4 Additional proofs in the Finite Markov Chain and McCall Model and Fixed-Point Theorem Family

1 One proposition and its proof

From the lecture about the Finite Markov Chain, we have the following proposition:

Proposition 1

If P is a stochastic matrix, then the k-th power P^k for all $k \in N$ is also a stochastic matrix.

Proof by induction.

Given the condition, we know that $P = (p_{i,j})_{m \times m}$ is a stochastic matrix.

By the definition of the stochastic matrix, we have

1. $p_{i,j} \geq 0$ for any $i, j \in \{1, \dots, m\}$;
2. $p_{i,1} + p_{i,2} + \cdots + p_{i,m} = \sum_{j=1}^{m} p_{i,j} = 1$ for $i \in \{1, \dots, m\}$.

When $n = 1$, $P^1 = P$ is a stochastic matrix.

Assume that $P^{k-1} = (p'_{i,j})_{m \times m}$ is a stochastic matrix, that is

1. $p'_{i,j} \geq 0$ for any $i, j \in \{1, \dots, m\}$;
2. $p'_{i,1} + p'_{i,2} + \cdots + p'_{i,m} = \sum_{j=1}^{m} p'_{i,j} = 1$ for $i \in \{1, \dots, m\}$.

then we are interested in

$$P^k = P^{k-1} \cdot P = (p''_{i,j})_{m \times m} \tag{1}$$

where

$$\begin{aligned} p''_{i,j} &= p'_{i,1} p_{1,j} + p'_{i,2} p_{2,j} + \cdots + p'_{i,n} p_{n,j} \\ &= \sum_{z=1}^{m} p'_{i,z} p_{z,j} \end{aligned} \tag{2}$$

Since we know that

$$p_{i,j} \geq 0 \; for \; any \; i, j \in \{1, \dots, m\} \tag{3}$$

and

$$p'_{i,j} \geq 0 \; for \; any \; i, j \in \{1, \dots, m\} \tag{4}$$

and the product of two nonnegative real numbers is a nonnegative real number, and the sum of finite many nonnegative real numbers is a nonnegative real number, then by (2), we have

$$p''_{i,j} \geq 0 \; for \; any \; i, j \in \{1, 2, \ldots, m\} \tag{5}$$

Since we have

$$p_{i,1} + p_{i,2} + \cdots + p_{i,m} = \sum_{j=1}^{m} p_{i,j} = 1 \; for \; i \in \{1, \ldots, m\} \tag{6}$$

and

$$p'_{i,1} + p'_{i,2} + \cdots + p'_{i,m} = \sum_{j=1}^{m} p'_{i,j} = 1 \; for \; any \; i \in \{1, \ldots, m\} \tag{7}$$

For any row $i \in \{1, \ldots, m\}$, by the definition of $p''_{i,j}$ we have

$$\begin{aligned} p''_{i,1} + p''_{i,2} + \cdots + p''_{i,m} &= \sum_{j=1}^{m} p''_{i,j} \\ &= \sum_{j=1}^{m} \sum_{z=1}^{m} p'_{i,z} p_{z,j} \\ &= \sum_{z=1}^{m} p'_{i,z} \sum_{j=1}^{m} p_{z,j} \\ &= \sum_{z=1}^{m} p'_{i,z} \cdot 1 \\ &= 1 \end{aligned} \tag{8}$$

By the definition of a stochastic matrix, we can conclude that the matrix P^k is also a stochastic matrix.

By induction, we can conclude that if P is a stochastic matrix, then the k-th power P^k for all $k \in N$ is also a stochastic matrix.

2 Brouwer's and Kakutani's Fixed-point theorem

Proposition 2 (Brouwer's Fixed-point theorem)

If X is a nonempty, compact, and convex subset of R^k for some integer k, and $f : X \to X$ is continuous, then there exists a fixed point of f, that is, for some $x^* \in X$, $f(x^*) = x^*$.

Proposition 3 (Kakutani's Fixed-Point Theorem)

If X is a nonempty compact, and convex subset of R^k for some integer k, and $F : X \to X$(Not a function) is nonempty and convex valued and upper semi-continuous, then there exists a fixed point of F, that is, for some $x^* \in X$, $f(x^*) = x^*$.

Remark

These two are equivalent, that is, Brouwer's FP theorem is implied by Kakutani's, and Kakutani's can be proved if one assumes Brouwer's.

Intuitive demonstration of Brouwer's FP Theorem

Suppose X is a circle (interior and boundary) in R^2, and $f : X \to X$ has no fixed point.

For each point $x \in X$, find $f(x)$ and draw a ray from $f(x)$ through x, label the point on the boundary hit by this ray as $g(x)$.

As long as $f(x) \neq x$, this is a well-defined process for defining $g(x)$.

If x moves a bit, then $f(x)$ moves only a bit. Hence, the ray moves only a bit and the intersection point moves only a bit. So intuitively, we can conclude that g is continous.

