = \left(\frac{-1}{2(b - z)}\left(1 \pm \sqrt{1 + 4(b - z)}\right), 1, 0\right), \quad b \neq z.$$

Fix the parameter $b = b_0$. For $z = b_0$ the corresponding eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -a$. We have three other cases to analyze:

- If $z < b_0$ then $4(b_0 - z) + 1 > 1$ and therefore the eigenvalues $\lambda_{2,3}$ are real with opposite signs. This implies that E_z is a saddle. Furthermore taking into account the corresponding eigenvectors $V_{2,3}$ the system has a normally hyperbolic saddle at E_z. This proves item (a) of the proposition.
- If $0 < z - b_0 < 1/4$ then the eigenvalues $\lambda_{2,3}$ are real and satisfy: $\lambda_{2,3} > 0$ if $a < 0$ and $\lambda_{2,3} < 0$ if $a > 0$. If $z - b_0 = 1/4$ then $\lambda_{2,3} = -a/2$. Furthermore taking into account the corresponding eigenvectors $V_{2,3}$ the system has a normally hyperbolic stable node at E_z if $a > 0$ and a normally hyperbolic unstable node at E_z if $a < 0$. Item (b) of the proposition is proved.
- If $z - b_0 > 1/4$ then the eigenvalues $\lambda_{2,3}$ are complex conjugate and satisfy: $\mathrm{Re}(\lambda_{2,3}) > 0$ if $a < 0$ and $\mathrm{Re}(\lambda_{2,3}) < 0$ if $a > 0$. Furthermore taking into account the corresponding eigenvectors $W_2 = \mathrm{Re}\,(V_2)$ and $W_3 = \mathrm{Im}\,(V_2)$ the system has a normally hyperbolic stable focus at E_z if $a > 0$ and a normally hyperbolic unstable focus at E_z if $a < 0$. This proves item (c) of the proposition. ■

Proposition 2.1 suggests possible existence of singularly degenerate heteroclinic cycles in system (1) when the parameters are $c = 0$, $a \neq 0$ and $b \in \mathbb{R}$. These cycles consist of invariant sets formed by a segment of equilibrium points together with orbits connecting two of the equilibria. For more details see [18] where firstly these cycles were studied, [13] where these cycles appear in the study of another Lorenz-like system and [19] where these cycles were studied in the Lorenz system.

Here one of these cycles is formed by one of the one-dimensional unstable manifolds of the normally hyperbolic saddle E_{z_0}, $z_0 < b_0$, which connects E_{z_0} with the normally hyperbolic focus E_{z_1}, $z_1 > b_0 + 1/4$, when $t \to \infty$ and the segment of equilibria between E_{z_0} and E_{z_1}. See Fig. 3.

Based on the analysis of Proposition 2.1 and numerical investigations one has the following conjecture.

Conjecture 2.2. *System (1) with suitable parameters $a > 0$, $b = b_0 > 0$ and $c = 0$ has infinitely many singularly degenerate heteroclinic cycles. Each of these cycles is formed by one of the one-dimensional unstable manifolds of the normally hyperbolic saddle E_{z_0}, $z_0 < b_0$, which connects E_{z_0} with the normally hyperbolic focus E_{z_1}, $z_1 > b_0 + 1/4$, when $t \to \infty$ and the segment of equilibria between E_{z_0} and E_{z_1}.*

See Section 4 where we study system (1) when the parameters are $a = 1$, $b = 1$ and c is positive and very small.

2.2.2. The case $c \neq 0$

If $c \neq 0$ then system (1) has either one or three isolated equilibria depending on the sign of the product bc. If $bc \leq 0$ then system (1) has only one equilibrium point at the origin $E_0 = (0, 0, 0)$. If $bc > 0$ then system (1) has three equilibrium points: $E_0 = (0, 0, 0)$ and $E_\pm = (\pm\sqrt{bc}, \pm\sqrt{bc}, b)$.

2.2.2.1 *Linear analysis at* E_0. Here we study the stability of the equilibrium E_0 from the linear point of view.

