

OPTIMAL CONSUMPTION, PORTFOLIO AND LIFE INSURANCE RULES FOR AN UNCERTAIN LIVED INDIVIDUAL IN A CONTINUOUS TIME MODEL

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A continuous time model for optimal consumption, portfolio and life insurance rules, for an investor with an arbitrary but known distribution of lifetime, is derived as a generalization of the model by Merton (1971). The classic Tobin–Markowitz separation theorem obtains with the mutual funds being identical to those obtained under the assumption of certain lifetime. The investor is found to have a ‘human capital’ component of wealth, which is independent of his preferences and risky market opportunities and represents the certainty equivalent of his future net (wage) earnings. Explicit solutions, which are linear in wealth, are found for the investor with constant relative risk aversion.

1. Introduction and conclusions

In this paper the work of Merton (1971) is generalized to include optimal consumption, investment and life insurance decisions for an investor with an arbitrary, but known, distribution of lifetime. Throughout this paper all markets are assumed to be perfect and frictionless with securities traded continuously. The price of all securities traded in these markets follows a stationary geometric Brownian motion so that prices at any future time have a log-normal distribution. Under these assumptions Merton has shown that portfolio separation will obtain for a decision-maker with known lifetime T wishing to maximize

$$E \int_0^T U(C, t) dt,$$

where U is the instantaneous utility function, C is consumption, and E is the expectation operator. By portfolio separation we mean a ‘mutual fund’ theorem can be proven, which states that each investor would be indifferent between choosing his portfolio by selecting from all available securities or choosing his

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portfolio by only buying shares in two mutual funds.¹ Furthermore, if a riskless security² exists then one of the mutual funds may be chosen to be the riskless security and the other a weighted combination of all the risky securities with the risky fund's price having a log-normal distribution.

It will be shown below that the separation theorem also obtains if the investor behaves to

$$\text{Max. } E[\int_0^T U(C, t) dt + B(Z, T)],$$

where now T is the uncertain time of death, Z is the legacy he leaves at death, B is his utility for bequest, and E is an expectation over time of death, as well as future distribution of prices. Furthermore, the composition of the risky mutual fund will be exactly the same as in Merton's case of known lifetime, i.e., all securities in the risky fund will be represented in the same proportions they would be if the investor had a certain lifetime. However, the uncertain lived individual will not, in general, invest the same proportion of his wealth in the risky fund as would the certain lived individual. Thus, within the context of the Merton model, we find that uncertain life time and life insurance in no way affect the investment opportunity set nor the efficient portfolios. This is in part due to the individual's expending (receiving) funds in order to buy (sell) life insurance.

The life insurance offered is of an instantaneous term³ variety where payment at a premium rate $P(t)$ dollars per unit time buys insurance of $P(t)/\mu(t)$ dollars, where $\mu(t)$ is specified in the insurance contract. The only general result concerning the bequest is that if an individual's imputed utility for wealth is strictly concave (i.e., he is risk-averse), then his legacy will be a strictly increasing function of wealth. In general this does not mean he will purchase insurance, but will either buy or sell it depending on his feelings for risk and his relative liking for consumption or legacy.

An interesting result obtained is a new 'separation' theorem which proves that for every investor there is an amount of 'human capital', independent of risky market opportunities and preferences, which is a certainty equivalent for the investor's future (wage) earnings stream which is assumed to be sure if the investor is alive. This 'human capital' term is calculated by discounting the future earnings stream until the maximum time of death at a discount rate equal to the sum of the risk-free rate plus the insurance rate applicable at that time. This rate is a generalization of a result by Yaari (1965) who shows that it is the correct discount rate to use for cash flows that will be received only if an individual does not die when life insurance is priced to be a fair game.

¹Of course the mutual funds will not be the same for any two investors unless they agree on the distribution of the securities.

²The riskless security has a sure return and will not default. Furthermore, all returns are stated in terms of net price level changes.

