Macroeconomics III

Finance and Development: A Tale of Two Sectors

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In what follows, we present a continuous-time version of Buera et al. (2011). We follow the overall advice in Moll (2014)[1]. This project is structured as follows. Section 1 and Section 2 derive the continuous time version of the model. Section 3 presents details on the implementation of a numerical solution. Section 4 discusses problems faced and concludes.

1 Recursive formulation of agent's problem

1.1 Discrete time

Note that we may write the agent's problem (in discrete time) as:

$$\begin{aligned} v(a, \mathbf{z}) &= \max_{\mathbf{c}, a' \geq 0} \; u(\mathbf{c}) + \beta\{\gamma v(a', \mathbf{z}) + (1-\gamma)\mathbb{E}_{\mathbf{z}^*}[v(a', \mathbf{z}^*)]\} \\ s.t. &\quad \mathbf{p} \cdot \mathbf{c} + a' \leq M(a, \mathbf{z}) + (1+r)a \end{aligned} \tag{1}$$

where the expectation is taken over the distribution of *new* ideas and $M(a, \mathbf{z})$ is defined by:

$$\begin{aligned} M(a, \mathbf{z}) &:= \max\{w, \Pi_s(a, z_s), \Pi_m(a, z_m)\} \\ \Pi_j(a, z_j) &= \max_{l \geq 0, k \in [0, \bar{k}_j(a, z_j)]} \; p_j z_j f(k, l) - Rk - wl - (1+r)p_j \kappa_j \qquad j \in \{s, m\} \end{aligned} \tag{2}$$

Let $s = M(a, \mathbf{z}) + ra - p \cdot \mathbf{c}$ denote savings. Since the utility function satisfies the Inada conditions, we know the budget constraint binds. We can then plug the restriction into the recursive formulation in (1) and rewrite the agent's problem as:

$$v(a, \mathbf{z}) = \max_{\mathbf{c}} \; u(\mathbf{c}) + \beta\left\{\gamma v(a+s, \mathbf{z}) + (1-\gamma)\mathbb{E}_{\mathbf{z}^*}[v(a+s, \mathbf{z}^*)]\right\} \tag{3}$$

[1]We also follow some of the approaches in Ahn and Moll's implementation of the Aiyagari model in continuous time for Python. Check `http://nbviewer.jupyter.org/github/QuantEcon/QuantEcon.notebooks/blob/master/aiyagari_continuous_time.ipynb`.

Given the above structure, we proceed in the following fashion: we first convert the occupational choice problem to continuous time, and then go back to find the Hamilton-Jacobi-Bellman (HJB) equation.

1.2 Occupational choice in continuous time

Fix a sector $j \in \{s, m\}$. Recall that, in discrete time, a capital level k is enforceable if, and only if:

$$\max_l \{p_j z_j f(k,l) - wl\} - Rk - (1+r)p_j\kappa_j + (1+r)a \geq (1-\phi)\left[\max_l \{p_j z_j f(k,l)\} + (1-\delta)k\right]$$

In Δ units of time, we have:

$$\begin{aligned}\Delta\left[\max_l \{p_j z_j f(k,l) - wl\}\right] - \Delta Rk - (1+\Delta r)\Delta p_j\kappa_j + (1+\Delta r)a \\ \geq (1-\phi)\left[\Delta\left[\max_l \{p_j z_j f(k,l)\}\right] + (1-\delta\Delta)k\right]\end{aligned}$$

Letting $\Delta \to 0$ yields:

$$k \leq \frac{a}{1-\phi} =: \hat{k}(a) \tag{4}$$

What is the intuition for this result? As time intervals shrink, the effect of profits on the incentive to default is negligible relatively to the effect of the default punishment (losing a). In the limiting case, only the latter matters. Do also note that this leads to a rental limit analogous to the one in Moll (2014).

In their definition of equilibrium, Buera et al. (2011) constrain bounds on capital rental contracts to be the most generous ones satisfying $\bar{k}_j(a, z_j) \leq k_j^u(z_j)$, where $k_j^u(z_j)$ is the *unconstrained* demand for capital (more on this below). We thus have that the bound on capital rental contracts in equilibrium in continuous time is:

$$\bar{k}_j(a, z_j) := \min\{\hat{k}(a), k_j^u(z_j)\} \tag{5}$$

(this structure will come in handy computationally later on). Now go back to the entrepreneur's problem. In Δ units of time, we have:

$$\Delta\Pi_j = \max_{l\geq 0, k\in[0,\bar{k}_j^{\Delta}(a,z_j)]} \Delta\{p_j z_j f(k,l) - wl - Rk\} - (1+\Delta r)\Delta p_j\kappa_j \tag{6}$$

Dividing by Δ and letting intervals shrink yields:

$$\Pi_j(a, z_j) = \max_{l\geq 0, k\in[0,\bar{k}_j(a,z_j)]} p_j z_j f(k,l) - wl - Rk - p_j\kappa_j \tag{7}$$

(interest accrued over $p_j\kappa_j$ is a second-order term). We therefore have that the continuous version of $M(a,\mathbf{z})$ is:

$$\begin{aligned} \Delta M(a,\mathbf{z}) &= \max\{\Delta w, \Delta\Pi_s, \Delta\Pi_m\} \quad \overset{\Delta\to 0}{\Longrightarrow} \\ M(a,\mathbf{z}) &= \max\{w, \Pi_s(a,z_s), \Pi_m(a,z_m)\} \end{aligned} \tag{8}$$

where $\Delta\Pi_s$ and $\Delta\Pi_m$ are given by (6) and $\Pi_s(a,z_s)$ and $\Pi_m(a,z_m)$ by (7). Finally, if we solve the unconstrained maximisation problem for the entrepreneur, we get:

$$\hat{k}_j^u(z_j) = \left[\frac{\alpha^{1-\theta}\theta^\theta p_j z_j}{w^\theta R^{1-\theta}}\right]^{\frac{1}{1-\alpha-\theta}}$$