Proposition 2.3. *Define the following subsets of* $\mathcal{W}$:

$$\mathcal{W}_1 = \{(a, b, c) : a \neq 0, b \in \mathbb{R}, c < 0\}, \qquad \mathcal{W}_2 = \{(a, b, c) : a \neq 0, b > 0, c \neq 0\},$$
$$\mathcal{W}_3 = \{(a, b, c) : a \neq 0, b = 0, c \neq 0\}, \qquad \mathcal{W}_4 = \{(a, b, c) : a > 0, b < 0, c > 0\},$$
$$\mathcal{W}_5 = \{(a, b, c) : a < 0, b < 0, c > 0\}.$$

The following statements hold:

1. *If* $(a, b, c) \in \mathcal{W}_1$ *then the equilibrium* E_0 *is unstable;*
2. *If* $(a, b, c) \in \mathcal{W}_2$ *then the equilibrium* E_0 *is unstable;*
3. *If* $(a, b, c) \in \mathcal{W}_3$ *then the equilibrium* E_0 *is not hyperbolic and it is unstable if either* $a < 0$ *or* $c < 0$;
4. *If* $(a, b, c) \in \mathcal{W}_4$ *then the equilibrium* E_0 *locally asymptotically stable;*
5. *If* $(a, b, c) \in \mathcal{W}_5$ *then the equilibrium* E_0 *is unstable.*

Proof. The characteristic polynomial of the Jacobian matrix of system (1) at E_0 has the form

$$p(\lambda) = (\lambda + c)(\lambda^2 + a\lambda - a^2 b).$$

Therefore the eigenvalues of the Jacobian matrix of system (1) at E_0 are given by

$$\lambda_1 = -c, \qquad \lambda_{\pm} = \frac{-a}{2}\left[1 \pm \sqrt{1 + 4b}\right].$$

If $c < 0$ then $\lambda_1 = -c > 0$. This proves item 1 of the proposition. If $b > 0$ then $1 - \sqrt{1 + 4b} < 0$ and this implies that $\lambda_- > 0$. Thus item 2 of the proposition is proved. If $b = 0$ then $\lambda_- = 0$ and $\lambda_+ = -a$. This proves item 3 of the proposition. If $a > 0$, $-1/4 \leq b < 0$ and $c > 0$ then $\lambda_1 < 0$ and $\lambda_{\pm}$ are real and negative. If $a > 0$, $b < -1/4$ and $c > 0$ then $\lambda_1 < 0$ and $\lambda_{\pm}$ are complex conjugate with negative real part. In both these cases E_0 is locally asymptotically stable. So item 4 of the proposition follows. The analysis of item 5 of the proposition is immediate from item 4. ∎

From item 4 of Proposition 2.3 we have the following question: how large is the basin of attraction of the equilibrium E_0?

The following proposition gives an answer to this question.

Proposition 2.4 (*See [17]*)**.** *If* $(a, b, c) \in \mathcal{W}_4$ *then the equilibrium* E_0 *is globally asymptotically stable.*

Proof. Define the analytical function $V : \mathbb{R}^3 \to \mathbb{R}$, $V(x, y, z) = -bx^2 + y^2 + az^2$. From elementary calculations one has $V(E_0) = 0$, $V(x, y, z) > 0$ if $(x, y, z) \neq E_0$ and $\dot{V}(x, y, z) = 2a(bx^2 - cz^2) < 0$ if $(x, y, z) \neq E_0$. Therefore V is a strict Lyapunov function for E_0 and the proposition is proved. ∎

2.2.2.2 *Linear analysis at* $E_{\pm}$. Consider the map

$$\mathcal{R} : \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathcal{R}(x, y, z) = (-x, -y, z).$$

It follows that $\mathcal{R}(f(\mathbf{x}, \boldsymbol{\zeta})) = f(\mathcal{R}(\mathbf{x}), \boldsymbol{\zeta})$, for all $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and $\boldsymbol{\zeta} = (a, b, c) \in \mathbb{R}^3$. Therefore system (1) is invariant under the change of coordinates $(x, y, z) \mapsto (-x, -y, z)$. So the stability of the equilibrium E_- can be obtained from the stability of E_+.