³There is no loss of generality in using term insurance since all available life insurance is a linear combination of one period (year) term insurance and a savings plan of some sort.

Lastly the assumption is made that the individual's utility for both consumption and bequest have constant relative risk aversion. In this case explicit solutions, which are linear in wealth, are found for optimal consumption, life insurance and portfolio rules.

Hakansson (1969) has also examined the problem analyzed herein with constant relative risk aversion utility, but in a discrete time environment. Unfortunately, he was unable to solve explicitly for the optimal level of life insurance but he was able to find conditions under which zero insurance is optimal.

Lastly, Hakansson finds that the optimal mix of risky securities will depend upon the investor's utility for consumption. We find the *level* of investment in risky assets will depend upon the utility for consumption, but not the mix of risky assets.

In an extension of Hakansson's discrete time formulation, Fischer (1973) also examines optimal lifetime insurance purchases. Fischer obtains a human capital theorem similar to the one proved in this paper, but only for utility functions with constant relative risk aversion.

2. The economic environment and asset dynamics

As Merton does, we assume that there are n securities and that the prices per share, $\{S_i(t)\}$, are generated by a non-stationary geometric Brownian motion process, i.e.,

$$dS_i(t)/S_i(t) = \alpha_i(t) dt + \sigma_i(t) dq_i(t), \quad i = 1, \dots, n, \quad (1)$$

where $dq_i(t)$ is the stochastic differential⁴ of $q_i(t)$ which is a standard normal random variable, $\alpha_i(t)$ is the constant⁵ instantaneous expected percentage change

⁴The variate dq is often referred to as Gaussian White Noise. The 'differentiation' of a normal random variable has rules different from those found in the calculus. The correct differentiation of functions of q is done according to a theorem known as Ito's Lemma. For a relatively easy to follow (but rigorous) explanation of stochastic integrals and stochastic calculus see Kushner (1967, especially ch. 10.5–10.7).

⁵By constant we mean non-stochastic functions of time and, in general, all parameters may be known functions of t . However, the gain from this generalization is illusory since no revision of the functions is allowed and all results are identical for the case where the parameters are true constants, except that the distributions would be linear and non-stationary for true constants. As an example we use Ito's Lemma and (1) to calculate

$$\begin{aligned} d \ln S_i(t) &= [dS_i(t)/S_i(t)] - \frac{1}{2}[dS_i(t)/S_i(t)]^2 \\ &= [\alpha_i(t) - \frac{1}{2}\sigma_i^2(t)] dt + \sigma_i(t) dq_i(t). \end{aligned}$$

Integrating we find that

$$\ln\{S_i(t)/S_i(0)\} = \int_0^t [\alpha_i(s) - \frac{1}{2}\sigma_i^2(s)] ds + \int_0^t \sigma_i(s) dq_i(s).$$

The first term in the right-hand side of this equation is simply a function of t , while the second term is a non-stationary normal random variable. Thus $S_i(t)$ has a non-stationary log-normal distribution. If α_i and σ_i are true constants, then

$$\ln\{S_i(t)/S_i(0)\} = (\alpha_i - \frac{1}{2}\sigma_i^2)t + \sigma_i q_i(t),$$

where $q_i(t)$ is normal with mean zero and variance t .

in price per unit time and $\sigma_i^2(t)$ is the constant instantaneous variance per unit time. Actually (1) is shorthand symbolism for the stochastic integral equation

$$S_i(t) = S_i(0) + \int_0^t S_i(s) \alpha_i(s) ds + \int_0^t S_i(s) \sigma_i(s) dq_i(s), \quad (1')$$

$$i = 1, \dots, n,$$

and (1) should *always* be interpreted as meaning (1'). The investor's wealth at time t (assuming he has not died previously) is given by the stochastic integral

