Asymmetric (Two Type) Procurement Auction Equilibrium

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The goal of this note is extend the baseline model discussed in Bolotnny and Vasserman (2019) to allow for two-dimensional types.

We begin by defining the primitives of the model in order of importance for the exercise.

Bidders indexed by i are described types consisting of two parameters: - A "cost" parameter: α^i - This captures the relative cost-efficiency of bidder i. - A "risk" parameter: γ^i - This captures the extent to which bidder i dislike risks.

The role of γ is mediated by a "utility" function that models how much a bidder values a given bet. In particular, we will use:

$$u(\pi) = 1 - \exp(-\gamma\pi)$$

To solidify what this means, suppose that a bidder faces a bet: win \$100 with probability 0.5 or \$0 with probability 0.5. If the bidder has $\gamma = 0.01$, then the expectation of his utility for the bet is:

$$0.5 \cdot u(100) + 0.5 \cdot u(0) = 0.5 \cdot (1 - \exp(-\frac{100}{100})) + 0.5 \cdot (1 - \exp(-\frac{0}{100}))$$

$$= 0.5 \cdot (1 - \exp(-1)) + 0.5 \cdot (0) \approx 0.316$$

By contrast, if he were able to get \$50 for sure, he would value this at

$$u(50) = 1 - \exp(-\frac{50}{100}) \approx 0.393$$

so you can see that the bidder "dislikes" risk. In general, the higher that γ is, the larger the discrepency between the value of a "bet" and a "sure thing" of equal expected value.

The rest of the parameters in the model explain the way that a bidder of type (α^i, γ^i) values a given bid that he is considering to submit to the auction.

This is described in the problem definision section just below.

For each of $t = 1, \dots, T$ materials (items) that procurement project will require, there is: - q_t^e: - The DOT engineer's estimate of the quantity of item t that will be needed for the project - q_t^b: - The bidder's estimated quantity of item t that will be used - σ_t^2 - The variance of the bidder's estimate of q_t^b

- c_t:
  - The market unit rate for item t

Note that I often use "DOT" to refer to the government org that runs the auction – DOT stands for Department of Transportation.

The Problem Definition

We will consider an auction in which there are 2 kinds of bidders, characterized by different values of γ: γ_1 and γ_2.

We are interested in solving for the "equilibrium bidding" function that maps each possible type $(\tilde{\alpha}, \gamma_i)$ to a "score" $s(\tilde{\alpha}, \gamma_i)$.

We can think the "score" function as a map:

$$s_i : [\underline{\alpha}, \overline{\alpha}] \times \gamma_i \to [\underline{S}, \overline{S}]$$

where each of $[\underline{\alpha}, \overline{\alpha}]$ and $[\underline{S}, \overline{S}]$ are subsets of $\mathbb{R}_+$.

Note that each γ-type is going to have its own monotonic score function. We have two γ types — γ_1 and γ_2 — and so we will need to solve for two score functions — $s_1(\alpha)$ and $s_2(\alpha)$.

In order for a prospective "score" function to be an equilibrium, it needs maximize the "expected utility" from participating in the auction for each possible bidder type $(\tilde{\alpha}, \tilde{\gamma})$ — let's call this $EU(\tilde{\alpha}, \tilde{\gamma}, s(\tilde{\alpha}, \tilde{\gamma}))$.

This is the product of the (expected) utility that the bidder would get from completing the project if he wins (given his bid) times the probability that he wins.

$$EU(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha})) = \mathrm{E}[u(\pi(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha})) \mid \text{win}] \times \Pr(\text{win} \mid s_i(\tilde{\alpha})\ )$$

Let's break this down. Suppose that bidder i drew a value α_i and bid according to the score function that we are trying to define, knowing that bidder j drew some other (unown to i) value α_j from the same IID distribution (e.g. the uniform distribution).

The rules of the auction are that the bidder with the **lowest** score wins the auction. Thus, we can write

$$\begin{aligned}
\Pr(\text{win} \mid s_i(\tilde{\alpha})\ ) &= \Pr(s_i(\tilde{\alpha}) < \text{ the other bidder's score}) \\
&= \Pr(s_i(\tilde{\alpha}) < s_j(\alpha_j)\ ) \\
&= \Pr(s_j^{-1}(s_i(\tilde{\alpha})) < \alpha_j\ ) \\
&= 1 - F(s_j^{-1}(s_i(\tilde{\alpha})))
\end{aligned}$$

where $F(a) = \frac{a - \underline{\alpha}}{\overline{\alpha} - \underline{\alpha}}$ is the CDF of the unifrom distribution.