Moreover, if x lies on the boundary of X, then $g(x) = x$.

But then g is a continous map from X to its boundary that is the identity on the boundary. And intuitively one cannot do this. You must "tear" X somewhere to map the whole circle onto its boundary in a way that leaves the boundary fixed.

The existence of a fixed point rule this thing(? what exactly) out because:

If $f(x) = x$, then it isn't clear how to define $g(x)$.

More to the point, slight movements in x around such a fixed point can mean a big movement in $g(x)$; the "proof" that g is continous depends on some separation between x and $f(x)$.

Remark

1. This is not a proof, but we can turn it into one:

A result in Borsuk's Lemma (*?*) shows that this sort of continous map is impossible (in R^2 and in higher dimensions)

2. More direct proofs are possible, and the most accessible are "constructive" or computational:

Using Sperner's Lemma (*?*), we can show how to find "approximate fixed points" to any degree of approximation desired, and then pass to a limit. (See Border (1990) or Scarf (1973)). We can prove Brouwer in this fashion, and use the Green's Theorem (*?*) as the basis of a proof.

Some extensions of Brouwer and Kakutani's FP Theorem

Some of FP theorems in the literature are in the same spirit as K and B's. e.g., In Brouwer, convexity per se of X is inessential.

There is a trivial extension of B & K's FP theorem:

Corollary 1

If X is homeomorphic to a convex and compact set $Y \subseteq R^k$, and $f : X \to X$ is continous, then there is some $x^* \in X$ such that $f(x^*) = x^*$.

Remark

Other extensions of B and K are less trivial. E.g., Eilenberg-Montgomery Fixed-Point Theorem.

3 Existence of stationary distribution of a stochastic matrix

Def. (Stationary distribution)

A distribution ψ^* on S is called stationary for P if

$$\psi^* = \psi^* P \tag{9}$$

Proposition 4. (Stochastic matrix vs. Existence of Stationary distribution)

Assume that the state space S is finite (if not more assumptions are required).

Every stochastic matrix P has at least one stationary distribution.

Proposition 5. (Equivalence of the Prop.4)

Without loss of generality, let $S = \{w_1, w_2, \ldots, w_m\}$ be the state space, $\Psi = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}_+^m | x_i \geq 0 \; and \; \sum_{i=1}^m x_i = 1\}$ be the set of all possible distributions, that is, $\psi_t \in \Psi$ for any t, and $P = (p_{i,j})_{m \times m}$ be a m-dimension stochastic matrix.

Suppose that $P : \psi \to \psi P$. Since $\psi = \Psi$ and $\psi P \subseteq \Psi$, w.l.g., we can write $P : \Psi \to \Psi$ such that $P(\psi_t) = \psi_{t+1}$, where $\psi_t = (x_1^t, x_2^t, \ldots, x_m^t)$, $\psi_{t+1} = (x_1^{t+1}, x_2^{t+1}, \ldots, x_m^{t+1})$

Then we can conclude that there exist some $\psi^* \in \Psi$ such that

$$P(\psi^*) = \psi^* \tag{10}$$

Analysis

If we can prove Ψ is a nonempty, compact, and convex subset of the Euclidean space R^m and $P: \Psi \to \Psi$ such that $P(\psi_t) = \psi_{t+1}$ is continuous, then by Brouwer's Fixed-point theorem, we prove the proposition, and then the existance of a stationary distribution for any stochastic matrix.

Proof.

Since at least $(0, .., 0, 1) \in \Psi$, so Ψ is not an empty set.

From the set Ψ, since $x_i \geq 0$ and $\sum_{i=1}^m x_i = 1$, we have $0 \leq x_i \leq 1$, so Ψ is bounded.

Let $f(x_1, x_2, \ldots, x_m) = \sum_{i=1}^m x_i - 1$, where $0 \leq x_i \leq 1$ for all i.

Since $f(x_1, x_2, \ldots, x_m)$ is continous in every x_i, where $0 \leq x_i \leq 1$ for all i, we know Ψ is closed.

Given Ψ is bounded and closed in Euclidean space $\mathbb{R}^m$, we know that Ψ is compact.

Also because the function $f(x_1, x_2, \ldots, x_m)$ is affine in every x_i, where $0 \leq x_i \leq 1$, for all i, the set Ψ is also convex.

Given a vector $\psi_t = (x_1^t, x_2^t, \ldots, x_m^t)$, we have

$$\begin{aligned} & P(\psi_t) = P(x_1^t, x_2^t, \ldots, x_m^t) \\ & = (p_1(x_1^t, x_2^t, \ldots, x_m^t), p_2(x_1^t, x_2^t, \ldots, x_m^t), \ldots, p_m(x_1^t, x_2^t, \ldots, x_m^t)) \\ & = (\sum_{i=1}^m x_i^t p_{i,1}, \sum_{i=1}^m x_i^t p_{i,2}, \ldots, \sum_{i=1}^m x_i^t p_{i,m}) = (x_1^{t+1}, x_2^{t+1}, \ldots, x_m^{t+1}) = \psi_{t+1} \end{aligned} \tag{11}$$

where $p_j(x_1^t, x_2^t, \ldots, x_m^t) = \sum_{i=1}^m x_i^t p_{i,m} = x_j^{t+1}$.

