mdg2197

Done in collaboration with Alisha

1. $\max \; 2x_1 - x_2 + x_3$

s.t.

$3x_1 + x_2 + x_3 + x_4 = 60$

$x_1 - x_2 + 2x_3 + x_5 = 10$

$x_1 + x_2 - x_3 + x_6 = 20$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$.

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$x = [0 \;\; 0 \;\; 0 \;\; 60 \;\; 10 \;\; 20]^T$

basis variables (4, 5, 6)

Tableau form

z	x_1	x_2	x_3	x_4	x_5	x_6	basic variable	ratio test
1	-2	+1	-1	0	0	0 = 0		
	3	1	1	1	0	0 = 60	$x_4 = 60$	
	1	-1	2	0	1	0 = 10	$x_5 = 10$	
	1	1	-1	0	0	1 = 20	$x_6 = 20$	

$BV = \{x_4, x_5, x_6\}$. $NBV = \{x_1, x_2, x_3\}$

Lets make x_1 basic

(B.V → basic variable
R.T → ratio test)

Z	x_1	x_2	x_3	x_4	x_5	x_6	B.V.	R.T.
1	-2	+1	-1	0	0	0 = 0		
	3	1	1	1	0	0 = 60	x_4	$x_1 \le 20$.
	1	-1	2	0	1	0 = 10	x_5	$x_1 \le 10$
	1	1	-1	0	0	1 = 20	x_6	$x_1 \le 20$.

⇒ pivoting from {4, 5, 6} → {4, 1, 6}

Z	x_1	x_2	x_3	x_4	x_5	x_6	B.V	R.T.
1	0	-1	3	0	2	0 = 20		
	0	-4	-5	1	3	0 = 30	$x_4 = 60$	
	1	-1	2	0	1	0 = 10	$x_1 = 10$	
	0	2	-3	0	1	-1 = 10	$x_6 = 20$	

here make x_2 as basic

Z	x_1	x_2	x_3	x_4	x_5	x_6	BV	RT
1	0	-1	3	0	2	0 = 20		
	0	4	-5	1	3	0 = 30	x_4	$x_2 \le 15/2$
	1	-1	2	0	1	0 = 10	x_1	$x_2 \ge -10$
	0	2	-3	0	1	1 = 10	x_6	$x_2 \le 5$

pivoting from {4, 1, 6} → {4, 1, 2}

Z	x_1	x_2	x_3	x_4	x_5	x_6	B.V	R.T.
1	0	0	1.5	0	2.5	0.5 = 25		
	0	0	1	1	1	-2 = 10		
	1	0	0.5	0	1.5	0.5 = 15		
	0	1	-1.5	0	0.5	0.5 = 5		

3

Hence the optimal solution is :-

$$X = \begin{bmatrix} 15 \\ 5 \\ 0 \\ 10 \\ 0 \\ 0 \end{bmatrix}$$

$Z = 25$

B.V = {4, 1, 2}   NBV = {3, 5, 6}

```
gurobi> m.addConstr(3*x1+x2+x3+x4,GRB.EQUAL,60,"c0")
<gurobi.Constr *Awaiting Model Update*>
gurobi> m.addConstr(x1-x2+2*x3+x5,GRB.EQUAL,10,"c1")
<gurobi.Constr *Awaiting Model Update*>
gurobi> m.addConstr(x1+x2-x3+x6,GRB.EQUAL,20,"c2")
<gurobi.Constr *Awaiting Model Update*>
gurobi> m.optimize()
Optimize a model with 4 rows, 7 columns and 16 nonzeros
Coefficient statistics:
  Matrix range     [1e+00, 3e+00]
  Objective range  [1e+00, 2e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [1e+01, 6e+01]
Presolve removed 1 rows and 4 columns
Presolve time: 0.01s
Presolved: 3 rows, 3 columns, 9 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    6.0000000e+01   1.999100e+01   0.000000e+00      0s
       4    2.5000000e+01   0.000000e+00   0.000000e+00      0s

Solved in 4 iterations and 0.03 seconds
Optimal objective  2.500000000e+01
gurobi> for v in m.getVars():
....... print(v.varName + ' = ' + str(v.x))
  File "<stdin>", line 2
    print(v.varName + ' = ' + str(v.x))
        ^
IndentationError: expected an indented block
gurobi> for v in m.getVars():
.......        print(v.varName + ' = ' + str(v.x))
.......
x1 = 0.0
x1 = 15.0
x2 = 5.0
x3 = 0.0
x4 = 10.0
x5 = 0.0
x6 = 0.0
gurobi> print(m.objVal)
25.0
```

2. max $2x_1 + 3x_2$

s.t

$$.5x_1 + .25x_2 \le 4$$
$$x_1 + 3x_2 \ge 20$$
$$x_1 + x_2 = 4$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