In words — we're assuming the bidders draw their values α_i and α_j IID from the uniform distribution with cdf $F(a)$. Each bidder sees his own α, and knows the bidding functions $s_i(\cdot)$ and $s_j(\cdot)$, but not the realization of his opponent's α. The probability that bidder i will win if he uses his bidding function $s_i(\alpha_i)$ — like he's supposed to, for this to be an equilibirum — is the probability that $s_i(\alpha_i)$ is smaller than $s_j(\cdot)$ evaluted at the (unknown to i) realization of α_j.

So (plugging in) for a proposed score functions $s_i(\cdot)$ and $s_j(\cdot)$, the expected utility of participation for bidder i can be written:

$$EU(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha})) = \mathrm{E}[u(\pi(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha}))) \mid \text{win}] \times [1 - F(s_j^{-1}(s_i(\tilde{\alpha})))]$$

Now the second part – the expected utility of profits conditional on winning. The math for this is the same as in the simpler case detailed in the paper, so I won't replicate it here — instead I'll present the formula and explain how it relates to the overall problem.

$$\mathrm{E}[u(\pi(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha}))) \mid \text{win}] = \left(1 - \exp\left(-\gamma_i \sum_{t=1}^{T} q_t^b \left(b_t^*(s_i(\tilde{\alpha})) - \tilde{\alpha} c_t\right) - \frac{\gamma_i \sigma_t^2}{2} \left(b_t^*(s_i(\tilde{\alpha})) - \tilde{\alpha} c_t\right)^2\right)\right)$$

This formula includes a new object: the unit bid $b_t^*(s_i(\tilde{\alpha}))$. What is this?

Although only the score determines who wins the auction, in reality, bidders don't actually submit "scores" directly – instead they submit a unit bid b_t for each item t in the procurement project. The "score" for a bidder submitting the vector $\{b_1, \ldots, b_T\}$ of bids, is then computed by multiplying by the DOT's estimate of the quantity of each item that will be needed and summing:

$$\text{score} = \sum_t b_t q_t^e.$$

But a result from the paper is that we don't have to think too hard about how to choose the unit bids in equilibrium — it is sufficient to solve for an equilibrium **score** function as we've discussed before, under the assumption that for any score (and $\tilde{\alpha}$ and γ_i) type), bidders will deterministically choose the unit bids $b_t^*(s_i(\tilde{\alpha}))$ to maximize $\mathrm{E}[u(\pi(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha}))) \mid \text{win}]$.

That is:

$$b_t^*(s_i(\tilde{\alpha})) = \arg\max_{\{b_t\}} \left[1 - \exp\left(-\gamma_i \textstyle\sum_{t=1}^T q_t^b \left(b_t - \tilde{\alpha} c_t\right) - \frac{\gamma_i \sigma_t^2}{2} \left(b_t - \tilde{\alpha} c_t\right)^2\right)\right]$$

$$\text{s.t.} \ \sum_{t=1}^T b_t q_t^e = s_i(\tilde{\alpha})$$

$$\text{and } b_t \geq 0 \text{ for each } t$$

Note: this really this should be $b_t^*(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha}))$ but I'm suppressing the other parameters since they're implied.

The optimization program above can be rewritten as a fairly standard constrained quadratic program:

$$b_t^*(s_i(\tilde{\alpha})) = \arg\max_{\{b_t\}} \left[\textstyle\sum_{t=1}^T \underbrace{q_t^b \left(b_t - \tilde{\alpha} c_t\right)}_{\text{Expectation of Profits}} \underbrace{-\frac{\gamma_i \sigma_t^2}{2} \left(b_t^i - \tilde{\alpha} c_t\right)^2}_{\text{Variance of Profits}}\right]$$

$$\text{s.t.} \ \sum_{t=1}^T b_t q_t^e = s_i(\tilde{\alpha})$$

$$\text{and } b_t \geq 0 \text{ for each } t$$

We need a numerical solver to solve this in general because there is no closed form way to know which non-negativity constraints will bind (e.g. which bids will be 0 at the optimum.) However, once we solve this, we know that the solution for any non-zero bid will follow:

