Chapter 8

Markowitz Portfolio Theory

8.1 Expected Returns and Covariance

The main question in portfolio theory is the following:
Given an initial capital $V(0)$, and opportunities (buy or sell) in N securities for investment, how would you allocate the capital so that the return on the portfolio is optimal in certain way?
More specifically, what we are looking for is a collection of weights:

$$w_1, w_2, \ldots, w_N$$

with $w_1 + w_2 + w_3 + \cdots + w_N = 1$, and $w_i V(0)$ invested in security i for $i = 1, 2, \ldots, N$, such that the return of the portfolio is optimal in certain sense. Our common sense suggests that you will need to take more risk if you seek high expected returns. On the other hand, investors are always risk averse in the sense that they demand the largest expected return, given the risk level, or the least possible risk, given the expected return. This leads to the concept of efficient portfolios. An *efficient portfolio* is a portfolio having simultaneously the smallest possible risk for its given level of expected return and the largest possible expected return for its given level of risk. The collection of all efficient portfolios is called the *efficient frontier*.
Given the weights $w_1, \ldots, w_N$, the number of shares to invest in security i is

$$n_i = \frac{w_i V(0)}{S_i(0)} \tag{8.1}$$

and the value of the portfolio at T is

$$V(T) = \sum_{i=1}^{N} n_i S_i(T)$$

so the return of the portfolio over $[0, T]$ is

$$R_{[0,T]} = \sum_{i=1}^{N} w_i R_i \tag{8.2}$$

where $R_i = (S_i(T) - S_i(0))/S_i(0)$ is the return of security i. We can see that the return of the portfolio is a linear combination of the returns of individual securities. The study of the portfolio return is therefore a study of various linear combinations of a collection of random variables.
Sometimes it is more convenient to use the log return:

$$r_i = \log\left(\frac{S_i(T)}{S_i(0)}\right) \tag{8.3}$$

which has the useful property: suppose $0 = t_0 < t_1 < t_2 < \cdots < t_m = T$, the *log return* over $[0, T]$

$$r = \log\left(\frac{S(T)}{S(0)}\right) = \log\left(\frac{S(t_1)}{S(t_0)} \cdot \frac{S(t_2)}{S(t_1)} \cdots \frac{S(t_m)}{S(t_{m-1})}\right) = r^{(1)} + r^{(2)} + \cdots + r^{(m)}$$

where $r^{(k)}$ is the log return over (t_{k-1}, t_k). It should be pointed out that for short period of time,

$$r(t, t+\Delta t) = \log\left(\frac{S(t+\Delta t)}{S(t)}\right) = \log\left(R(t, t+\Delta t) + 1\right) \approx R(t, t+\Delta t) \tag{8.4}$$

So quite often we do not distinguish between these two returns, as long as the time period is short.

8.2 Assumptions of Markowitz Theory

Before we begin the discussion on the Markowitz theory, we state some assumptions for the market:

- Investors are rational.
- The supply and demand equilibrium is instantly achieved.
- There are no arbitrage opportunities.
- Access to information is available to all participants.
- Price moves are efficient.
- The market is liquid.
- There is no transaction cost.
- There are no taxes.
- Everyone has the same opportunity of borrowing and lending.

We also establish some facts about the return vector R. There is a financial implication relating to the concept of linear independence: there is no redundant security in the portfolio, that is, there is no security in the portfolio that has a return as a linear combination of returns from other securities in the portfolio. The consequence of this is that the covariance matrix V is invertible. Since V is always semi-positive definite, the additional property that V is invertible implies that V is positive definite.
How do we describe the portfolio risk? We should consider the variance of the return of the portfolio, which consists of weighted sum of the securities variances and the weighted sum of securities covariances, in the form of $\mathbf{w}^T V \mathbf{w}$, where $\mathbf{w} = (w_1, w_2, \ldots, w_N)^T$ is the weight vector.