$$W(t) = W_0 - \int_0^t C(s) ds - \int_0^t P(s) ds + \int_0^t Y(s) ds + \sum_{i=1}^n \left[\int_0^t \frac{w_i(s) W(s)}{S_i(s)} dS_i(s) \right], \quad (2)$$

where $W(0) = W_0$ is his wealth at time 0; $Y(s)$ is his non-stochastic rate of earnings (presumably wages) at time s ; $w_i(s)$ is the fraction of his wealth invested in security i at time s so that

$$\sum_{i=1}^n w_i \equiv 1.$$

Thus, $w_i(s)W(s)/S_i(s)$ is the number of shares of security i that the investor holds at time s . Substituting (1) into (2) for dS_i/S_i and writing the result as a stochastic differential equation, we have⁶

$$dW(t) = -C(t) dt - P(t) dt + Y(t) dt + \sum_{i=1}^n w_i(t) W(t) [\alpha_i(t) dt + \sigma_i(t) dq_i(t)]. \quad (3)$$

If one of the n assets is riskless, say the n th one, so that $\sigma_n = 0$ and $\alpha_n = r$, the instantaneous riskless rate of return, then (3) is rewritten as

$$dW = -C dt - P dt + Y dt + \sum_{i=1}^m w_i W [(\alpha_i - r) dt + \sigma_i dq_i] + rW dt, \quad (3')$$

where $m = n - 1$ and w_1, \dots, w_m are unconstrained since we have substituted

$$w_n = 1 - \sum_{i=1}^m w_i.$$

⁶Merton derives the equation for wealth dynamics by carefully taking limits of a discrete time process and using Ito's Lemma. Our equation is of course identical to his except for the life insurance premium $P(t)dt$.

Before proceeding with the stochastic dynamic programming formulation we will first specify the probability law governing the investor's lifetime. We assume that he will die at a time $T \in [0, \bar{T}]$ and that T is specified by a probability density function π , such that

$$\begin{aligned}\pi(t) &\geq 0 \quad \text{for all } t \in [0, \bar{T}], \\ \pi(t) &> 0 \quad \text{for all } t \in (0, \bar{T}), \\ \int_0^{\bar{T}} \pi(t) dt &= 1,\end{aligned}\tag{4}$$

and $\pi(t)$ is continuous in the neighborhood of \bar{T} .⁷ Let $G(t)$ be the right-hand cumulative probability distribution, i.e.,

$$G(t) = \int_t^{\bar{T}} \pi(\tau) d\tau, \quad 0 \leq t \leq \bar{T}.\tag{5}$$

Denote by $\pi(T; t)$ the conditional probability density for death at time T conditional upon the investor being alive at time t , so that

$$\pi(T; t) = \pi(T)/G(t), \quad 0 \leq t \leq T \leq \bar{T}, \quad t \neq \bar{T}.\tag{6}$$

In particular we will define $\lambda(t)$ by

$$\lambda(t) = \pi(t; t), \quad 0 \leq t < \bar{T}.\tag{7}$$

When $\pi(\bar{T}) > 0$, it is easily seen that for $t \rightarrow \bar{T}$, $\lambda(t) \rightarrow \infty$. If $\pi(\bar{T}) = 0$, then, as Yaari (1965) does for each $t \in [0, \bar{T})$, we define τ_t by

$$\pi(\tau_t) = \sup_{\tau \in (t, \bar{T}]} \pi(\tau) > \pi(\bar{T}) = 0.$$

Now $G(t) \leq (\bar{T} - t)\pi(\tau_t)$, so that

$$\lambda(t) \geq \pi(t)/\{(\bar{T} - t)\pi(\tau_t)\}.\tag{8}$$

Now since $\pi(t)$ is continuous, it must be monotone decreasing in the neighborhood of \bar{T} , so that $\pi(t)/\pi(\tau_t) = 1$ and hence $\lambda(t)$ diverges.