Standard form

max $2x_1 + 3x_2$

s.t

$$0.5x_1 + 0.25x_2 + x_3 = 4$$
$$x_1 + 3x_2 - x_4 = 20$$
$$x_1 + x_2 = 4$$
$$x_1, x_2, x_3, x_4 \ge 0$$

tableue,

Z	x_1	x_2	x_3	x_4	RHS	B.V	RT
1	-2	-3	0	0	0		
	0.5	0.25	1	0	4		
	1	3	0	-1	20		
	1	1	0	0	4		

B.V = {3, 4},

Z	x_1	x_2	x_3	x_4	RHS	B.V.	RT
1	0	-1	0	0	8		
	0	-0.25	1	0	2	$x_3 = 20$	
	0	-2	0	1	-16	$x_4 = -4$	
	1	1	0	0	4		

As x_4 cannot be negative {3, 4} is not feasible lets take B.V {3} NBV {1,2 4}.

z	x_1	x_2	x_3	x_4	RHS	B.V	R.T.
1	0	0	0	-0.5	16		
	0	0	1	-0.125	4	$x_3 = 4$	
	0	1	0	-0.5	8	$x_2 = 8$	
	1	1	0	0	4	$x_1 = 4$	

$$X = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix} \qquad z = 16$$

```
gurobi> m1.setObjective(2*x11+3*x21,GRB.MAXIMIZE)
gurobi> m1.addConstr(0.5*x11+0.25*x21<=4,"c0")
<gurobi.Constr *Awaiting Model Update*>
gurobi> m1.addConstr(x11+3*x21>=20,"c1"),
(<gurobi.Constr *Awaiting Model Update*>,)
gurobi> m1.addConstr(x11+x21,GRB.EQUAL,4,"c2")
<gurobi.Constr *Awaiting Model Update*>
gurobi> m1.optimize()
Optimize a model with 3 rows, 4 columns and 6 nonzeros
Coefficient statistics:
  Matrix range     [3e-01, 3e+00]
  Objective range  [2e+00, 3e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [4e+00, 2e+01]
Presolve removed 0 rows and 2 columns
Presolve time: 0.01s

Solved in 0 iterations and 0.01 seconds
Infeasible or unbounded model
```

3. | z | $-x_1$ | _ | _ | _ | _ | $-x_6$ | = | 7 |
|---|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | _ | _ | = | _ |
| | _ | _ | x_3 | _ | _ | $+2x_6$ | = | 5 |
| | x_1 | _ | _ | _ | _ | _ | = | _ |

(a). if {1, 3, 5} are basic variables,

$z = * x_2 + * x_3 + * x_4$

+ if the basic variables are 1, 3, 5 then,

Tableau:-

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	B.V.
0	1	-1	a_2	a_5	a_7	a_{10}	-1	7	
1		0	0	0	0	a_{11}	a_{14}	a_{15}	
2		a_1	a_3	1	a_8	a_{12}	2	5	
3		1	a_4	a_6	a_9	a_{13}	0	a_{16}	

4 if B.V = {1, 3, 5}, NBV = {2, 4, 6}

for 1 to be basic, $A_i = 0$ for $i \neq B$

hence making Row 0 of $x_1 = 0$, Row 0 + Row 1

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	0	$a_2 + a_4$	a_5	$a_7 + a_9$	$a_{10} + a_{13}$	-1	$7 + a_{16}$
	0	0	0	0	$a_{11} = 1$	a_{14}	$= a_{15}$
	$a_1 = 0$	a_3	1	a_8	$a_{12} = 0$	2	= 5
	1	a_4	$a_6 = 0$	a_9	$a_{13} = 0$	0	$= a_{16}$

hence $x_3 = 5$ & $x_1 = a_{16}$

$x_5 = a_{15}$

Scanned by CamScanner

hence if a_{15} and a_{16} are non-negative ie $a_{15} \geq 0$ & $a_{16} \geq 0$ only then $\{1, 3, 5\}$ can be basic variables.

(ii). Basic variables $= \{2, 3, 4\}$.
N.BV $= \{1, 5, 6\}$.
hence the tableu should look like

	Z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	B.V
0	1	-1	0	0	0	a_{10}	-1	7	
1		0	0	0	0				
2				1	0		2	= 5	
3				0	1		0		

This is not possible as :-
① Row 2 of $x_3 = 1$, hence for the other two basic varie 2 & 4, either of them should have row 1 & row 3 to be 1,
② now as all the coefficients of row 1 are 0 for $\{2, 3, 4\}$ they cannot form a basis as they are not linearly independant

(iii). Taking the case of basis with $\{1, 3, 5\}$, the $\bar{x}$ is feasible till a_{15} & a_{16} are greater or equal to zero,
$\bar{x}$ is not optimal as the objective function has

$\therefore$ $z - x_1 + \_ + \_ + \_ - x_6 = 7$ /

Hence we can optimise the solution (assuming maximization) till all the elements of row 0 are positive.