8.3 Two-Security Portfolio Theory

Assumptions: two securities with $\mu_1 \neq \mu_2, \sigma_1 < \sigma_2$ and correlation $-1 < \rho < 1$, and we use

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad V = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \tag{8.5}$$

We can calculate the portfolio return and variance

$$\mu_P = w_1\mu_1 + w_2\mu_2 = \boldsymbol{\mu}^T \mathbf{w} \tag{8.6}$$

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2 = \mathbf{w}^T V \mathbf{w} \tag{8.7}$$

Suppose we target a portfolio with portfolio return μ_P, the minimization problem is to minimize σ_P subject to $\mathbf{w}^T\mathbf{e} = 1$, where $\mathbf{e} = (1, 1)^T$ and $\mathbf{w}^T\boldsymbol{\mu} = \mu_P$. This problem is quite easy to solve as $\mathbf{w}$ can be determined from the two constraints first:

$$\mathbf{w} = \begin{pmatrix} w_\mu \\ 1 - w_\mu \end{pmatrix}, \quad w_\mu = \frac{\mu_P - \mu_2}{\mu_1 - \mu_2} \tag{8.8}$$

which leads to the portfolio variance satisfying

$$\sigma_P^2 = A\mu_P^2 + B\mu_P + C \tag{8.9}$$

for some constants A, B, and C. It can be shown that $A > 0$ and $C > 0$, therefore if we trace all these pairs (σ_P, μ_P), we will find a hyperbola. With this information we will try to determine the efficient frontier from this hyperbola. This is a hyperbola with opening to the right, and we can find the turning point, which corresponds to the absolute minimum for σ_P:

$$\sigma_G = \sqrt{\frac{\det C}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}, \quad \mu_G = \frac{(\sigma_2^2 - \rho\sigma_1\sigma_2)\mu_1 + (\sigma_1^2 - \rho\sigma_1\sigma_2)\mu_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{8.10}$$

In figure 8.1, we use an example where $\mu_1 = 0.1,\ \mu_2 = 0.15,\ \sigma_1 = 0.2,\ \sigma_2 = 0.4$, and $\rho = 0.5$, and plot the Markowitz efficient frontier. The minimum-variance portfolio has $\mu_G = 0.1$ and $\sigma_G = 0.2$ which is the portfolio with $w_1 = 1$ and $w_2 = 0$.
We can make the following remarks:

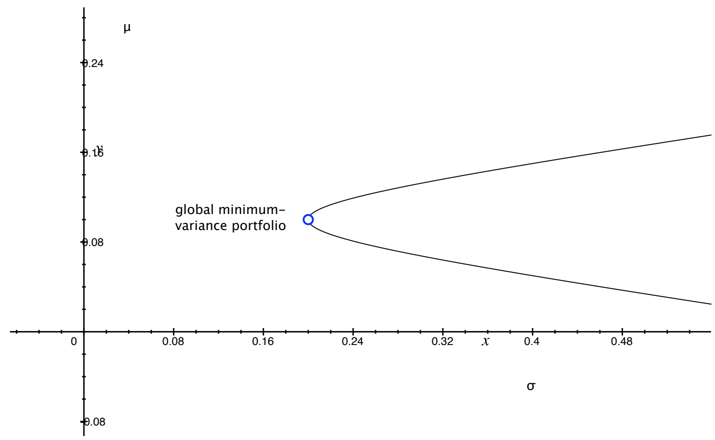

Figure 8.1: Markowitz Efficient Frontier for Two Security Portfolios

- The efficient frontier consists of the upper branch of the hyperbola, including the turning point.
- We can in general reduce risk through diversification if we have reliable correlation information.
- Riskless portfolios exist only in the case $\rho = \pm 1$.
- In the case $\rho = 0$, we can create portfolios with

$$0 < \sigma_G \leq \min\{\sigma_1, \sigma_2\}$$

  and this portfolio requires no short selling ($0 < w_G < 1$).

8.4 Efficient Frontier for N-Securities with Short Selling

Assumptions:

- $\sigma_i > 0$ for $i = 1, \ldots, N$.
- Expected returns μ_i's are distinct, and risk levels σ_i's are distinct.
- Unlimited short sales are allowed.