For the sake of completeness we state the following lemma (Routh–Hurwitz stability criterion) whose proof can be found in [20], p. 58.

Lemma 2.5. *The polynomial* $L(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$ *with real coefficients has all roots with negative real parts if and only if the numbers* p_1, p_2, p_3 *are positive and the inequality* $p_1 p_2 > p_3$ *is satisfied.*

We have the following proposition.

Proposition 2.6. *Define the following subsets of* $\mathcal{W}$:

$$\mathcal{V}_1 = \{(a, b, c) : a \neq 0, b > 0, c > 0\}, \qquad \mathcal{V}_2 = \{(a, b, c) : a \neq 0, b < 0, c < 0\},$$
$$\mathcal{V}_{11} = \{(a, b, c) \in V_1 : a < 0, b > 0, c > 0\},$$
$$\mathcal{V}_{12} = \{(a, b, c) \in V_1 : a > 0, b > 0, 0 < c < a(b - 1)/(b + 1)\},$$
$$\mathcal{V}_{13} = \{(a, b, c) \in V_1 : a > 0, b > 0, c > a(b - 1)/(b + 1)\},$$
$$\mathcal{H} = \{(a, b, c) \in V_1 : a > 0, b > 0, c = a(b - 1)/(b + 1)\}.$$

So one has $\mathcal{V}_1 = \mathcal{V}_{11} \cup \mathcal{V}_{12} \cup \mathcal{V}_{13} \cup \mathcal{H}$. *The following statements hold:*

1. *If* $(a, b, c) \in \mathcal{V}_{11}$ *then the equilibrium* E_+ *is unstable;*
2. *If* $(a, b, c) \in \mathcal{V}_{12}$ *then the equilibrium* E_+ *is unstable;*

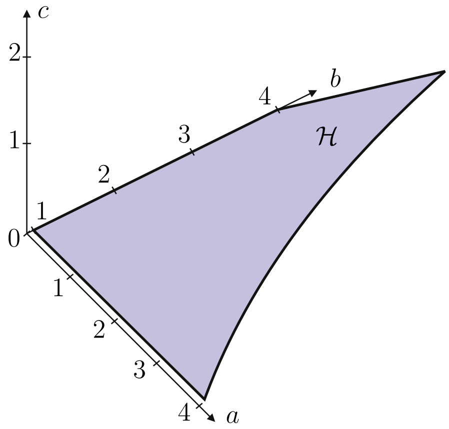

Fig. 4. The Hopf surface $\mathcal{H}$ of the equilibrium E_+: $a > 0, b > 1$ and $c = c_*$ (5).

3. *If* $(a, b, c) \in \mathcal{V}_{13}$ *then the equilibrium* E_+ *is locally asymptotically stable;*
4. *If* $(a, b, c) \in \mathcal{H}$ *then the equilibrium* E_+ *is unstable;*
5. *If* $(a, b, c) \in \mathcal{V}_2$ *then the equilibrium* E_+ *is unstable.*

Proof. The characteristic polynomial of the Jacobian matrix of system (1) at E_+ has the form

$$p(\lambda) = \lambda^3 + (a + c)\lambda^2 + ac(b + 1)\lambda + 2a^2bc.$$

Suppose $(a, b, c) \in \mathcal{V}_1$. If $(a, b, c) \in \mathcal{V}_{11}$ then the coefficient $ac(b + 1)$ of $p(\lambda)$ is negative. From Lemma 2.5 it follows that the equilibrium E_+ is unstable. This proves item 1 of the proposition. From Lemma 2.5 the equilibrium E_+ is locally asymptotically stable if the coefficients of the characteristic polynomial satisfy

$$a + c > 0, \qquad ac(b + 1) > 0, \qquad a^2bc > 0, \qquad (a + c)ac(b + 1) > 2a^2bc. \tag{4}$$