Now that the investor's probability distribution for death has been given, we can specify the life insurance rates. We assume that paying premiums at the rate $P(t)$ buys life insurance in the amount $P(t)/\mu(t)$ for the next (very) short period. Thus the investor's total legacy should he die at time t with wealth W is

$$Z(t) = W(t) + \{P(t)/\mu(t)\}.\tag{9}$$

⁷We will not be overly concerned with the regularity conditions on non-stochastic exogenous functions of time; in general we will assume them to be continuous in the neighborhood of \bar{T} , unless otherwise noted.

We assume that

$$\mu(t) = \lambda(t) + \eta(t), \quad (10)$$

where $\eta(t)$ is any non-negative *integrable* function continuous in the neighborhood of \bar{T} , i.e.,

$$H(t) \equiv \int_{\bar{T}}^t \eta(\tau) d\tau < \infty, \quad t \in [0, \bar{T}], \quad (11)$$

and

$$\lim_{t \rightarrow \bar{T}} \{\eta(t)/\lambda(t)\} = 0.$$

Hence

$$\lim_{t \rightarrow \bar{T}} \{\mu(t)/\lambda(t)\} = 1.$$

The investor wishes to choose optimal controls (i.e., consumption, life insurance and portfolio functions) to maximize his expected lifetime utility for consumption and legacy. That is, to

$$\text{Max.}_{(C, P, w)} E'_0 \left[\int_0^T U(C(s), s) ds + B(Z(T), T) \right], \quad (12)$$

where E'_0 is the conditional expectation operator over all random variables including T conditional upon $W(0) = W_0$; U is assumed to be strictly concave in C and B is assumed strictly concave in Z . The maximization (12) is subject to the constraints that $W(0) = W_0$ and the budget constraint (3) or, if a riskless asset exists (3').

3. The stochastic dynamic programme formulation

We will use stochastic dynamic programming to derive the optimal controls. Define

$$J(W, t) \equiv \text{Max.}_{(C, P, w)} E_t \left[\int_t^{\bar{T}} \pi(T; t) \left[\int_t^T U(C, s) ds + B(Z(T), T) \right] dT \right], \quad (13)$$

where E_t is the conditional expectation operator over all random variables except T , conditional upon $W(t) = W$. Now if the integral (13) is absolutely convergent then we may integrate by parts to find that

$$J(W, t) = \text{Max.}_{(C, P, w)} E_t \frac{\int_t^{\bar{T}} [\pi(T)B(Z(T), T) + G(T)U(C, T)] dT}{G(t)}. \quad (14)$$

Define

$$\begin{aligned} \emptyset(C, P, w; W, t) \equiv & \lambda(t)B(Z(t), t) + U(C, t) - \lambda(t)J \\ & + J_t + \left[\sum_{i=1}^n w_i \alpha_i W + Y - C - P \right] J_w \\ & + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} w_i w_j W^2 J_{ww}, \end{aligned} \quad (15)$$

given $w_i(t) = w_i$, $C(t) = C$, $W(t) = W$, $Y(t) = Y$, $P(t) = P$ and $Z(t) = Z$; and where $J_t = \partial J / \partial t$, $J_w = \partial J / \partial W$, $J_{ww} = \partial^2 J / \partial W^2$, and $\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$, where ρ_{ij} is the instantaneous correlation coefficient between dq_i and dq_j ; i.e., $E(dq_i dq_j) = \sigma_{ij} dt$.⁸ The optimal controls C^* , P^* and w^* are given by the following theorem.