Given $w_1, w_2, \ldots, w_N$, we can express the portfolio return and risk as

$$\mu_P(\mathbf{w}) = \mathbf{w}^T \boldsymbol{\mu}, \quad \sigma_P(\mathbf{w}) = \sqrt{\mathbf{w}^T V \mathbf{w}}$$

The following will be needed in the calculations:

$$\begin{aligned}
A &= \mathbf{e}^T V^{-1}\mathbf{e} > 0 \\
B &= \boldsymbol{\mu}^T V^{-1}\mathbf{e} \\
C &= \boldsymbol{\mu}^T V^{-1}\boldsymbol{\mu} > 0 \\
AC &- B^2 > 0
\end{aligned}$$

The minimization problem we want to solve is

$$\text{minimize} \quad \frac{1}{2}\mathbf{w}^T V \mathbf{w} \tag{8.11}$$

$$\text{subject to} \quad \mathbf{w}^T\mathbf{e} = 1 \text{ and } \mathbf{w}^T\boldsymbol{\mu} = \mu \tag{8.12}$$

We cannot determine $\mathbf{w}$ from these two constraints like in the 2-security case. In fact, there will be infinitely many portfolios satisfying these two constraints. To solve this problem, we use the Langrange multiplier approach. The Lagrange function is

$$\mathcal{L}(\mathbf{w}, \lambda) = f(\mathbf{w}) + \lambda_1(1 - \mathbf{w}^T\mathbf{e}) + \lambda_2(\mu - \mathbf{w}^T\boldsymbol{\mu}) = f(\mathbf{w}) + \boldsymbol{\lambda}^T\mathbf{h}(\mathbf{w}) \tag{8.13}$$

where

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 1 - \mathbf{w}^T\mathbf{e} \\ \mu - \mathbf{w}^T\boldsymbol{\mu} \end{pmatrix} \tag{8.14}$$

To solve the Lagrange multiplier problem, we differentiate the Lagrange function with respect to each component of $\mathbf{w}$ and $\boldsymbol{\lambda}$. The derivatives with respect to $\mathbf{w}$ yields

$$\frac{\partial \mathcal{L}}{\partial w_j} = 0 \quad \Rightarrow \quad V\mathbf{w} = \lambda_1\mathbf{e} + \lambda_2\boldsymbol{\mu} \quad \Rightarrow \mathbf{w} = \lambda_1 V^{-1}\mathbf{e} + \lambda_2 V^{-1}\boldsymbol{\mu} \tag{8.15}$$

The other two derivatives give

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = 0 \quad \Rightarrow \quad 1 - \mathbf{w}^T\mathbf{e} = 0 \tag{8.16}$$

and

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = 0 \quad \Rightarrow \quad \mu - \mathbf{w}^T\boldsymbol{\mu} = 0 \tag{8.17}$$

Using Eq.(8.15) in these two equations, we obtain the following 2×2 system for λ_1 and λ_2:

$$(\mathbf{e}^T V^{-1}\mathbf{e})\lambda_1 + (\boldsymbol{\mu}^T V^{-1}\mathbf{e})\lambda_2 = 1 \tag{8.18}$$

$$(\mathbf{e}^T V^{-1}\boldsymbol{\mu})\lambda_1 + (\boldsymbol{\mu}^T V^{-1}\boldsymbol{\mu})\lambda_2 = 1 \tag{8.19}$$

which gives

$$\lambda_1 = \frac{C - \mu B}{AC - B^2}, \quad \lambda_2 = \frac{\mu A - B}{AC - B^2} \tag{8.20}$$

Therefore the solution for $\mathbf{w}$ is the minimum-variance portfolio weight vector

$$\mathbf{w}_\mu = \left(\frac{C - \mu B}{AC - B^2}\right) V^{-1}\mathbf{e} + \left(\frac{\mu A - B}{AC - B^2}\right) V^{-1}\boldsymbol{\mu} \tag{8.21}$$

The efficient frontier is determined from the hyperbola

$$\sigma_P^2(\mu) = \frac{A\mu^2 - 2B\mu + C}{AC - B^2} \tag{8.22}$$

Similar to the 2-security case, the turning point gives the global minimum-variance portfolio:

$$(\sigma_G, \mu_G) = \left(\frac{1}{\sqrt{A}}, \frac{B}{A}\right), \quad \mathbf{w}_G = \frac{V^{-1}\mathbf{e}}{A} \tag{8.23}$$

The set

$$F_{P,N} = \left\{(\sigma_P(\mathbf{w}), \mu_P(\mathbf{w})) : \mathbf{w}^T\mathbf{e} = 1\right\}$$

is called the *feasible set*, where each point corresponds to a portfolio with the constraints met.
In the case $N = 2$, the feasible set is just a one-parameter set that is the hyperbola itself. In the cases $N \geq 3$, it is represented by the region enclosed and including the hyperbola show in the following graph.