The last inequality in (4) can be written as $c > a(b - 1)/(b + 1) > 0$. So if $(a, b, c) \in \mathcal{V}_{12}$ then E_+ is unstable and if $(a, b, c) \in \mathcal{V}_{13}$ then E_+ is locally asymptotically stable. This proves item 2 and 3 of the proposition. Item 4 of the proposition will be proved in the next section (see Theorem 3.1) with the calculation of the first Lyapunov coefficient at the equilibrium E_+ for parameters in $\mathcal{H}$. In fact, when $(a, b, c) \in \mathcal{H}$ the equilibrium E_+ is a Hopf point, that is, the characteristic polynomial of the Jacobian matrix of system (1) at E_+ has eigenvalues of the form

$$\lambda_1 = -\frac{2ab}{b + 1} < 0, \qquad \lambda_{2,3} = \pm i a\sqrt{b - 1}.$$

It remains only to verify item 5 of the proposition, that is $(a, b, c) \in \mathcal{V}_2$. If $a < 0$ then $a + c < 0$ and this implies that E_+ is unstable according to the first inequality of (4). So $a > 0$ and $ac < 0$. The last inequality of (4) can be written as

$$(a + c)(b + 1) < 2ab \Leftrightarrow a + ab + c + bc < 2ab \Leftrightarrow a + c < ab - bc = b(a - c).$$

Thus $b(a - c) > a + c > 0$. As $b < 0$ one has $a - c < 0$ and this implies that $a < c < 0$. This proves item 5 of the proposition. ■

Define

$$c_* = a(b - 1)/(b + 1) > 0. \tag{5}$$

If $(a, b, c) \in \mathcal{H}$ then the Jacobian matrix of (3) at E_+ has one negative real eigenvalue λ_1 and a pair of purely imaginary eigenvalues $\lambda_{2,3}$. The set $\mathcal{H}$ of the above proposition is called the Hopf surface of the equilibrium E_+. See Fig. 4. From the Center Manifold Theorem, at a Hopf point a two dimensional center manifold is well-defined, it is invariant under the flow generated by (1) and can be continued with arbitrary high class of differentiability to nearby parameter values (see [21], p. 152). This center manifold is attracting since $\lambda_1 < 0$. So it is enough to study the stability of E_+ for the flow restricted to the family of parameter-dependent continuations of the center manifold.

3. Bifurcation analysis in system (1)

The beginning of this section is a review of the projection method described in [21] for the calculation of the first Lyapunov coefficient associated to Hopf bifurcations, denoted by l_1. This method can be extended to the calculation of the other Lyapunov coefficients. See [21] for the calculation of the second Lyapunov coefficient and [22,23] for the calculation of the third and the fourth Lyapunov coefficients, respectively.

Consider the differential equation

$$\mathbf{x}' = f(\mathbf{x}, \boldsymbol{\zeta}), \tag{6}$$

where $\mathbf{x} \in \mathbb{R}^3$ and $\boldsymbol{\zeta} \in \mathbb{R}^3$ are respectively vectors representing phase variables and control parameters. Assume that f is of class C^∞ in $\mathbb{R}^3 \times \mathbb{R}^3$. Suppose that (6) has an equilibrium point $\mathbf{x} = \mathbf{x_0}$ at $\boldsymbol{\zeta} = \boldsymbol{\zeta_0}$ and, denoting the variable $\mathbf{x} - \mathbf{x_0}$ also by $\mathbf{x}$, write

$$F(\mathbf{x}) = f(\mathbf{x}, \boldsymbol{\zeta_0}) \tag{7}$$

as

$$F(\mathbf{x}) = A\mathbf{x} + \frac{1}{2}B(\mathbf{x}, \mathbf{x}) + \frac{1}{6}C(\mathbf{x}, \mathbf{x}, \mathbf{x}) + O(\|\mathbf{x}\|^4), \tag{8}$$

where $A = f_{\mathbf{x}}(0, \boldsymbol{\zeta_0})$ and, for $i = 1, 2, 3$,

$$B_i(\mathbf{x}, \mathbf{y}) = \sum_{j,k=1}^{3} \left.\frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k}\right|_{\xi=0} x_j\, y_k, \qquad C_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{j,k,l=1}^{3} \left.\frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}\right|_{\xi=0} x_j\, y_k z_l.$$

Suppose that $(\mathbf{x_0}, \boldsymbol{\zeta_0})$ is an equilibrium point of (6) where the Jacobian matrix A has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$, and admits no other eigenvalue with zero real part. Let T^c be the generalized eigenspace of A corresponding to $\lambda_{2,3}$. By this it is meant the largest subspace invariant by A on which the eigenvalues are $\lambda_{2,3}$.