Theorem 1.⁹ If the $S_i(t)$ are generated by a geometric Brownian motion, i.e., they are log-normally distributed, U is strictly concave in C , and B is strictly concave in Z , then there exists a set of optimal controls C^ , P^* and w^* satisfying (3), $\sum_{i=1}^n w_i = 1$ and $J(W, \bar{T}) = B(Z(\bar{T}), \bar{T})$ and these controls are such that*

$$0 = \emptyset(C^*, P^*, w^*; W, t) \geq \emptyset(C, P, w; W, t), \quad t \in [0, \bar{T}].$$

Theorem 1 implies that

$$0 = \text{Max.}_{\{C, P, w\}} \{ \emptyset(C, P, w; W, t) \}. \quad (16)$$

We form the Lagrangian

$$L = \emptyset + \Psi \left[1 - \sum_{i=1}^n w_i \right],$$

and differentiate to find the first-order conditions

$$0 = L_C(C^*, P^*, w^*) = U_C(C^*, t) - J_w, \quad (17)$$

$$0 = L_{w_k}(C^*, P^*, w^*) = -\Psi + J_w \alpha_k W + J_{ww} \sum_{j=1}^n \sigma_{kj} w_j^* W^2, \quad (18)$$

$$k = 1, \dots, n,$$

⁸We note that $\Omega = [\sigma_{ij}]$ is $n \times n$ if there is no risk-free asset and is $m \times m$ if there is a risk-free asset and assume that it is non-singular in either case.

⁹This theorem is almost identical to Merton's Theorem I. For a heuristic proof based upon taking limits of a discrete time process, see Merton (1969) or Drefus (1965). For a more formal proof, see Kushner (1971, ch. 11.3). For a rigorous proof, see Kushner (1967, ch. IV, Theorem 7).

$$0 = L_P(C^*, P^*, w^*) = \frac{\lambda}{\mu} B_Z \left(W + \frac{P^*}{\mu}, t \right) - J_W, \quad (19)$$

$$0 = L_\psi(C^*, P^*, w^*) = 1 - \sum_{i=1}^n w_i^*. \quad (20)$$

The second-order conditions are

$$L_{CC} = U_{CC} < 0,$$

$$L_{w_k w_j} = \sigma_{kj} W^2 J_{WW},$$

$$L_{PP} = \{\lambda/\mu^2\} B_{ZZ} < 0,$$

all other second partials being zero. Thus a sufficient condition for a maximum is $J_{WW} < 0$ which we will henceforth assume. Total differentiation of (17) with respect to W yields $\partial C^*/\partial W > 0$ and of (19) yields $\partial Z^*/\partial W > 0$, where $Z^* = W + P^*/\mu$ is the optimal of legacy.

We now observe that our eq (18), which determines the optimal portfolio of securities, is identical to Merton's (20) [except for the last term in (20) which is zero under the assumption of log-normality of prices]. Thus without further derivation we may conclude that Merton's generalization of the Tobin-Markowitz separation theorem is valid under the assumption of uncertain lifetime.

Theorem 2 (Merton). *Given n assets with prices S_i which are log-normally distributed, then (1) there exists a unique (up to a non-singular transformation) pair of mutual funds which are linear combination of these assets such that, independent of preferences, wealth distribution, probability distribution of lifetime or life insurance opportunities individuals will be indifferent between choosing from a linear combination of these two funds or a linear combination of the original assets. (2) If S_f is the price per share of either fund then S_f is log-normally distributed.*

As Merton shows the proportion¹⁰ of values each mutual fund would invest in each risky security depends only on the physical distribution of returns, i.e., $\{\alpha_i, \sigma_{ij}\}$, and hence these proportions for the uncertain lived investor are the same as they are for his certain lived counterpart. Moreover, if one of the assets is risk-free then the following corollary is valid.

¹⁰See Merton (1971, p. 384) for the proportions of each asset held in each fund. Our notation is the same as Merton's for ease of comparison. Also Merton shows (p. 399), for the case of exponential distribution of lifetime, that the only effect is to add a time discount factor to the utility function.

