A Note on the Two Type Procurement Auction Equilibrium

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The goal of this note is extend the baseline model discussed in Bolotnny and Vasserman (2019) to allow for two-dimensional types.

We begin by defining the primitives of the model in order of importance for the exercise.

Bidders indexed by i are described types consisting of two parameters:

- A "cost" parameter: α^i
  - This captures the relative cost-efficiency of bidder i.
- A "risk" parameter: γ^i
  - This captures the extent to which bidder i dislike risks.

The role of γ is mediated by a "utility" function that models how much a bidder values a given bet. In particular, we will use:

$$u(\pi) = 1 - \exp(-\gamma\pi)$$

To solidify what this means, suppose that a bidder faces a bet: win \$100 with probability 0.5 or \$0 with probability 0.5. If the bidder has $\gamma = 0.01$, then the expectation of his utility for the bet is:

$$0.5 \cdot u(100) + 0.5 \cdot u(0) = 0.5 \cdot (1 - \exp(-\frac{100}{100})) + 0.5 \cdot (1 - \exp(-\frac{0}{100}))$$

$$= 0.5 \cdot (1 - \exp(-1)) + 0.5 \cdot (0) \approx 0.316$$

By contrast, if he were able to get \$50 for sure, he would value this at

$$u(50) = 1 - \exp(-\frac{50}{100}) \approx 0.393$$

so you can see that the bidder "dislikes" risk. In general, the higher that γ is, the larger the discrepency between the value of a "bet" and a "sure thing" of equal expected value.

The rest of the parameters in the model explain the way that a bidder of type (α^i, γ^i) values a given bid that he is considering to submit to the auction.

This is described in the problem definision section just below.

For each of $t = 1, \ldots, T$ materials (items) that procurement project will require, there is:

- q_t^e:
  - The DOT engineer's estimate of the quantity of item t that will be needed for the project
- q_t^b:
  - The bidder's estimated quantity of item t that will be used
- σ_t^2
  - The variance of the bidder's estimate of q_t^b
- c_t:
  - The market unit rate for item t

Note that I often use "DOT" to refer to the government org that runs the auction – DOT stands for Department of Transportation.

The Problem Definition

We are interested in solving for the "equilibrium bidding" function that maps each possible type $(\tilde{\alpha}, \tilde{\gamma})$ to a "score" $s(\tilde{\alpha}, \tilde{\gamma})$.

We can think the "score" function as a map:

$$s : [\underline{\alpha}, \overline{\alpha}] \times [\underline{\gamma}, \overline{\gamma}] \to [\underline{S}, \overline{S}]$$

where each of $[\underline{\alpha}, \overline{\alpha}]$, $[\underline{\gamma}, \overline{\gamma}]$ and $[\underline{S}, \overline{S}]$ are subsets of $\mathbb{R}_+$.

In order for a prospective "score" function to be an equilibrium, it needs maximize the "expected utility" from participating in the auction for each possible bidder type $(\tilde{\alpha}, \tilde{\gamma})$ — let's call this $EU(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))$.

This is the product of the (expected) utility that the bidder would get from completing the project if he wins (given his bid) times the probability that he wins.

$$EU(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma})) = \mathrm{E}[u(\pi(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))) \mid \text{win}] \times \Pr(\text{win} \mid s(\tilde{\alpha}, \tilde{\gamma})\ )$$

Let's break this down. The rules of the auction are that the bidder with the **lowest** score wins the auction. Thus, we can write

$$\Pr(\text{win} \mid s(\tilde{\alpha}, \tilde{\gamma})\ ) = \Pr(s(\tilde{\alpha}, \tilde{\gamma}) < \text{ all other bidders' scores})$$

Let's assume there are only two bidders to make things simple (so that "all other bidders" is just the one other bidder). Our construction requires that **every** type of bidder uses the same score function $s(\cdot, \cdot)$.

Thus the probability that $s(\tilde{\alpha}, \tilde{\gamma})$ is lower than the other bidder's score is the probability that the other bidder j drew a type (α^j, γ^j) such that $s(\tilde{\alpha}, \tilde{\gamma}) < s(\alpha^j, \gamma^j)$.

How might we find this distribution? We will assume (as a primitive of the model – not something that we compute) that each of $\tilde{\alpha}$ and $\tilde{\gamma}$ are drawn independently according to some probability distributions. For instance a simple case would have them distributed uniformly on their domains:

$$\alpha \sim U[\underline{\alpha}, \overline{\alpha}] \text{ and } \gamma \sim U[\underline{\gamma}, \overline{\gamma}].$$

Let's write $f_\alpha(\tilde{\alpha})$ and $F_\alpha(\tilde{\alpha})$ for the pdf and cdf of the α distribution evaluated at $\tilde{\alpha}$, respectively, and $f_\gamma(\tilde{\gamma})$ and $F_\gamma(\tilde{\gamma})$ as the pdf and cdf for γ.