8.5 Efficient Frontier N-Securities without Short Selling

If short selling is not allowed, the weight vector has additional constraints so the space of weights become

$$W_N^* = \left\{\mathbf{w} : \mathbf{w}^T\mathbf{e} = 1,\ w_i \geq 0, i = 1, \ldots, N\right\}$$

The optimization problems with this nonlinear constraint are much more difficult than the original optimization problem with linear constraints, and in most situations some nonlinear programming methods are needed to solve this problem.

8.6 Mutual Fund Theorem

In Eq.(8.20), we wonder if there are points (σ, μ) on the efficient frontier that correspond to particular λ_1 and λ_2 values. In particular, we ask what happens when $\lambda_1 = 0$ or $\lambda_2 = 0$. In the first case, we have

$$\mu = \frac{C}{B} = \mu_D, \quad \mathbf{w} = \frac{V^{-1}\boldsymbol{\mu}}{B} = \mathbf{w}_D, \quad \sigma^2 = \frac{C}{B^2} = \sigma_D^2$$

For reasons that become clear later, we call this portfolio the *diversified portfolio* so the solution above is denoted by a subscript D. In the second case when we consider the portfolio that corresponds to $\lambda_2 = 0$, we have

$$\mu = \frac{B}{A} = \mu_G, \quad \mathbf{w} = \frac{V^{-1}\mathbf{e}}{A} = \mathbf{w}_G, \quad \sigma^2 = \sigma_G^2$$

that means that $\lambda_2 = 2$ corresponds to the global minimum variance portfolio. Are these two portfolios so special that we can obtain any other portfolio on the frontier as a linear combination of these two portfolios? A straightforward calculation shows that given any required expected return μ, we can find

$$a_\mu = \frac{A(C - \mu B)}{AC - B^2}$$

such that the portfolio with the following weight vector

$$\mathbf{w}_\mu = a_\mu \mathbf{w}_G + (1 - a_\mu)\mathbf{w}_D$$

will have the desired expected return μ. This is called a mutual fund theorem or separation theorem.

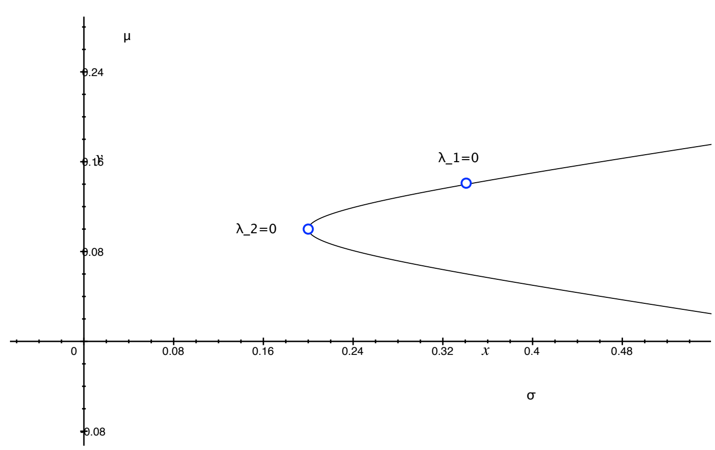

Figure 8.2: Two Mutual Fund Portfolios

As it turns out, this can be achieved with any two portfolios on the frontier so the more general mutual fund theorem states: *Any minimum variance portfolio* $\mathbf{w}$ *can be expressed in terms of any two distinct minimum variance portfolios*

$$\mathbf{w} = s_1 \mathbf{w}_a + s_2 \mathbf{w}_b$$

where $\mathbf{w}_a \neq \mathbf{w}_b$*, and* s_1 *and* s_2 *satisfying* $s_1 + s_2 = 1$ *can be calculated by certain formula similar to the formula for* a_μ*.*

The significance of the mutual fund theorem is that if you want to construct a portfolio with a required μ, you don't have to build from scratch to pick all the individual securities according to the weight $\mathbf{w}_\mu$. All you have to do is to invest in two minimum variance portfolios (mutual funds) with the weight vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ corresponding to two points on the frontier, and calculate the allocation s_1 and s_2 similar to a_μ. In another word, you would let people do the work for you.

Bibliography

[1] A. O. Petters and X. Dong, *An Introduction to Mathematical Finance and Applications.* Springer Undergraduate Texts in Mathematics and Technology, 2016.