Now write $\Pi(z_j,k) = \max_{l\geq 0} p_j z_j k^\alpha l^\theta - wl - Rk - p_j\kappa_j$. Applying the envelope theorem, one sees $\Pi(z_j,k)$ is increasing in k for $k \leq \hat{k}_j^u(z_j)$. We thus have that the demand for capital of an entrepreneur in sector j is just:

$$k^d(z_j,a) = \min\left\{\frac{a}{1-\phi}, \left[\frac{\alpha^{1-\theta}\theta^\theta p_j z_j}{w^\theta R^{1-\theta}}\right]^{\frac{1}{1-\alpha-\theta}}\right\} \tag{9}$$

and the demand for labour is:

$$l_u^d(z_j,a) = \left[\frac{p_j z_j (k^d(z_j,a))^\alpha \theta}{w}\right]^{\frac{1}{1-\theta}} \tag{10}$$

Given a guess of r and w, profits are easily computed using the above.

1.3 Recursive formulation in continuous time

Let $\lambda = (1-\gamma)$ be the hazard rate at which new ideas arrive. Write $\Delta s = \Delta M(a,\mathbf{z}) + \Delta\{ra - p\cdot c\}$, where $\Delta M(a,\mathbf{z})$ is given by (8). In Δ units of time, the problem can be written as

$$\begin{aligned} v(a,\mathbf{z}) = \max_{\mathbf{c}} \Delta u(\mathbf{c}) + &\\ (1-\rho\Delta)\{&\overbrace{\exp(-\lambda\Delta)}^{\text{No new ideas}} v(a+\Delta s,\mathbf{z}) + \overbrace{\exp(-\lambda\Delta)\lambda\Delta}^{\text{One new idea}} \mathbb{E}_{\mathbf{z}^*}[v(a+\Delta s,\mathbf{z}^*)] + \\ &\underbrace{\frac{\exp(-\lambda\Delta)(\lambda\Delta)^2}{2}}_{\text{Two new ideas}} \max\{\mathbb{E}_{\mathbf{z}^{*\mathbf{1}}}[v(a+\Delta s,\mathbf{z}^{*\mathbf{1}})], \mathbb{E}_{\mathbf{z}^{*\mathbf{2}}}[v(a+\Delta s,\mathbf{z}^{*\mathbf{2}})]\} + \ldots\} \end{aligned} \tag{11}$$

where the first term inside braces of the RHS captures the event that no new ideas arrive, the second one captures the event that one new idea arrives, the third event captures the event that two new ideas arrive, and so on. Notice that these probabilities sum up to one[2]. Also, we indexed the "latent" random vectors of ideas to indicate that these are different draws. However, since

[2]Since we are assuming that the arrival process of entrepreneur ideas is a Poisson process with arrival rate λ, we have that the expected number of arrivals in Δ units of time is $\lambda\Delta$. The probability of i ideas arriving in Δ units

$\{\mathbf{z_t^*}\}_{t=1}^{\infty}$ is iid, the expected value is the same, so we could have dropped the maximum operator in the higher order terms of the RHS and have written

$$\begin{aligned} v(a,\mathbf{z}) = \max_{\mathbf{c}} \Delta u(\mathbf{c}) \, + \\ (1-\rho\Delta)\{\overbrace{\exp(-\lambda\Delta)}^{\text{No new ideas}} v(a+\Delta s,\mathbf{z}) + \overbrace{\exp(-\lambda\Delta)\lambda\Delta}^{\text{One new idea}} \mathbb{E}_{\mathbf{z}^*}[v(a+\Delta s,\mathbf{z}^*)] \, + \\ \underbrace{\frac{\exp(-\lambda\Delta)(\lambda\Delta)^2}{2}}_{\text{Two new ideas}} \mathbb{E}_{\mathbf{z}^*}[v(a+\Delta s,\mathbf{z}^*)] + \ldots\} \end{aligned} \quad (12)$$

If we subtract $(1-\rho\Delta)v(a,\mathbf{z})$ from both sides, we have that

$$\begin{aligned} \rho\Delta v(a,\mathbf{z}) = \max_{\mathbf{c}} \Delta u(\mathbf{c}) \, + \\ (1-\rho\Delta)\{\exp(-\lambda\Delta)[v(a+\Delta s,\mathbf{z}) - v(a,\mathbf{z})] + \exp(-\lambda\Delta)\lambda\Delta[\mathbb{E}_{\mathbf{z}^*}[v(a+\Delta s,\mathbf{z}^*)] - v(a,\mathbf{z})] \, + \\ \frac{\exp(-\lambda\Delta)(\lambda\Delta)^2}{2}[\max\{\mathbb{E}_{\mathbf{z}^{*\mathbf{1}}}[v(a+\Delta s,\mathbf{z}^{*\mathbf{1}})], \mathbb{E}_{\mathbf{z}^{*\mathbf{2}}}[v(a+\Delta s,\mathbf{z}^{*\mathbf{2}})]\} - v(a,\mathbf{z})] + \ldots\} \end{aligned} \quad (13)$$

Dividing both sides by Δ and letting intervals shrink, we obtain

$$\rho v(a,\mathbf{z}) = \max_{\mathbf{c}} u(\mathbf{c}) + \frac{\partial v(a,\mathbf{z})}{\partial a} s + \lambda \left[\mathbb{E}_{\mathbf{z}^*}[v(a,\mathbf{z}^*)] - v(a,\mathbf{z})\right] \quad (14)$$