So (plugging in) for a proposed score function $s(\cdot)$, the expected utility of participation can be written:

$$EU(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma})) = \mathrm{E}[u(\pi(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))) \mid \text{win}] \times \int_{\underline{\alpha}}^{\overline{\alpha}} \int_{\underline{\gamma}}^{\overline{\gamma}} \left[\mathbf{1}\{s(\tilde{\alpha}, \tilde{\gamma}) < s(\alpha^j, \gamma^j)\}\right] f(\alpha^j) f(\gamma^j) d\alpha^j d\gamma^j$$

Note that the probability of winning critically depends on the choice of the scoring function. For a given scoring function, the probability of winning is easy to compute.

Now the second part – the expected utility of profits conditional on winning. The math for this is the same as in the simpler case detailed in the paper, so I won't replicate it here — instead I'll present the formula and explain how it relates to the overall problem.

$$\mathrm{E}[u(\pi(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))) \mid \text{win}] = \left(1 - \exp\left(-\tilde{\gamma} \sum_{t=1}^{T} q_t^b \left(b_t^*(s(\tilde{\alpha}, \tilde{\gamma})) - \tilde{\alpha} c_t\right) - \frac{\tilde{\gamma}\sigma_t^2}{2} \left(b_t^*(s(\tilde{\alpha}, \tilde{\gamma})) - \tilde{\alpha} c_t\right)^2\right)\right)$$

This formula includes a new object: the unit bid $b_t^*(s(\tilde{\alpha}, \tilde{\gamma}))$. What is this?

Although only the score determines who wins the auction, in reality, bidders don't actually submit "scores" directly – instead they submit a unit bid b_t for each item t in the procurement project. The "score" for a bidder submitting the vector $\{b_1, \ldots, b_T\}$ of bids, is then computed by multiplying by the DOT's estimate of the quantity of each item that will be needed and summing:

$$\text{score} = \sum_t b_t q_t^e.$$

But a result from the paper is that we don't have to think too hard about how to choose the unit bids in equilibrium — it is sufficient to solve for an equilibrium **score** function as we've discussed before, under the assumption that for any score (and $(\tilde{\alpha}, \tilde{\gamma})$ type), bidders will deterministically choose the unit bids $b_t^*(s(\tilde{\alpha}, \tilde{\gamma}))$ to maximize $\mathrm{E}[u(\pi(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))) \mid \text{win}]$.

That is:

$$b_t^*(s(\tilde{\alpha}, \tilde{\gamma})) = \arg\max_{\{b_t\}} \left[1 - \exp\left(-\tilde{\gamma} \textstyle\sum_{t=1}^T q_t^b \left(b_t - \tilde{\alpha} c_t\right) - \frac{\tilde{\gamma}\sigma_t^2}{2} \left(b_t - \tilde{\alpha} c_t\right)^2\right)\right]$$

$$\text{s.t.} \quad \sum_{t=1}^T b_t q_t^e = s(\tilde{\alpha}, \tilde{\gamma})$$

$$\text{and } b_t \geq 0 \text{ for each } t$$

Note: this really this should be $b_t^*(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))$ but I'm suppressing the other parameters since they're implied.

The optimization program above can be rewritten as a fairly standard constrained quadratic program:

$$b_t^*(s(\tilde{\alpha}, \tilde{\gamma})) = \arg\max_{\{b_t\}} \left[\textstyle\sum_{t=1}^T \underbrace{q_t^b \left(b_t - \tilde{\alpha} c_t\right)}_{\text{Expectation of Profits}} \underbrace{- \frac{\tilde{\gamma}\sigma_t^2}{2} \left(b_t^i - \tilde{\alpha} c_t\right)^2}_{\text{Variance of Profits}}\right]$$

$$\text{s.t.} \quad \sum_{t=1}^T b_t q_t^e = s(\tilde{\alpha}, \tilde{\gamma})$$

$$\text{and } b_t \geq 0 \text{ for each } t$$

We need a numerical solver to solve this in general because there is no closed form way to know which non-negativity constraints will bind (e.g. which bids will be 0 at the optimum.) However, once we solve this, we know that the solution for any non-zero bid will follow:

$$b_t^*(s(\tilde{\alpha}, \tilde{\gamma})) = \tilde{\alpha} \cdot c_t + \frac{1}{\tilde{\gamma}} \cdot \frac{q_t^b}{\sigma_t^2} + \frac{q_t^e}{\sigma_t^2 \sum_{p: b_p^* > 0} \left[\frac{\left(q_p^e\right)^2}{\sigma_p^2}\right]} \left(s - \sum_{p: b_p^* > 0} \left[\tilde{\alpha} \cdot c_p^o q_p^e + \frac{1}{\tilde{\gamma}} \cdot \frac{q_p^b}{\sigma_p^2} q_p^e\right]\right)$$

If all of the item bids are above 0 at the optimum, then the summations in the formula above are over all items $t = 1, \ldots, T$. Otherwise, the summations are over the items that have unit bids above 0 at the optimum.