Let $p, q \in \mathbb{C}^3$ be vectors such that

$$Aq = i\omega_0 q, \qquad A^\top p = -i\omega_0 p, \qquad \langle p, q\rangle = \sum_{i=1}^{3} \bar{p}_i q_i = 1, \tag{9}$$

where $A^\top$ is the transpose of the matrix A. Any vector $y \in T^c$ can be represented as $y = wq + \bar{w}\bar{q}$, where $w = \langle p, y\rangle \in \mathbb{C}$. The two dimensional center manifold associated to the eigenvalues $\lambda_{2,3} = \pm i\omega_0$ can be parameterized by the variables w and $\bar{w}$ by means of an immersion of the form $\mathbf{x} = H(w, \bar{w})$, where $H : \mathbb{C}^2 \to \mathbb{R}^3$ has a Taylor expansion of the form

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2\le j+k\le 3} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O(|w|^4), \tag{10}$$

with $h_{jk} \in \mathbb{C}^3$ and $h_{jk} = \bar{h}_{kj}$. Substituting this expression into (6) we obtain the following differential equation

$$H_w w' + H_{\bar{w}} \bar{w}' = F(H(w, \bar{w})), \tag{11}$$

where F is given by (7). The complex vectors h_{ij} are obtained solving the system of linear equations defined by the coefficients of (11), taking into account the coefficients of F (see Remark 3.1 of [22], p. 27), so that system (11), on the chart w for a central manifold, writes as follows

$$w' = i\omega_0 w + \frac{1}{2}\, G_{21} w|w|^2 + O(|w|^4),$$

with $G_{21} \in \mathbb{C}$.

The *first Lyapunov coefficient* l_1 is defined by

$$l_1 = \frac{1}{2}\mathrm{Re}\; G_{21}, \tag{12}$$

where $G_{21} = \langle p, \mathcal{H}_{21}\rangle$, $\mathcal{H}_{21} = C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11})$, $h_{11} = -A^{-1}B(q, \bar{q})$, $h_{20} = (2i\omega_0 I_3 - A)^{-1}B(q, q)$ and I_3 is the unit 3×3 matrix.

A Hopf point is called *transversal* if the parameter dependent complex eigenvalues cross the imaginary axis with non-zero derivative. In a neighborhood of a transversal Hopf point with $l_1 \neq 0$ the dynamic behavior of the system (6), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to the following complex normal form $w' = (\eta + i\omega)w + l_1 w|w|^2$, where $w \in \mathbb{C}$, η, ω and l_1 are real functions having derivatives of arbitrary higher order, which are continuations of 0, ω_0 and the first Lyapunov coefficient at the Hopf point. See [21,22] for details. When $l_1 < 0$ ($l_1 > 0$) one family of stable (unstable) periodic orbits can be found on this family of manifolds, shrinking to an equilibrium point at the Hopf point.

In the rest of this section we study the stability of E_+ for parameters in $\mathcal{H}$. We have the following theorem.

Theorem 3.1. *Consider system (1). The first Lyapunov coefficient at E_+ for parameter values in $\mathcal{H}$ is given by*

$$l_1(a, b, c_*) = \frac{a(b-1)\left(6 + b\left(b^3 + b^2 + b + 9\right)\right)}{b(1 + b(2 + b(-6 + b(-9 + b(8 + b(b+7))))))}. \tag{13}$$

As $b > 1$ then $l_1(a, b, c_*) > 0$ for all parameters $(a, b, c) \in \mathcal{H}$ and system (1) has a transversal Hopf point at E_+ for $(a, b, c) \in \mathcal{H}$. More specifically, the Hopf point at E_+ is unstable (weak repelling focus) and for each $c > c_*$, but close to c_*, there exists an unstable limit cycle near the asymptotically stable equilibrium point E_+. See Fig. 5.