2 Stationary distribution

The vector of entrepreneurial ideas (or talent) $\mathbf{z}^* = (z_S^*, z_M^*)'$ is drawn from a distribution $\mu(\mathbf{z}^*)$. These entrepreneurial ideas "die" with constant hazard rate of $1-\gamma$ and, in this case, a new vector is independently drawn from $\mu(\mathbf{z}^*)$. In other words, we can write (using time subscripts)

$$\mathbf{z_{t+1}} = \gamma \mathbf{z_t} + (1-\gamma)\mathbf{z^*_{t+1}}$$

where we make the distinction between the observed vector of ideas at some period t, $\mathbf{z_t}$ and the vector of ideas that is drawn from $\mu(\mathbf{z}^*)$, $\mathbf{z_t^*}$. Notice that since the sequence $\{\mathbf{z_t^*}\}_{t=1}^{\infty}$ is iid, we can rewrite

$$\mathbf{z_{t+1}} = \gamma \mathbf{z_t} + (1-\gamma)\mathbf{z}^* \quad (15)$$

At the beginning of any period, an individual's state is described by his wealth a and vector of talent $\mathbf{z}$. Let $G_t(a,\mathbf{z}) \coloneqq \mathbb{P}(a_t \leq a, \mathbf{z_t} \leq \mathbf{z})$ denote the joint CDF of the state random vector and

of time is $\mathbb{P}(k=i) = \frac{\exp(-\lambda\Delta)(\lambda\Delta)^i}{i!}$. It is straightforward to see that

$$\sum_{i=0}^{\infty} \mathbb{P}(k=i) = \sum_{i=0}^{\infty} \frac{\exp(-\lambda\Delta)(\lambda\Delta)^i}{i!} = 1$$

let $g_t(a, z)$ be the associated PDF. Next, let $\lambda_t(a, \mathbf{z}) \coloneqq \mathbb{P}(a_t \leq a, \mathbf{z_t} = \mathbf{z})$. Notice that

$$\frac{\partial \lambda_t}{\partial a} = g_t(a, \mathbf{z})$$

or, equivalently,

$$\lambda_t = \int_0^a g_t(\tilde{a}, \mathbf{z})\ d\tilde{a}$$

Now, using (15), we can write

$$\begin{aligned} \lambda_{t+1}(a, \mathbf{z}) = \gamma \left[\int \mathbb{P}(a_{t+1} \leq a, a_t = \tilde{a}, \mathbf{z_t} = \mathbf{z})\ d\tilde{a} \right] \\ + (1-\gamma) \left[\int \int \mathbb{P}(a_{t+1} \leq a, a_t = \tilde{a}, \mathbf{z_{t+1}} = \mathbf{z}, \mathbf{z_t} = \tilde{\mathbf{z}})\ d\tilde{\mathbf{z}} d\tilde{a} \right] \end{aligned} \tag{16}$$

Notice that $\mathbb{P}(a_{t+1} \leq a, a_t = \tilde{a}, \mathbf{z_t} = \mathbf{z}) = \mathbb{P}(a_{t+1} \leq a | a_t = \tilde{a}, \mathbf{z_t} = \mathbf{z}) \mathbb{P}(a_t = \tilde{a}, \mathbf{z_t} = \mathbf{z})$ and that $\mathbb{P}(a_{t+1} \leq a, a_t = \tilde{a}, \mathbf{z_{t+1}} = \mathbf{z}, \mathbf{z_t} = \tilde{\mathbf{z}}) = \mathbb{P}(a_{t+1} \leq a, \mathbf{z_{t+1}} = \mathbf{z} | a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}}) \mathbb{P}(a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}})$. Since for a given wealth a and vector of ideas $\mathbf{z}$ the policy function $a'(a, \mathbf{z})$ uniquely determines the wealth of the following period, we have that

$$\mathbb{P}(a_{t+1} \leq a | a_t = \tilde{a}, \mathbf{z_t} = \mathbf{z}) = \mathbb{P}(a_{t+1} \leq a, \mathbf{z_{t+1}} = \mathbf{z} | a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}}) = \begin{cases} 1 & \text{if } a'(a, \mathbf{z}) \leq a \\ 0 & \text{if } a'(a, \mathbf{z}) > a \end{cases} \tag{17}$$

Therefore, we have that the first integral in (16) can be written as

$$\int \mathbb{P}(a_{t+1} \leq a | a_t = \tilde{a}, \mathbf{z_t} = \mathbf{z}) \mathbb{P}(a_t = \tilde{a}, \mathbf{z_t} = \mathbf{z})\ d\tilde{a} = \int_{\{\tilde{a} : a'(\tilde{a}, \mathbf{z}) \leq a)\}} \mathbb{P}(a_t = a, \mathbf{z_t} = \mathbf{z})\ d\tilde{a} \tag{18}$$

Let the flow of savings be given by $s_t = M(a_t, \mathbf{z_t}) + r a_t + \mathbf{p} \cdot \mathbf{c_t}$. The policy function is then given by $s(a, \mathbf{z}) = M(a, \mathbf{z}) + ra - \mathbf{p} \cdot \mathbf{c}(a, \mathbf{z})$. If the policy function $a'(a, \mathbf{z})$ is strictly increasing, then we have that[3]

$$\{\tilde{a} : a'(\tilde{a}, \mathbf{z}) \leq a\} \Leftrightarrow \{\tilde{a} : \tilde{a} \leq a - s(a, \mathbf{z})\} \tag{19}$$

Thus, we can simplify (18) to

$$\int_{\{\tilde{a} : \tilde{a} \leq a - s(a, \mathbf{z})\}} \mathbb{P}(a_t = a, \mathbf{z_t} = \mathbf{z})\ d\tilde{a} = \lambda_t(a - s(a, \mathbf{z}), \mathbf{z}) \tag{20}$$