We can find the partial derivative with respect to s, $\frac{\partial b_t(s)}{\partial s}$, from this equation:

$$\frac{\partial b_t(\tilde{s})}{\partial s} = \frac{q_t^e}{\sigma_t^2 \sum_{p: b_p^* > 0} \left[\frac{\left(q_p^e\right)^2}{\sigma_p^2}\right]} \quad \text{if} \quad b_t(\tilde{s}) > 0$$

and $\frac{\partial b_t(\tilde{s})}{\partial s} = 0$ if $b_t(\tilde{s}) = 0$.

Going back to the expected utility for participating in an auction, let's go back to a more high level expression:

$$EU(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma})) = \underbrace{\mathrm{E}[u(\pi(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))) \mid \text{win}]}_{\text{Value of winning}} \times \underbrace{\Pr(s(\tilde{\alpha}, \tilde{\gamma}) < s(\alpha^j, \gamma^j))}_{\text{Probability of winning}}$$

and to simplify notation, let's write this as:

$$EU(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma})) \equiv V(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma})) \times P(s(\tilde{\alpha}, \tilde{\gamma})).$$

We would like to find the function $s(\cdot, \cdot)$ such every bidder type $(\tilde{\alpha}, \tilde{\gamma})$ will be maximally happy with his bid $s(\tilde{\alpha}, \tilde{\gamma})$ — meaning that he could not profit by bidding a different score if his opponent is using $s(\cdot, \cdot)$ to detrmine *her* bid.

A quick inspection of the $EU(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))$ function shows that it is in fact concave in s (as a value), and so a sufficient condition for the optimality of $s(\cdot, \cdot)$ is that the first order condition of $EU(\cdot)$ holds:

$$\frac{\partial EU(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))}{\partial \tilde{s}} = 0.$$

This first order condition is what will define the differential equaiton that we need to solve in order to find a function $s(\cdot, \cdot)$ that will satisfy the conditions to be an "equilibrium".

Taking the derivative from our expression above:

$$\frac{\partial EU(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))}{\partial \tilde{s}} = \frac{\partial}{\partial \tilde{s}} V(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma})) \times P(s(\tilde{\alpha}, \tilde{\gamma})) + V(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma})) \times \frac{\partial}{\partial \tilde{s}} P(s(\tilde{\alpha}, \tilde{\gamma})) = 0.$$

Note that we can compute $\frac{\partial}{\partial \tilde{s}} V(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))$ fairly easily – this is exactly the same as in the one dimensional type case:

Let's simplify notation one more time and write:

$$CE(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))) = \sum_{t=1}^{T} q_t^b \left(b_t^*(s(\tilde{\alpha}, \tilde{\gamma})) - \tilde{\alpha} c_t\right) - \frac{\tilde{\gamma} \sigma_t^2}{2} \left(b_t^*(s(\tilde{\alpha}, \tilde{\gamma})) - \tilde{\alpha} c_t\right)^2$$

so that

$$V(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma})) = 1 - \exp[-\tilde{\gamma} CE(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))]$$

$$\frac{\partial}{\partial \tilde{s}} V(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma})) = \tilde{\gamma} \frac{\partial}{\partial \tilde{s}} CE(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))) \times \exp[-\tilde{\gamma} CE(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))],$$

where

$$\frac{\partial}{\partial \tilde{s}} CE(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))) = \sum_{t=1}^{T} \left[\frac{\partial b_t^*(s(\tilde{\alpha}, \tilde{\gamma}))}{\partial s} \left(q_t^b - \tilde{\gamma} \sigma^2 (b_t^*(s(\tilde{\alpha}, \tilde{\gamma})) - \tilde{\alpha} c_t) \right) \right].$$

As noted before, we solve for $b_t^*(s(\tilde{\alpha}, \tilde{\gamma}))$ and $\frac{\partial b_t^*(s(\tilde{\alpha}, \tilde{\gamma}))}{\partial s}$ numerically, and so $\frac{\partial}{\partial \tilde{s}} CE(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma})))$ and consequently $\frac{\partial}{\partial \tilde{s}} V(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))$ are well defined and computable for any function $s(\cdot, \cdot)$.

The challenge is in figuring out how to figure out $P(s(\tilde{\alpha}, \tilde{\gamma}))$. For a given guess of $s(\cdot, \cdot)$ this is easily done – just integrate over the distributions of α and γ. But this is where my knowledge of implementing differential equation solvers is sparse.