Procurement Auction Equilibrium Derivation w/ Risk Aversion

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1 Two Bidders, Multiple Items

We now consider an auction with two bidders, A and B, in which multiple items are being auctioned off. The auction is characterized by the variables:

- Ex-post (realized/true) quantities of the items needed: $\mathbf{q^b} = \{q_1^b, \dots, q_T^b\}$
- The DOT's estimate of the quantities needed: $\mathbf{q^e} = \{q_1^e, \dots, q_T^e\}$
- The DOT's estimate of the unit cost of the item: $\mathbf{c^o} = \{c_1^o, \dots, c_T^o\}$
- Bidder i's cost for item t is $\alpha^i c_t^o$

Both bidders observe their unit costs but observe a **common, noisy** signal of the ex-post quantity $\mathbf{q}^a$:

$$q_t^b = q_t^a + \epsilon \text{ where } \epsilon \sim \mathcal{N}(0, \sigma^2) \text{ is IID}$$

The bidders are risk averse, however, with a standard CARA utility function and a common CARA coefficient of risk aversion γ:

$$u_i(\pi) = 1 - exp(-\gamma(\pi)).$$

Moreover, winner of the auction is the bidder with the lowest *score*, based on the *DOT's* quantity estimates:

$$s(\cdot) = \sum_{t=1}^{T} b_t q_t^e.$$

Although the bidders anticipate that their signal of the true quantities is noisy, we assume that it is correct on average.

We therefore construct an equilibrium in which bidders employ a bidding rule that is monotonic in the expected (ex-post) total cost of the project, which we will denote as each bidder's pseudotype.[1] [2]

$$\alpha > \alpha' \Rightarrow s(\alpha) > s(\alpha'),$$

where α^i, the bidder i's cost multiplier type.

Each bidder i maximizes his expected utility from the auction:

$$E[u_i(s(\alpha^i), \alpha^i)] = E_\epsilon \left[1 - \exp\left(-\gamma \sum_{t=1}^{T} (q_t^b - \epsilon_t) \cdot (b_t(s) - \alpha^i c_t^o)\right)\right] \cdot \text{Prob}(s(\alpha^i) < s(\alpha^j)),$$

where the opposing bidder's type α^j is a random variable distributed with some cdf $F(\alpha)$: for instance if $\alpha \sim U[a, b]$, then the cdf is $F(\alpha) = \frac{\alpha}{b-a}$ and the pdf is $f(\alpha) = 1$.

Under the monotonicity assumption, we have that

$$\text{Prob}(s^i < s^j) = \text{Prob}(s(\alpha^i) < s(\alpha^j)) = \text{Prob}(\alpha^i < \alpha^j) = 1 - F(\alpha^i).$$

Now to simplify the expected utility formula, we use the fact that $E[\exp(-\gamma(b-c)\epsilon)] = \exp(\frac{\gamma^2\sigma^2(b-c)^2}{2})$:

[1] Note that because the winner of the auction is determined by a single-dimensional score, we need a monotonicity rule that is single dimensional. This presents a problem when item costs are not monotonically ranked. For this version, we reconcile this by assuming that bidders cost types **are** monotonically ranked given the characteristics of the auction. That is, item-level costs for each bidder i are given by $c_t^i = \alpha^i c_o^t$ where c_o^t is a common item cost for item i and α^i is a bidder-specific multiplier.

[2] We assume that the ex-post quantites are set exogenously and cannot be changed by the bidders, and that bidders' unit costs are constant (and do not vary with quantity). Therefore, the bidders' ex-post cost of the project can be described as a single dimensional sum. Moreover, since only the score determines the winner of the auction (regardless of how the unit bids are distributed) the single dimensional mapping of ex-post cost type to score is well defined.

$$
\begin{aligned}
E[u_i(s(\alpha^i),\alpha^i)] =& E\left[1-\exp\left(-\gamma\sum_{t=1}^{T}(q_t^b-\epsilon_t)(b_t(s(\alpha^i))-\alpha^i c_t^o)\right)\right](1-F(\alpha^i)) && (1)\\
=& \left[1-\exp\left(-\gamma\sum_{t=1}^{T}(q_t^b-\frac{\gamma(b_t(s(\alpha^i))-\alpha^i c_t^o)}{2}\sigma^2)(b_t(s(\alpha^i))-\alpha^i c_t^o)\right)\right](1-F(\alpha^i)). && (2)
\end{aligned}
$$

For notational convenience, we will drop the superscripts on α^i since only one bidder is considered at a time. **Note:** We should replace this with $\tilde{\alpha}$ to highlight that this is a particular value realization – however to avoid making a mistake I will leave the note in its previous more confusing form for now.