Now notice that the second integral in (16) can be written as

$$\int \int \mathbb{P}(a_{t+1} \leq a, \mathbf{z_{t+1}} = \mathbf{z} | a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}}) \mathbb{P}(a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}})\ d\tilde{\mathbf{z}} d\tilde{a} \tag{21}$$

As argued above, for a given wealth a and vector of ideas $\mathbf{z}$ the policy function $a'(a, \mathbf{z})$ uniquely determines the wealth of the following period. Combining this with the fact that $\{\mathbf{z}_\mathbf{t}^*\}_{t=1}^{\infty}$ is iid, we

[3] Recall that we had, from the recursive formulation constraint, $\mathbf{p} \cdot \mathbf{c} + a' \leq M(a, \mathbf{z}) + (1+r)a$. Rearranging, we are left with $a' \leq s + a$.

can write

$$\mathbb{P}(a_{t+1} \leq a, \mathbf{z_{t+1}} = \mathbf{z} | a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}}) \mathbb{P}(a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}}) = \\ \mathbb{P}(a_{t+1} \leq a | a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}}) \mathbb{P}(\mathbf{z_{t+1}} = \mathbf{z} | a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}}) \mathbb{P}(a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}})$$

The first term on the RHS was given in (17). In a given period $t+1$ the draw from the distribution $\mu(\mathbf{z}^*)$ is independent from both the observed vector of ideas z_t and the chosen wealth a_t in the previous period. This allows us to write $\mathbb{P}(\mathbf{z_{t+1}} = \mathbf{z} | a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}}) = \mathbb{P}(\mathbf{z_{t+1}} = \mathbf{z}) = \mu(\mathbf{z})$. We can then rewrite (21) as

$$\begin{aligned} \int \int \mathbb{1}\{(a'(\tilde{a}, \tilde{\mathbf{z}}) \leq a\} \mu(\mathbf{z}) \mathbb{P}(a_t = \tilde{a}, \mathbf{z_t} = \tilde{\mathbf{z}}) \, d\tilde{\mathbf{z}} d\tilde{a} &= \mu(\mathbf{z}) \int \int \mathbb{1}\{(a'(\tilde{a}, \tilde{\mathbf{z}}) \leq a\} g_t(\tilde{a}, \tilde{\mathbf{z}}) \, d\tilde{\mathbf{z}} d\tilde{a} \\ &= \mu(\mathbf{z}) \mathbb{E}_{\tilde{\mathbf{z}}} \left[\mathbb{E}_{\tilde{a}|\tilde{\mathbf{z}}} \left[\mathbb{1}\{(a'(\tilde{a}, \tilde{\mathbf{z}}) \leq a\} | \tilde{\mathbf{z}} \right] \right] \end{aligned} \tag{22}$$

Letting $g_t^{a|\mathbf{z}}(a, \mathbf{z})$ denote the conditional PDF of wealth given a vector of ideas, notice that

$$\begin{aligned} \mathbb{E}_{\tilde{a}|\tilde{\mathbf{z}}} \left[\mathbb{1}\{(a'(\tilde{a}, \tilde{\mathbf{z}}) \leq a\} | \tilde{\mathbf{z}} \right] &= \int \mathbb{1}\{(a'(\tilde{a}, \tilde{\mathbf{z}}) \leq a\} g_t^{a|\mathbf{z}}(\tilde{a}, \tilde{\mathbf{z}}) \, d\tilde{a} \\ &= \int_0^{a - s(\tilde{a}, \tilde{\mathbf{z}})} g_t^{a|\mathbf{z}}(\tilde{a}, \tilde{\mathbf{z}}) \, d\tilde{a} \\ &= \mathbb{P}(\tilde{a} \leq a - s(\tilde{a}, \tilde{\mathbf{z}}) | \tilde{\mathbf{z}}) \end{aligned}$$

where, in the last equality, we used the fact depicted in (19). We can then rewrite (22) as

$$\begin{aligned} \mu(\mathbf{z}) \mathbb{E}_{\tilde{\mathbf{z}}} \left[\mathbb{E}_{\tilde{a}|\tilde{\mathbf{z}}} \left[\mathbb{1}\{(a'(\tilde{a}, \tilde{\mathbf{z}}) \leq a\} | \tilde{\mathbf{z}} \right] \right] &= \mu(\mathbf{z}) \int \mathbb{P}(\tilde{a} \leq a - s(\tilde{a}, \tilde{\mathbf{z}}) | \tilde{\mathbf{z}}) \mathbb{P}(\tilde{\mathbf{z}}) \, d\tilde{\mathbf{z}} \\ &= \mu(\mathbf{z}) \int \mathbb{P}(\tilde{a} \leq a - s(\tilde{a}, \tilde{\mathbf{z}}), \mathbf{z} = \tilde{\mathbf{z}}) \, d\tilde{\mathbf{z}} \end{aligned} \tag{23}$$

2.1 From discrete to continuous time

Plugging (20) and (23) into (16), we obtain the following condition

$$\lambda_{t+1}(a, \mathbf{z}) = \gamma \left[\lambda_t(a - s(a, \mathbf{z}), \mathbf{z}) \right] + (1 - \gamma) \left[\mu(\mathbf{z}) \int \mathbb{P}(\tilde{a} \leq a - s(\tilde{a}, \tilde{\mathbf{z}}), \mathbf{z} = \tilde{\mathbf{z}}) \, d\tilde{\mathbf{z}} \right]$$

.