$$b_t^*(s_i(\tilde{\alpha})) = \tilde{\alpha} \cdot c_t + \frac{1}{\gamma_i} \cdot \frac{q_t^b}{\sigma_t^2} + \frac{q_t^e}{\sigma_t^2 \sum_{p: b_p^* > 0} \left[\frac{\left(q_p^e\right)^2}{\sigma_p^2}\right]} \left(s - \sum_{p: b_p^* > 0} \left[\tilde{\alpha} \cdot c_p^o q_p^e + \frac{1}{\gamma_i} \cdot \frac{q_p^b}{\sigma_p^2} q_p^e\right]\right)$$

If all of the item bids are above 0 at the optimum, then the summations in the formula above are over all items $t = 1, \ldots, T$. Otherwise, the summations are over the items that have unit bids above 0 at the optimum.

We can find the partial derivative with respect to s, $\frac{\partial b_t(s)}{\partial s}$, from this equation:

$$\frac{\partial b_t(\tilde{s})}{\partial s} = \frac{q_t^e}{\sigma_t^2 \sum_{p: b_p^* > 0} \left[\frac{\left(q_p^e\right)^2}{\sigma_p^2}\right]} \quad \text{if} \quad b_t(\tilde{s}) > 0$$

and $\frac{\partial b_t(\tilde{s})}{\partial s} = 0$ if $b_t(\tilde{s}) = 0$.

Going back to the expected utility for participating in an auction, let's go back to a more high level expression:

$$EU(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha})) = \underbrace{\mathrm{E}[u(\pi(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha}))) \mid \text{win}]}_{\text{Value of winning}} \times \underbrace{[1 - F(s_j^{-1}(s_i(\tilde{\alpha})))]}_{\text{Probability of winning}}$$

and to simplify notation, let's write this as:

$$EU(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha})) \equiv V(s_i(\tilde{\alpha})) \times [1 - F(s_j^{-1}(s_i(\tilde{\alpha})))].$$

We would like to find the function $s_i(\alpha)$ such that bidder i will be maximally happy with his bid $s_i(\alpha_i)$ no matter what draw of α_i he gets — meaning that he could not profit by bidding a different score if his opponent is using $s_j(\alpha)$ to detrmine *her* bid.

A quick inspection of the $EU(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha}))$ function shows that it is in fact concave in s (as a value), and so a sufficient condition for the optimality of $s(\cdot, \cdot)$ is that the first order condition of $EU(\cdot)$ holds:

$$\frac{\partial EU(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha}))}{\partial \tilde{s}} = 0.$$

This first order condition is what will define the differential equaiton that we need to solve in order to find a function $s(\cdot, \cdot)$ that will satisfy the conditions to be an "equilibrium".

Taking the derivative from our expression above:

$$\frac{\partial EU(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha}))}{\partial \tilde{s}} = \frac{\partial}{\partial \tilde{s}} V(s_i(\tilde{\alpha})) \times [1 - F(s_j^{-1}(s_i(\tilde{\alpha})))] + V(s_i(\tilde{\alpha})) \times \frac{\partial}{\partial \tilde{s}} [1 - F(s_j^{-1}(s_i(\tilde{\alpha})))] = 0. \tag{1}$$

Note that we can compute $\frac{\partial}{\partial \tilde{s}} V(s_i(\tilde{\alpha}))$ fairly easily – this is exactly the same as in the one dimensional type case:

Let's simplify notation one more time and write:

$$CE(s_i(\tilde{\alpha})) = \sum_{t=1}^{T} q_t^b \left(b_t^*(s_i(\tilde{\alpha})) - \tilde{\alpha} c_t\right) - \frac{\gamma_i \sigma_t^2}{2} \left(b_t^*(s_i(\tilde{\alpha})) - \tilde{\alpha} c_t\right)^2$$

so that

$$V(s_i(\tilde{\alpha})) = 1 - \exp[-\gamma_i CE(s_i(\tilde{\alpha}))]$$

$$\frac{\partial}{\partial \tilde{s}} V(s_i(\tilde{\alpha})) = \gamma_i \frac{\partial}{\partial \tilde{s}} CE(s_i(\tilde{\alpha}))) \times \exp[-\gamma_i CE(s_i(\tilde{\alpha}))],$$

where

$$\frac{\partial}{\partial \tilde{s}} CE(s_i(\tilde{\alpha})) = \sum_{t=1}^{T} \left[\frac{\partial b_t^*(s_i(\tilde{\alpha}))}{\partial \tilde{s}} \left(q_t^b - \gamma_i \sigma^2 (b_t^*(s_i(\tilde{\alpha},)) - \tilde{\alpha} c_t) \right) \right].$$