Given previous conditions, we know the function $p_j(\psi_t) = x_j^{t+1}$ is continuous in Ψ, and hence, $P(\psi_t) = \psi_{t+1}$ should also be continuous on Ψ.

By Brouwers's Fixed-Point theorem, we can conclude that there exists some $\psi^* \in \Psi$ such that

$$P(\psi^*) = \psi^* \tag{12}$$

Which is what we want to prove for the proposition.

Since the proposition is equivalent to the existence theorem, we can also conclude that every stochastic matrix P has at least one stationary distribution.

Appendix: Fixed-point theorem

Before we prove the existence of stationary distribution, we need to introduce fixe-point theorem first.

A1 The simplest Fixed-point theorem

Proposition 6

Suppose $f : [0, 1] \to [0, 1]$ is continuous. Then there exists a fixed point of f, meaning some point x^* such that $f(x^*) = x^*$.

Proof

Define the function $\phi : [0, 1] \to \mathbb{R}$ by $\phi(x) = f(x) - x$.

The function ϕ is continous, and $\phi(0) = f(0) - 0 = f(0) \geq 0$ while $\phi(1) = f(1) - 1 \leq 0$.

By Intermediate-value theorem, there exists some point $x^* \in [0, 1]$ s.t.

$$\phi(x^*) = 0 \tag{13}$$

Since $\phi(x) = f(x) - x$, we have

$$f(x^*) - x^* = 0 \; or \; f(x^*) = x^* \tag{14}$$

Remark

This is the simplest of all fixed-point theorems.

A2 General Forms of the Fixed-point theorem

More generally, fixed-point theorems have two basic forms.

Proposition 7 (FP form for function)

If f is a function with domain X and range X [and then conditions on X and f are given], then there exists some $x^* \in X$ such that $f(x^*) = x^*$.

Proposition 8 (FP form for correspondence)

If F is a correspondence with domain X and range X (that is, $F(x) \subseteq X$ for all $x \in X$), [and then conditions on X and F are given], then there exists some $x^* \in X$ such that $x^* \in F(x^*)$.

Remarks

The fixed-point theorem will sometimes say about the structure of the set of fixed points.

A3 Fixed-point theorem and Economics

In Economics, Fixed-point thoerems are used to prove that an equilibrium to some system or other exists.

e.g., we can use fixed-point theory to prove the existence of a Walrasian equilibrium for a general equilibrium economy.

Economists use a variety of fixed-point theorems for these purposes.

A4 Brouwer's and Kakutani's Fixed-point theorems

Proposition 9 (Brouwer's Fixed-point theorem)

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Other f-p theorems

Some others are distinctly different from Brouwer and Kakutani's.

Two of the most important are Banach's Fixed-Point Theorem (also known as the Contraction-Mapping Theorem), and Tarksi's Fixed-Point Theorem.

Both concern fixed points of functions, but have relatively easy extensions to correspondences.

A5 Banach's and Nadler's Fixed-Point Theorems

Because Banach's is sometimes used in context where the domain (and range) of the function are more complex than finite-dimensional Euclidean space, we will give it in greater generality.

Def. (Contraction mapping)

The function f is called a **contraction mapping** if, for some $\alpha < 1$, $f : X \to X$ satisfies:

$$d(f(x), f(x')) \leq \alpha d(x, x') \; for \; all \; x, x' \in X \tag{15}$$

Proposition 11 (Banach's; Function)

If $f : X \to X$ is contraction mapping defined on a complete metric space (X, d), then f has a unique fixed point x^*, a single point satisfying $f(x^*) = x^*$.

Proof.

Take any point from X and label it x^0. Let $x^1 = f(x^0)$ and inductively, $x^{n+1} = f(x^n)$.

Let $A = d(x^0, x^1)$, and inductively, $d(x^n, x^{n+1}) \leq \alpha^n A$.

Def. (Contractive correspondence)

The correspondence F is **contractive** if, for some $\alpha < 1$, $F : X \to Z$ satisfies $d_H(F(x), F(x')) \leq \alpha d(x, x')$ for all $x, x^0 \in X$.

Proposition 12 (Nadler's; Correspondence)

If F is a contractive, nonempty-valued, and compact-valued correspondence defined on a complete metric space, then there is some $x^* \in X$ such that $x^* \in F(x^*)$.

Reference

1. David M. Kreps:"Microeconomics Foundations I: Choice and Competitive Markets", 2013.