In Δ units of time, we have

$$\lambda_{t+\Delta}(a, \mathbf{z}) = \\ (1 - (1 - \gamma)\Delta) \left[\lambda_t(a - \Delta s(a, \mathbf{z}), \mathbf{z}) \right] + ((1 - \gamma)\Delta) \left[\mu(\mathbf{z}) \int \mathbb{P}(\tilde{a} \leq a - \Delta s(\tilde{a}, \tilde{\mathbf{z}}), \mathbf{z} = \tilde{\mathbf{z}}) \, d\tilde{\mathbf{z}} \right]$$

Adding and subtracting the term $\lambda_t(a, \mathbf{z})$ in the LHS and rearranging, we get

$$\lambda_{t+\Delta}(a, \mathbf{z}) - \lambda_t(a, \mathbf{z}) + \lambda_t(a, \mathbf{z}) - \lambda_t(a - \Delta s(a, \mathbf{z}), \mathbf{z}) = \\ ((1 - \gamma)\Delta) \left[\mu(\mathbf{z}) \int \left(\mathbb{P}(\tilde{a} \leq a - \Delta s(\tilde{a}, \tilde{\mathbf{z}}), \mathbf{z} = \tilde{\mathbf{z}}) \, d\tilde{\mathbf{z}} \right) - \lambda_t(a - \Delta s(a, \mathbf{z}), \mathbf{z}) \right]$$

Multiplying by $\frac{s(a,\mathbf{z})}{s(a,\mathbf{z})}$ the last two terms on the LHS and rearranging, we get

$$\frac{\lambda_{t+\Delta}(a,\mathbf{z})-\lambda_t(a,\mathbf{z})}{\Delta}+\frac{\lambda_t(a-\Delta s(a,\mathbf{z}),\mathbf{z})-\lambda_t(a,\mathbf{z})}{-\Delta s(a,\mathbf{z})}s(a,\mathbf{z})=$$
$$(1-\gamma)\left[\mu(\mathbf{z})\int\left(\mathbb{P}(\tilde{a}\leq a-\Delta s(\tilde{a},\tilde{\mathbf{z}}),\mathbf{z}=\tilde{\mathbf{z}})\;d\tilde{\mathbf{z}}\right)-\lambda_t(a-\Delta s(a,\mathbf{z}),\mathbf{z})\right]$$

Taking limits as $\Delta\to 0$, we obtain

$$\dot{\lambda}_t(a,\mathbf{z})+g_t(a,\mathbf{z})s(a,\mathbf{z})=(1-\gamma)\left[\mu(\mathbf{z})\mathbb{P}(a_t\leq a)-\lambda_t(a,\mathbf{z})\right] \tag{24}$$

Differentiating (24) with respect to a, and letting $f_t(a)$ denote the PDF of wealth at period t, we obtain[4]

$$\dot{g}_t(a,\mathbf{z})+\frac{\partial\left[g_t(a,\mathbf{z})s(a,\mathbf{z})\right]}{\partial a}=(1-\gamma)\left[\mu(\mathbf{z})f_t(a)-g_t(a,\mathbf{z})\right]$$

Thus, at a stationary state ($\dot{g}_t(a,\mathbf{z})=0$), we have

$$g(a,\mathbf{z})=\mu(\mathbf{z})f(a)-\frac{1}{1-\gamma}\frac{\partial\left[g(a,\mathbf{z})s(a,\mathbf{z})\right]}{\partial a}$$

Distribution of random vector of ideas

Let $H_t(\mathbf{z})$ denote the CDF of *observed* vector of ideas $\mathbf{z}$ at period t and $h_t(\mathbf{z})$ the associated PDF. Also, let $\mathcal{M}(\mathbf{z}):=\mathbb{P}(\mathbf{z}^*\leq\mathbf{z})$ denote the CDF of the "latent" random vector of ideas $\mathbf{z}^*$. We then have that

$$\begin{aligned}H_{t+1}(\mathbf{z})&=\gamma\mathbb{P}(\mathbf{z_t}\leq\mathbf{z})+(1-\gamma)\mathcal{M}(\mathbf{z})\\&=\gamma H_t(\mathbf{z})+(1-\gamma)\mathcal{M}(\mathbf{z})\end{aligned}$$

In Δ units of time, we have

$$H_{t+\Delta}(\mathbf{z})=\left(1-(1-\gamma)\Delta\right)H_t(\mathbf{z})+\left((1-\gamma)\Delta\right)\mathcal{M}(\mathbf{z})$$

Rearranging, we get

$$\frac{H_{t+\Delta}(\mathbf{z})-H_t(\mathbf{z})}{\Delta}=(1-\gamma)\left[\mathcal{M}(\mathbf{z})-H_t(\mathbf{z})\right]$$

Taking limits, we obtain

$$\dot{H}_t(\mathbf{z})=(1-\gamma)\left[\mathcal{M}(\mathbf{z})-H_t(\mathbf{z})\right]$$

Differentiating with respect to $\mathbf{z}$ we obtain

$$\dot{h}_t(\mathbf{z})=(1-\gamma)\left[\mathcal{M}(\mathbf{z})-h_t(\mathbf{z})\right]$$

This means that, at a stationary state ($\dot{h}_t(\mathbf{z})=0$), we have $h_t(\mathbf{z})=\mu(\mathbf{z})$.

[4] We used Young's Theorem on the first term of the RHS:

$$\frac{\partial\dot{\lambda}_t(a,\mathbf{z})}{\partial a}=\frac{\partial}{\partial a}\left[\frac{\partial\lambda_t(a,\mathbf{z})}{\partial t}\right]=\frac{\partial}{\partial t}\left[\frac{\partial\lambda_t(a,\mathbf{z})}{\partial a}\right]=\frac{\partial}{\partial t}g_t(a,\mathbf{z})=\dot{g}_t(a,\mathbf{z})$$

Summing up

We are left with the conditions

$$\dot{h}_t(\mathbf{z}) = 0 \quad \Rightarrow \quad h_t(\mathbf{z}) = \mu(\mathbf{z}) \tag{25}$$

$$\dot{g}_t(a,\mathbf{z}) = 0 \quad \Rightarrow \quad g(a,\mathbf{z}) = \mu(\mathbf{z})f(a) - \frac{1}{1-\gamma}\frac{\partial\left[g(a,\mathbf{z})s(a,\mathbf{z})\right]}{\partial a} \tag{26}$$