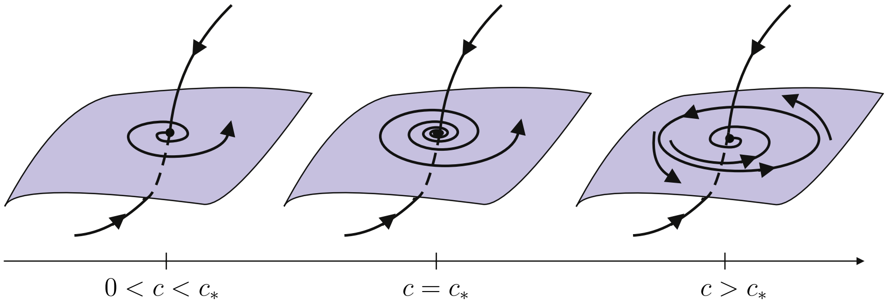

Fig. 5. Bifurcation diagrams of system (1): if $0 < c < c_*$ then E_+ is unstable; if $c = c_*$ then E_+ is an weak unstable focus; if $c > c_*$ then E_+ is stable and an unstable limit cycle appears from the Hopf bifurcation.

Proof. For parameters on the Hopf surface $\mathscr{H}$ one has:

$$\lambda_1 = -\frac{2ab}{b+1}, \qquad \lambda_{2,3} = \pm \mathrm{i}\omega_0, \qquad \omega_0 = a\sqrt{b-1}, \quad b > 1,$$
$$q = \left(\frac{\sqrt{\frac{a(b-1)b}{b+1}}}{b - \mathrm{i}\sqrt{b-1} - 1}, \mathrm{i}\sqrt{\frac{ab}{b+1}}, 1\right),$$
$$p = (p_1, p_2, p_3), \qquad p_1 = \frac{b\left(b^2-1\right)}{2\sqrt{ab(b+1)}\left(-\mathrm{i}b^2 + 2\sqrt{b-1}b + \mathrm{i}\right)},$$
$$p_2 = \frac{\left(\sqrt{b-1} + 2\mathrm{i}\right)\sqrt{ab\left(b^2-1\right)}}{4a\sqrt{b-1}b - 2\mathrm{i}a\left(b^2-1\right)}, \qquad p_3 = \frac{\left(-\mathrm{i}b + \sqrt{b-1} + \mathrm{i}\right)(b+1)}{-2\mathrm{i}b^2 + 4\sqrt{b-1}b + 2\mathrm{i}},$$
$$B(\mathbf{x}, \mathbf{y}) = (0, -a(x_1\, y_3 + x_3\, y_1), x_1\, y_2 + x_2\, y_1), \qquad C(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (0, 0, 0).$$

The expressions of the complex vectors h_{11}, h_{20} and the complex number G_{21} are too long to be put in print. Performing the calculations in (12) one has the first Lyapunov coefficient given by (13). The first Lyapunov coefficient (13) is positive since the numerator of (13) is positive ($b > 1$) and the denominator is also positive ($b > 1$).

It remains only to verify the transversality condition of the Hopf bifurcation. In order to do so, consider the family of differential equation (1) regarded as dependent on the parameter c. The real part, $\eta = \eta(c)$, of the pair of complex eigenvalues at the critical parameter $c = c_*$ verifies

$$\eta'(c_*) = \mathrm{Re}\left\langle p, \left.\frac{\mathrm{d}A}{\mathrm{d}c}\right|_{c=c_*} q\right\rangle = \frac{2b^2}{b(b(b+5)-1)-1} - \frac{1}{2} < 0,$$

since the first quotient is smaller that $1/2$. In the above expression A is the Jacobian matrix of system (1) at E_+. Therefore, the transversality condition at the Hopf point holds. ∎

4. Chaotic behavior of system (1) for small c

In this section we give some numerical evidence for the chaotic behavior of system (1) when the parameters a and b are fixed and c is positive and very small. The idea is to find evidence for the existence of chaotic attractors for system (1) which bifurcate from the singular degenerate heteroclinic cycles when the parameter $c = 0$. See Section 2.2.1, Fig. 3 and Proposition 2.1.