As noted before, we solve for $b_t^*(s_i(\tilde{\alpha}))$ and $\frac{\partial b_t^*(s_i(\tilde{\alpha}))}{\partial s}$ numerically, and so $\frac{\partial}{\partial \tilde{s}} CE(s_i(\tilde{\alpha})))$ and consequently $\frac{\partial}{\partial \tilde{s}} V(s_i(\tilde{\alpha}))$ are well defined and computable for any function $s_i(\cdot)$.

Now we can also take the derivative of the second part directly — but it will be a function of $s_j^{-1}(\tilde{s})$:

$$\frac{\partial}{\partial \tilde{s}} [1 - F(s_j^{-1}(s_i(\tilde{\alpha})))] = [-f(s_j^{-1}(s_i(\tilde{\alpha})))] \times \frac{\partial s_j^{-1}(s_i(\tilde{\alpha}))}{\partial \tilde{s}}$$

Defining the inverse function and writing down the ODE sytem

For convenience, we will write the inverse functions

$$\varphi_j(\tilde{s}) \equiv s_j^{-1}(\tilde{s}) \text{ and } \varphi_i(\tilde{s}) \equiv s_i^{-1}(\tilde{s})$$

Equilibrium bidding funcitons will alwyas be monotonic (at least for this class of problems) and so there is a 1:1 relationship between $s_i(\tilde{\alpha})$ and its inverse $\varphi_i(\tilde{s})$:

$$\varphi_i(s_i(\tilde{\alpha})) = \alpha \text{ and } s_i(\varphi_i(\tilde{s})) = \tilde{s}.$$

We can thus rewrite the first order condition from Equation (1):

$$\frac{\partial EU(\tilde{\alpha}, \gamma_i, s_i(\tilde{\alpha}))}{\partial \tilde{s}} = \frac{\partial}{\partial \tilde{s}} V(s_i(\tilde{\alpha})) \times [1 - F(\varphi_j(s_i(\tilde{\alpha})))] + V(s_i(\tilde{\alpha})) \times [-f(\varphi_j(s_i(\tilde{\alpha})))] \times \frac{\partial \varphi_j(s_i(\tilde{\alpha}))}{\partial \tilde{s}} = 0.$$

Or equivalently, for each possible s_i and s_j:

$$\frac{\partial \varphi_j(s_i)}{\partial \tilde{s}} = \frac{1 - F(\varphi_j(s_i))}{f(\varphi_j(s_i))} \times \frac{\frac{\partial}{\partial \tilde{s}} V(s_i)}{V(s_i)}$$

and by symmetry:

$$\frac{\partial \varphi_i(s_j)}{\partial \tilde{s}} = \frac{1 - F(\varphi_i(s_j))}{f(\varphi_i(s_j))} \times \frac{\frac{\partial}{\partial \tilde{s}} V(s_j)}{V(s_j)}$$

This fully defines the system of ODE problem up to a few boundary conditions:

1. The ODEs for i and j have to be solved together
2. Both boundary scores have to be the same:

$$\varphi_i(\overline{S}) = \varphi_j(\overline{S}) = \overline{\alpha} \text{ and } \varphi_i(\underline{S}) = \varphi_j(\underline{S}) = \underline{\alpha}$$

or equivalently:

$$\overline{S} = s_i(\overline{\alpha}) = s_j(\overline{\alpha}) \text{ and } \underline{S} = s_i(\underline{\alpha}) = s_j(\underline{\alpha})$$

3. There is a known initial condition that (supposing that $\gamma_j < \gamma_i$):

$$\overline{S} = s_i(\overline{\alpha})$$

must be the smallest value of s s.t. $V(s, \overline{\alpha}, \gamma_i) = 0$

Note: I have to double check the condition for $\overline{S}$ above, in (3) is truly correct.