3 Numerical computation

We now present details on the numerical implementation of a solution. Let $\mathbf{A} = \{a_1, a_2, \ldots, a_N\}$, where $a_1 = 0$, be the grid for wealth and let $\mathbf{Z} = \{\mathbf{z_1}, \mathbf{z_2}, \ldots, \mathbf{z_M}\}$ be a grid (discretisation) for the vectors of entrepreneurial ideas. We may define the grid of states as a (MN) $\times$ 1 vector:

$$S \coloneqq \begin{pmatrix} (a_1, \mathbf{z_1}) \\ (a_2, \mathbf{z_1}) \\ \vdots \\ (a_N, \mathbf{z_1}) \\ (a_1, \mathbf{z_2}) \\ (a_2, \mathbf{z_2}) \\ \vdots \\ (a_N, \mathbf{z_2}) \\ \vdots \\ (a_1, \mathbf{z_M}) \\ (a_2, \mathbf{z_M}) \\ \vdots \\ (a_N, \mathbf{z_M}) \end{pmatrix}$$

Do also define the following objects:

$$
S_Z \coloneqq \begin{pmatrix} \mathbf{z_1} \\ \mathbf{z_1} \\ \vdots \\ \mathbf{z_1} \\ \mathbf{z_2} \\ \mathbf{z_2} \\ \vdots \\ \mathbf{z_2} \\ \vdots \\ \mathbf{z_M} \\ \mathbf{z_M} \\ \vdots \\ \mathbf{z_M} \end{pmatrix} \qquad S_A \coloneqq \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \\ a_1 \\ a_2 \\ \vdots \\ a_N \\ \vdots \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}
$$

Next, let $\bar{\mathbf{D}}$ be the N $\times$ N numerical differentiation matrix with respect to a, i.e.:

$$
\bar{\mathbf{D}} \coloneqq \begin{pmatrix} -\frac{1}{da} & \frac{1}{da} & 0 & 0 & \dots & 0 \\ -\frac{1}{2da} & 0 & \frac{1}{2da} & 0 & \dots & 0 \\ 0 & -\frac{1}{2da} & 0 & \frac{1}{2da} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -\frac{1}{da} & \frac{1}{da} \end{pmatrix}
$$

where $da = a_{i+1} - a_i$ for every $i \in \{1, \dots, N\}$. With this, define the following MN $\times$ MN *partial* differentiation matrix:

$$
D_{AZ} \coloneqq \mathbb{I}_{\mathrm{M}} \otimes \bar{\mathbf{D}} = \begin{pmatrix} \bar{\mathbf{D}} & 0 & \dots & 0 \\ 0 & \bar{\mathbf{D}} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \bar{\mathbf{D}} \end{pmatrix}
$$

Further, consider the following M $\times$ 1 auxiliary vectors

$$
\mu(\mathbf{Z}) = \begin{pmatrix} \mu(\mathbf{z_1}) \\ \mu(\mathbf{z_2}) \\ \vdots \\ \mu(\mathbf{z_M}) \end{pmatrix} \qquad i_M = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}
$$

Finally, consider the MN $\times$ MN matrix

$$N \coloneqq \mu(Z){i_M}' \otimes \mathbb{I}_\mathrm{N} = \begin{pmatrix} \mu(\mathbf{z_1})\mathbb{I}_\mathrm{N} & \dots & \mu(\mathbf{z_1})\mathbb{I}_\mathrm{N} \\ \mu(\mathbf{z_2})\mathbb{I}_\mathrm{N} & \dots & \mu(\mathbf{z_2})\mathbb{I}_\mathrm{N} \\ \vdots & \vdots & \vdots \\ \mu(\mathbf{z_M})\mathbb{I}_\mathrm{N} & \dots & \mu(\mathbf{z_M})\mathbb{I}_\mathrm{N} \end{pmatrix}$$

With these objects at hand, we will be able to provide a numerical solution to the problem of finding the stationary equilibrium. The algorithm we propose goes as follows (we normalise $p_m = 1$):

- Define an excess demand function in the following way: for a given vector (r, w, p_s)

  1. Solve the occupational choice problem and compute $M(a, z)$ for every state.
  2. Compute the value function $v(a, \mathbf{z})$ and the associated policy functions.
  3. Compute the stationary distribution $g(a, z)$.
  4. Compute and return the excess demand in three of the four markets in the model.

- Given an initial guess (r^0, w^0, p_s^0), use a *tâtonnement*-based approach in order to reach equilibrium: update prices based on the excess demand until ||excess_demand|| < tol or the maximum number of iterations is reahced.

Below, we provide practical details on the implementation of the proposed algorithm. Throughout, we fix a tuple (r, w, p_s).

3.1 Computing the value function

Given the vector (r, w, p_s), one can easily compute $M(a, \mathbf{z})$. For each $(a, \mathbf{z}) \in S$, compute the profits in each sector and compare these with the wage. We write $M(S)$ for the resulting $(MN) \times 1$ vector.