In Fig. 6 is illustrated the phase portrait of system (1) when the parameter values are $a = 1, b = 1$ and $c = 0.08$. In Fig. 7 are presented the projections of this phase portrait in the planes xy and xz. These figures suggest the existence of an chaotic attractor which appears from the bifurcation of the singularly degenerate heteroclinic cycles of system (1). See also Fig. 3 when the parameter values are $a = 1, b = 1$ and $c = 0$.

The choice of the value 0.08 for the parameter c is based on the analysis of the Lyapunov exponents. The Lyapunov exponent spectrum of system (1) in terms of the parameter c for fixed parameters $a = 1$ and $b = 1$ is depicted in Fig. 8. The parameter c is varied in the interval [0, 0.2]. This figure corroborates with the idea of the existence of chaotic behaviors in system (1) which appear from the bifurcation of the singularly degenerate heteroclinic cycles. In Fig. 8 we consider a transient with 1000 iterations and use 10,000 iterations for the calculation of the Lyapunov exponents according to [24,25]. Note that for $c = 0.08$ one has the following Lyapunov exponents: $\lambda_1 = 0.02921002$, $\lambda_2 = 0.00001420$ and $\lambda_3 = -1.10906736$. For the same parameters one also has the following Kaplan–Yorke dimension: $D_{KY} = 2.13642$.

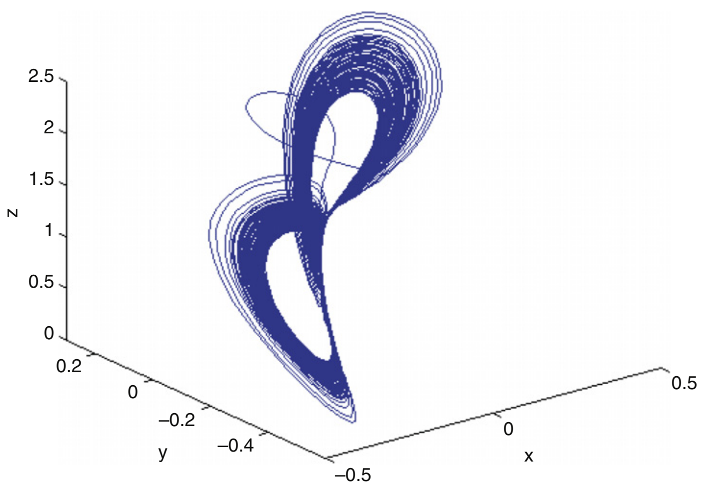

Fig. 6. Phase portrait of system (1) for parameters $a = 1$, $b = 1$ and $c = 0.08$. Time of integration: [0, 1000]. Initial conditions: (0, 0.5, 1.5), (1, −0.2, 1) and (−0.3, −0.3, 2). Stepsize 0.01. The figure suggests the existence of an chaotic attractor which appears from the bifurcation of the singularly degenerate heteroclinic cycles of system (1). See two projections of this phase portrait in Fig. 7.

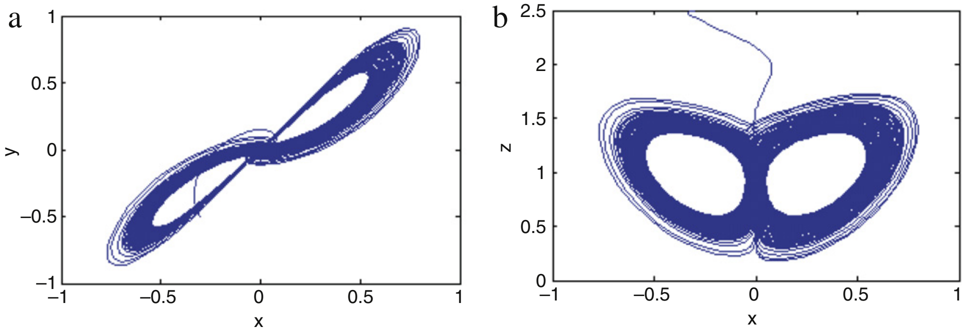

Fig. 7. Projections of the phase portrait of system (1) (see Fig. 6) on the planes *xy* (a) and *xz* (b).