4. $\bar{x}$ is optimal
starting with {1,3,5} → B.V {2,4,6} → NBV.
Eg tableue with concrete values

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	B.V	R.T.
0	1	-1	1	0	2	0	-1	7		
1		0	0	0	0	1	1	10		
2		0	2	1	4	0	2	5		
3		1	1	0	2	0	2	15		

Row 0 + ROW3 for making 1 B.V

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	B.V	R.T
	0	2	0	4	0	1	22		
	0	0	0	0	1	1	10	$x_5 = 10$	
	0	2	1	4	0	2	5	$x_3 = 5$	
	1	1	0	2	0	2	15	$x_1 = 15$	

Hence in this tableau with these values, the solution is optimal with $z = 22$.

5. In the tableau designed above, we only have one unique optimal solution, as the coefficients of all N.B.V {2,4,6} are non-zero in the objective function.

6. $\bar{X}$ is infeasible,
assume B.V = {1, 3, 5} NBV = {2, 3, 4}.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	BV	RT
1	-1	a_2	a_5	a_7	a_{10}	-1	7		
	0	0	0	0	a_{11}	a_{14}	a_{15}		
	a_1	a_3	1	a_8	a_{12}	2	5		
	1	a_4	a_6	a_9	a_{13}	0	a_{16}		

for x_1 to be basis,

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	BV
	0	a_2+a_4	$a_5=0$	a_7+a_9	$0=a_{10}$	-1	$7+a_{16}$	
	0	0	0	0	$1=a_{11}$	a_{14}	a_{15}	$x_5=a_{15}$
	$a_1=0$	a_3	1	a_8	$0=a_{12}$	2	5	$x_3=5$
	1	a_4	$a_6=0$	a_9	$0=a_{13}$	0	a_{16}	$x_1=a_{16}$.

if this is infeasible then, $a_{15}, a_{16} \leq 0$ either one of them or any one of them
for eg. [taking from part 4]

	z	x_1	x_2	x_3	x_4	x_5	x_6	
0		0	2	0	4	0	1	= 22
1		0	0	0	0	1	1	= 10
2		2	2	1	4	0	2	= 5
3		1	1	0	2	0	2	= 15

to make ROW 2 of $x_1 = 0$,

	Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS	B.V	RT
0		0	2	0	4	0	1	22		
1		0	0	0	0	1	1	10		
2		0	0	1	0	0	-2	-25		
3		1	1	0	2	0	2	15		

Hence now this is now infeasible

4. given $\max c^T x : Ax \le b$
→ but this is infeasible
→ we augment by adding d_i where d_i is a fixed number.
→ hence, the objective function to maximise. profit minus cost of augmentation:

$$\max (c_i x_i - d_i d_i)$$
$$\text{s.t.} \quad A x_i \le b_i + d_i \quad (\text{for } i^{th} \text{ constraint})$$
$$\text{s.t} \quad d_i \le k_i$$

5. Given :- $P = \{x : Ax = b, x \ge 0\}$
⇒ $c^T x$ → let this be the primary function
⇒ main objective function:-

$$\max \quad c^T x \quad \text{s.t} \quad Ax = b.$$

let the solution of this be c' where

$c_i' = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \end{bmatrix}$ where a_1, a_2 are the optimal coefficients for corresponding x_i.

⇒ $d^T x$ → let this be the secondary function
⇒ second objective of this is :-

$$\max \quad d^T x. \quad \text{s.t} \quad Ax = b$$

Now we already has c_i denoting optimal values of x_i i.e the decision variables, hence we try to find optimal values of all the remaining decision variables

⇒ hence if we have, m b.v from 1^{st} objective function, then we find value of m decision variables x hence, for 2^{nd} objective function we can keep these values constant hence x will be a matrix of n − m dimension.

2^{nd} method → weights,

$\max_x \ d_1 c^T x + d_2 d^T x$ s.t. $Ax = b$

here $d_1 > d_2$ as $c^T x$ is the primary objective function

6. (1) $\max c^T x : Ax = b \quad x \geq 0$

(2) $\max c^T x : Ax \leq b$

→ we have, $\max c^T x$ st. $Ax \leq b$

• but x_i need not be non-negative,

∴ x_i can be written as difference of two positive numbers $x_i^+ - x_i^-$ such that

$x_i^+ - x_i^- \leq 0$ & put the constraints:-

$x_i^+, x_i^- \geq 0$

→ hence we get:-

$\max c^T x : Ax \leq b$ and $x \geq 0$.

now if we add slack variables then we get ①,

$\max c^T x : Ax = b$ & $x \geq 0$

Conversely, if we are given ①,

$\max c^T x : Ax = b \quad x \geq 0$

and if x_i is $x_i \leq 0$ which is expressed as $x_i^+ - x_i^-$ s.t $x_i^+, x_i^- \geq 0$, then, ① can be written as,

$\max c^T x : Ax = b$

and if we remove the slack variables, we get ②,

hence conversely also this is true,

Hence proved