Now, we need to choose an initial guess for $v(S)_{(MN)\times 1}$. Choosing a constant as an initial guess would not be satisfactory for, in this case, we would obtain a zero vector for the numerical derivative. We adopt a "staying-put" approach: the initial guess for the value function is given by the discounted indirect utility (i.e. $\frac{u(\mathbf{c}^{\mathrm{sp}})}{\rho}$) obtained by solving the individual's static maximisation problem:

$$\max_{\mathbf{c}} \; u(\mathbf{c}) \quad s.t. \quad \mathbf{p} \cdot \mathbf{c} \leq M(a, \mathbf{z}) + ra$$

Since the utility function satisfies the Inada conditions, the budget constraint binds, and we have that

$$\begin{aligned} c_m^{\mathrm{sp}} &= (M(a, \mathbf{z}) + ra) \left[1 + p_s \left(\frac{\psi}{(1-\psi)p_s} \right)^{\epsilon} \right] \\ c_s^{\mathrm{sp}} &= M(a, \mathbf{z}) + ra - \frac{c_m}{p_s} \end{aligned}$$

Now, given a guess $v^n(S)$, one may compute the associated consumption policy function associated. Indeed, note that the FOCs in (14) give:

$$\nabla_{c_m} u(\mathbf{c}) = \frac{\partial v(a,\mathbf{z})}{\partial a}$$
$$\nabla_{c_s} u(\mathbf{c}) = \frac{\partial v(a,\mathbf{z})}{\partial a} p_s$$

Using our functional form, we get:

$$(\psi c_s^{(1-1/\epsilon)} + (1-\psi)c_m^{(1-1/\epsilon)})^{\frac{1/\epsilon-\sigma}{1-1/\epsilon}}(1-\psi)c_m^{(-1/\epsilon)} = \frac{\partial v(a,\mathbf{z})}{\partial a}$$
$$(\psi c_s^{(1-1/\epsilon)} + (1-\psi)c_m^{(1-1/\epsilon)})^{\frac{1/\epsilon-\sigma}{1-1/\epsilon}}\psi c_s^{(-1/\epsilon)} = \frac{\partial v(a,\mathbf{z})}{\partial a} p_s$$

which in its turn yields $c_s = c_m \left\{\frac{p_s(1-\psi)}{\psi}\right\}^{-\epsilon}$. Plugging this back into the above leads to a simple way of finding each consumption as a function solely of $\frac{\partial v(a,\mathbf{z})}{\partial a}$[5]:

$$\begin{aligned} c_m &= \left[\frac{\partial v}{\partial a}\right]^{-\frac{1}{\sigma}} \left\{\left[\psi^{\epsilon}(1-\psi)^{1-\epsilon}p_s^{1-\epsilon} + (1-\psi)\right]^{\frac{1-\epsilon\sigma}{\epsilon-1}}(1-\psi)\right\}^{\frac{1}{\sigma}} \\ c_s &= c_m \left\{\frac{p_s(1-\psi)}{\psi}\right\}^{-\epsilon} \end{aligned} \tag{27}$$

Thus, given a guess $v_N(S)$, one gets:

$$c_m^N(S) = [D_{AZ}.v_N(S)]^{-\frac{1}{\sigma}} \left\{\left[\psi^{\epsilon}(1-\psi)^{1-\epsilon}p_s^{1-\epsilon} + (1-\psi)\right]^{\frac{1-\epsilon\sigma}{\epsilon-1}}(1-\psi)\right\}^{\frac{1}{\sigma}}$$
$$c_s^N(S) = c_m^N(S) \left\{\frac{p_s(1-\psi)}{\psi}\right\}^{-\epsilon}$$

Let $c^N(S) = \begin{bmatrix} c_m^N(S) & c_s^N(S) \end{bmatrix}$. Denote by $u(c^N)$ the corresponding utility across different states. We can write $S^N = \text{diag}\left\{M(S) + rS_A - c^N(S)\begin{pmatrix}1 & p_s\end{pmatrix}'\right\}$. We then may find $v_{N+1}(S)$ implicitly by solving:

$$\rho v^{N+1}(S) = u(c^N) + S^N.D_{AZ}.V^{N+1}(S) + (1-\gamma)\{N'V^{N+1}(S) - V^{N+1}(S)\} \tag{28}$$

Rearranging the above equation, we get:

$$\rho v^{N+1}(S) = u(c^N) + \left[S^N.D_{AZ} + (1-\gamma)\left(N' - \mathbb{I}_{\text{MN}}\right)\right] v^{N+1}(S) \tag{29}$$

We iterate until convergence. We can also add a step of size Δ so the algorithm does not overshoot[6].

[5]When $\epsilon = 1$, utility is given by $u(\mathbf{c}) = (\sigma-1)(\psi\ln(c_s) + (1-\psi)\ln(c_m))$. Then, from the FOCs, we obtain $c_s = \frac{(\sigma-1)\psi}{p_s}\left(\frac{\partial v(a,\mathbf{z})}{\partial a}\right)^{-1}$ and $c_m = (\sigma-1)(1-\psi)\left(\frac{\partial v(a,\mathbf{z})}{\partial a}\right)^{-1}$.

[6]In this case, one would solve the following:

$$\left(\rho + \frac{1}{\Delta}\right) v^{N+1}(S) = u(c^N) + \frac{1}{\Delta}V^N(S) + \left[S^N.D_{AZ} + (1-\gamma)\left(N' - \mathbb{I}_{\text{MN}}\right)\right] v^{N+1}(S)$$

In practice , we use an upwind scheme for the computation of derivatives, as suggested in Achdou et al. (2017). The idea is to use the forward difference approximation whenever the drift of the state variable is positive and the backward difference whenever it is negative. This approach is preferred, as using centralised differences may not ensure convergence [7].