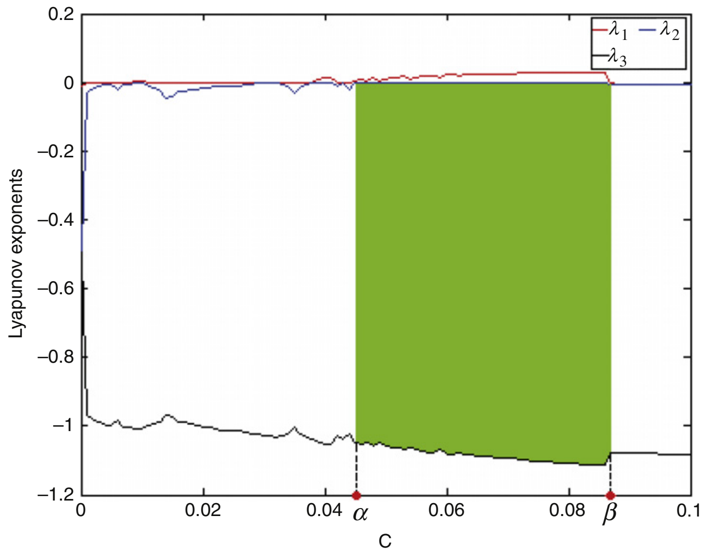

Fig. 8. Lyapunov exponent spectrum of system (1) in terms of the parameter c for fixed parameters $a = 1$ and $b = 1$. The parameter c is varied in the interval [0, 0.2]. The figure suggests the existence of chaotic behaviors in system (1) which appear from the bifurcation of the singularly degenerate heteroclinic cycles. The red line, blue one and black one respectively stand for the Lyapunov exponents λ_1, λ_2 and λ_3. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

With the analysis performed here one can find the chaotic phenomenon in the green region of Fig. 8. In this region system (1) possesses one positive Lyapunov exponent along with one zero and one negative Lyapunov exponents. The chaotic attractor emerges as $c = \alpha = 0.045011$, with further increase of c, chaotic attractor no longer exists while $c > \beta = 0.085976$.

Note that system (1) has three equilibria when the parameters are $a = 1$, $b = 1$ and $c = 0.08$. The equilibrium E_0 is unstable since $(a, b, c) = (1, 1, 0.08) \in \mathcal{W}_2$ (see Proposition 2.3). The equilibria $E_{\pm}$ are locally asymptotically stable since $(a, b, c) = (1, 1, 0.08) \in \mathcal{V}_{13}$ (see Proposition 2.6). In other words the chaotic behavior of system (1) coexists with two locally asymptotically stable equilibria. As far as we know this analysis was performed at the first time in [26].

5. Concluding remarks

This paper starts reviewing the linear stability analysis which accounts for the characterization, in the space of parameters, of the Lyapunov stability of the equilibria of system (1). It continues with the extension of the analysis to the first order, codimension one points, based on the calculation of the first Lyapunov coefficient (see Theorem 3.1).

With the analytic data provided in the analysis performed here, the bifurcation diagrams are established. Figs. 4 and 5 provide a qualitative synthesis of the dynamical conclusions achieved at the parameter values where the system (1) has the most complex isolated equilibrium points.

We have also computed the stability of equilibria along the straight line $\mathcal{L}$ for the parameter value $c = 0$ (see Proposition 2.1). These results suggest the existence of infinitely many singularly degenerate heteroclinic cycles (see Conjecture 2.2). We intend to address this problem in a future work.

We give some numerical evidences – via Lyapunov exponents – for the chaotic behavior of system (1) when the parameters a and b are fixed and c is positive and very small.

This article is finished displaying a difference between system (1) and the Lorenz system: for the non-classical parameters $\sigma = 0$, $b = -1$ and $r < -1$ the Lorenz system has a codimension two bifurcation at the nontrivial equilibria $Q_{\pm}$. In fact, for these parameter values the eigenvalues of the Jacobian matrix are $\lambda_1 = 0$ and $\lambda_{2,3} = \pm i\sqrt{-r}$ (fold-Hopf bifurcation [21]). As we saw this type of bifurcation does not appear in system (1).

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