In order for the bidding functions to construct an equilibrium, it must be that each bidder's bid function is profit maximizing in response to his opponent:

$$
\frac{\partial \tilde{u}_i(\tilde{s},\alpha)}{\partial \tilde{s}}|_{\tilde{s}=s(\alpha)} = 0.
$$

Computing the derivative for the FOC:

$$
\begin{aligned}
&\frac{\partial \tilde{u}(\tilde{s},\alpha)}{\partial \tilde{s}}|_{\tilde{s}=s(\alpha)} =\\
&\quad =\frac{\partial}{\partial \tilde{s}}|_{\tilde{s}=s(\alpha)}\left[1-\exp\left(-\gamma\sum_{t=1}^{T}(q_t^b-\frac{\gamma(b_t(\tilde{s})-\alpha^i c_t^o)}{2}\sigma^2)(b_t(\tilde{s})-\alpha^i c_t^o)\right)\right](1-F(s^{-1}(\tilde{s}))\\
&\quad =-\frac{f(s^{-1}(\tilde{s}))}{s'(s^{-1}(\tilde{s}))}+\frac{f(s^{-1}(\tilde{s}))}{s'(s^{-1}(\tilde{s}))}\exp\left(-\gamma\sum_{t=1}^{T}(q_t^b-\frac{\gamma(b_t(\tilde{s})-\alpha^i c_t^o)}{2}\sigma^2)(b_t(\tilde{s})-\alpha^i c_t^o)\right)+\\
&\qquad \exp(\ldots)\sum_{t=1}^{T}\left[\left(\gamma q_t^b-\gamma^2\sigma^2(b_t(\tilde{s})-\alpha c_t^o)\right)\frac{\partial b_t(\tilde{s})}{\partial s}\right](1-F(s^{-1}(\tilde{s}))\\
&\qquad =0.
\end{aligned}
$$

Substituting $\tilde{s}=s(\alpha)$ and rearranging:

$$
s'(\alpha)\sum_{t=1}^{T}\left[\left(\gamma q_t^b-\gamma^2\sigma^2(b_t(s(\alpha))-\alpha c_t^o)\right)\frac{\partial b_t(s(\alpha))}{\partial s}\right]=\frac{f(\alpha)}{1-F(\alpha)}\frac{1-\exp(\cdot)}{\exp(\cdot)}, \qquad (3)
$$

where $\exp(\cdot) = \exp\left(-\gamma \sum_{t=1}^{T}(q_t^b - \frac{\gamma(b_t(s(\alpha))-\alpha^i c_t^o)}{2}\sigma^2)(b_t(s(\alpha)) - \alpha^i c_t^o)\right)$.

Note that in order for this bidding rule to be a true best response, the bidding function $\mathbf{b}(s)$ has to be optimal (given α and $s(\alpha)$) – in the sense of maximizing the expected utility of winning. That is, given a score s, the bidding function must solve:

$$\max_{\mathbf{b}(s)} \left[-\exp\left(\sum_{t=1}^{T} \frac{\gamma^2\sigma^2}{2}(b_t(s) - \alpha c_t^o)^2 - \gamma q_t^b (b_t(s) - \alpha c_t^o) \right) \right] \tag{4}$$

$$\text{s.t. } \sum_{t=1}^{T} b_t(s) q_t^e = s$$

$$\text{and } b_t \geq 0 \text{ for each } t$$

With a bit of simplification, we can rewrite this optimization problem as a relatively easy constrained quadratic program:

$$CE(s,\alpha) \equiv \max_{\mathbf{b}(s)} \left[-\gamma \sum_{t=1}^{T} q_t^b (b_t(s) - \alpha c_t^o) - \frac{\gamma\sigma^2}{2}(b_t(s) - \alpha c_t^o)^2 \right] \tag{5}$$

$$\text{s.t. } \sum_{t=1}^{T} b_t(s) q_t^e = s$$

$$\text{and } b_t \geq 0 \text{ for each } t$$

In order to solve the differential equation (6), we can numerically solve for the optimal bid function and use the numerical derivative for $\frac{\partial b_t(s)}{\partial s}$.

We need a numerical solver because there is no closed form way to know which non-negativity constraints will bind (e.g. which bids will be 0 at the optimum.) However, once we solve this, we know that the solution for any non-zero bid will follow:

$$b_t^*(s,\alpha) = \alpha^i c_t^o + \frac{q_t^b}{\gamma\sigma_t^2} + \frac{q_t^e}{\sigma_t^2 \sum_{p:b_p^*>0} \left[\frac{\left(q_p^e\right)^2}{\sigma_p^2} \right]} \left(s - \sum_{p:b_p^*>0} \left[\alpha^i c_p^o q_p^e + \frac{q_p^b}{\gamma\sigma_p^2} q_p^e \right] \right) \quad \text{if } b_t^* > 0$$

We can find the partial derivative with respect to s, $\frac{\partial b_t(s)}{\partial s}$, from this equation.

Once we do this, we can completely define our differential equation:

$$s'(\alpha) = \frac{f(\alpha)}{1-F(\alpha)} \frac{1-\exp(CE(s(\alpha),\alpha))}{\exp(CE(s(\alpha),\alpha))} \cdot \frac{1}{\sum_{t=1}^{T}\left[\left(\gamma q_t^b - \gamma^2\sigma^2(b_t^*(s(\alpha)) - \alpha c_t^o)\right)\frac{\partial b_t^*(s(\alpha))}{\partial s}\right]} \tag{6}$$

Finally note that once can rewrite

$$\frac{1-\exp(x)}{\exp(x)} = \frac{1}{\exp(-x)-1},$$

which we can use to further simplify the equation above.

1.0.1 Boundary Condition

The uniqueness of the equilibrium bidding function derived above relies on pinning down a unique boundary condition: that the highest possible cost type $\overline{\alpha}$ expects zero utility at every possible state of the world, regardless of whether or not it wins the auction. In order to guarantee this, we define $s(\overline{\alpha})$ to be such that

$$CE(s(\overline{\alpha}),\overline{\alpha}) = 0.$$

We can find this numerically by searching over a grid of potential scores $\tilde{s}$ and finding the one that satisfies the zero certainty equivalent condition above.