3.2 Computing the stationary distribution

We now present the method we use to numerically compute the stationary distribution. We seek, in this section, to represent (26) in matrix form. We may define the joint PDF $g(a,t)$ over the grid of states as a (MN) $\times$ 1 vector:

$$g(S) \coloneqq \begin{pmatrix} g(a_1, \mathbf{z_1}) \\ g(a_2, \mathbf{z_1}) \\ \vdots \\ g(a_N, \mathbf{z_1}) \\ g(a_1, \mathbf{z_2}) \\ g(a_2, \mathbf{z_2}) \\ \vdots \\ g(a_N, \mathbf{z_2}) \\ \vdots \\ g(a_1, \mathbf{z_M}) \\ g(a_2, \mathbf{z_M}) \\ \vdots \\ g(a_N, \mathbf{z_M}) \end{pmatrix}$$

How do we obtain the partial derivative term on the RHS of (26)? First, let us define the following MN $\times$ MN diagonal matrix

$$S_{AZ} \coloneqq \operatorname{diag}\{s(a,\mathbf{z}) : (a,\mathbf{z}) \in S\} = \begin{pmatrix} s(a_1, \mathbf{z_1}) & 0 & 0 & \dots & 0 \\ 0 & s(a_2, \mathbf{z_1}) & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & s(a_N, \mathbf{z_M}) \end{pmatrix}$$

With this at hand, we can obtain the MN $\times$ 1 vector of partial derivatives on the RHS of (26): just compute $D_{AZ}.S_{AZ}.g(S) = \left[\mathbb{I}_{\mathrm{M}} \otimes \bar{\mathbf{D}}\right] S_{AZ}.g(S)$. Finally, we need to compute the first term on the RHS of (26), which can be written as the following MN $\times$ 1 vector

[7] We did experiment with using simple centralised differences and could not attain convergence. The upwind scheme is useful as it ensures the Barles and Souganidis (1991) monotonicity. See Achdou et al. (2017).

$$\mu(S_Z)f(S_A) \coloneqq \begin{pmatrix} \mu(\mathbf{z_1})f(a_1) \\ \mu(\mathbf{z_1})f(a_2) \\ \vdots \\ \mu(\mathbf{z_1})f(a_N) \\ \mu(\mathbf{z_2})f(a_1) \\ \mu(\mathbf{z_2})f(a_2) \\ \vdots \\ \mu(\mathbf{z_2})f(a_N) \\ \vdots \\ \mu(\mathbf{z_M})f(a_1) \\ \mu(\mathbf{z_M})f(a_2) \\ \vdots \\ \mu(\mathbf{z_M})f(a_N) \end{pmatrix}$$

We can now obtain $\mu(S_Z).f(S_A)$ by computing $N.g(S) = [\mu(Z).i_M{}' \otimes \mathbb{I}_{\mathrm{N}}]\,.g(S)$. To finish our computation, consider the following matrix

$$L \coloneqq N - \frac{1}{1-\gamma} D_{\mathrm{AZ}}.S_{\mathrm{AZ}} = \mu(Z).i_{\mathrm{M}}{}' \otimes \mathbb{I}_{\mathrm{N}} - \frac{1}{1-\gamma}[\mathbb{I}_{\mathrm{M}} \otimes \mathbf{D}]S_{\mathrm{AZ}}$$

We then have that $g(S) = L.g(S)$, which implies $(L - \mathbb{I}_{\mathrm{MN}})g(S) = 0$. Thus, we can obtain $g(S)$ by finding the eigenvector associated with the eigenvalue zero of the matrix $(L - \mathbb{I}_{\mathrm{MN}})$ and normalising it to one.

4 Final remarks

Simulation of the model is left to `Replicate.py`. This file implements class *Buera* in `Buera.py`, which solves the model for given guesses of r, w and p_s and computes excess demand in asset, labour and service markets. The class also allows searching for equilibrium using the *tâtonnement*-based algorithm described in Section 3.

The algorithm runs really fast. Indeed, when we use a grid with 1,000 points for the asset and five nodes for the Pareto distribution (a total of $1,000 \times 5 \times 5 = 25,000$ states), we are able to compute excess demand in markets in less than one second.

Nonetheless, our solution method is partially flawed: the *tâtonnement*-based approach does not ensure convergence to the equilibrium. Buera et al. (2011) do not describe their algorithm, so we were not able to implement a version of it. Moreover, since firms are credit-constrained, we are not able to obtain, like in the Aiyagari model, representative firms whose FOCs allow us to pin down w and p_s as function of r[8]. We are therefore left with a three-dimensional root-finding problem, and since the numerical excess demand function is not smooth, derivative-based methods don't work properly.

Finally, it should be noted that, once convergence is achieved, the reproduction of Table 2 and Figure 4 is a somewhat simpler task. Table 2 uses the closed-form results in Propositions 2 and 3 in order to calibrate most parameters – except for parameters ϕ and β, which are calibrated using the simulated moments of the numerical solution. As for Figure 4, one could obtain, for instance, the dashed line by reallocating capital within a sector so that that the marginal productivity of capital is equalised among active entrepreneurs. This would involve solving a non-linear system of equations: let N_j be the number of active entrepreneurs in a given sector. We would need to find an allocation $\{k_j^i\}_{i\in\{1,2,\ldots,N_j\}}$ satisfying: (a) marginal products $MPK(K_j^i)$ are equal within the sector ($N_j - 1$ equations); (b) the total amount of capital invested in the sector remains the same (one restriction). We could use `fsolve` in order to solve this just-identified system, and then would recalculate output in the sector and find the TFP as a residual[9].

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[8] Note that the models in Moll (2014) and Cagetti and De Nardi (2006) introduce an unconstrained sector (public firms in Moll (2014); non-entrepreneurial sector in Cagetti and De Nardi (2006)) whose aggregate production function yields FOCs that reduce the dimensionality of the problem. In Moll (2014), the problem boils down to searching for only one price, the equilibrium interest rate; so the bisection method applies.

Do also note that the closed-form results in Proposition 2 of Buera et al. (2011) only apply to the perfect-credit benchmark ($\phi = 1$) and are themselves approximate. Indeed, we tried to use them in order to reduce the problem into only finding the equilibrium interest rate, but could not achieve convergence in that case either.

[9] In a working paper version of the article, the authors say they calculate sectoral outputs as $A_j = Y_j K_j^{-\frac{1}{3}} L_j^{-\frac{2}{3}}$. See Buera et al. (2008).

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