Corollary (Merton). If one of the assets is risk-free (say the n th) then one mutual fund can be chosen to contain only the risk-free security and the other to contain only the risky assets in the proportions

$$\delta_k = \frac{\sum_{i=1}^m v_{ki}(\alpha_j - r)}{\sum_{i=1}^m \sum_{j=1}^m v_{ij}(\alpha_j - r)}, \quad k = 1, \dots, m, \quad (21)$$

where

$$[v_{ij}] = [\sigma_{ij}]^{-1}.$$

We will henceforth assume that there is a riskless asset so that we can work without loss of generality with only two assets, one risk-free and the other a mutual fund of risky assets with a log-normally distributed price $S(t)$ defined by

$$dS/S = \alpha dt + \sigma dq, \quad (22)$$

where

$$\begin{aligned} \alpha &\equiv \sum_{k=1}^m \delta_k \alpha_k, \\ \sigma^2 &\equiv \sum_{k=1}^m \sum_{j=1}^m \delta_k \delta_j \sigma_{kj}, \\ dq &\equiv \sum_{k=1}^m (\delta_k \sigma_k dq_k / \sigma). \end{aligned} \quad (23)$$

Hence we may define $w^*(t) \equiv w_j^*(t)/\delta_j$ to be the optimal fraction of wealth to be invested in the risky fund at time t . Now from (18) with $k = n$ we find that $\Psi = J_W r W$. Substituting for w_j^* and Ψ in (18), then multiplying (18) by δ_k and summing from $k = 1, \dots, m$, we get

$$0 = J_W(\alpha - r) + J_{WW} w^* \sigma^2 W^2, \quad W \neq 0. \quad (18')$$

Let $R(W, t) \equiv -J_{WW}/J_W$ be the investors (imputed) risk aversion for wealth and assume that $R > 0$. We find that

$$w^* W = \frac{\alpha - r}{\sigma^2} \frac{1}{R}. \quad (24)$$

Differentiating (24) with respect to W yields

$$\frac{\partial(w^* W)}{\partial W} = \frac{\alpha - r}{\sigma^2} \frac{R_W}{R^2}.$$

Thus the optimal *dollar amount* invested in the risky fund will be an increasing function of wealth if and only if the imputed risk aversion in wealth is decreasing. However, even if $R_w < 0$ it is not necessarily true that the optimal *fraction of wealth* will increase since

$$\frac{\partial w^*}{\partial W} = \frac{\alpha - r}{\sigma^2} \frac{R + WR_w}{(RW)^2},$$

which may be either positive or negative if $R_w < 0$. On the other hand if $R_w > 0$, i.e., positive increasing risk aversion, then both the dollar amount and fraction of wealth invested in the risky fund decrease with increasing wealth.

Finally we substitute $w_i^* = w^* \delta_i$ into (16) and then (18') into the result to find that

$$\begin{aligned} 0 = & \lambda B\left(W + \frac{P^*}{\mu}, t\right) + U(C^*, t) + J_t - \lambda J \\ & + J_w [Wr + Y - C^* - P^*] - \frac{1}{2} \frac{J_w^2}{J_{ww}} \left(\frac{\alpha - r}{\sigma}\right)^2. \end{aligned} \quad (16')$$

Thus (16'), (17), (18'), (19) and (20), i.e., $w_n^* = 1 - w^*$, are a complete set of partial differential equations for $C^*(W, t)$, $P^*(W, t)$, $w^*(W, t)$, $w_n^*(W, t)$ and $J(W, t)$, the five required functions. We note that (17) and (19) respectively, are the normal marginality requirements that the marginal utility of current consumption equals the marginal utility of wealth (future consumption) which in turn equals the expected marginal utility of legacy. Furthermore, we may regard C^* and P^* as, respectively, the levels of consumption and premium payment net of any minimum levels. This can be seen by letting $f(t)$ be the minimum (subsistence) level of consumption at time t and letting $n(t)$ be the minimum level of legacy at time t . [Note that $n(t)$ may be negative if non-purchased insurance, such as social security and employee benefit policies, exceed minimum requirements.] Then defining

$$C'(t) \equiv C^*(t) - f(t),$$

$$P'(t) = P^*(t) - \mu(t)n(t),$$

$$Z'(t) \equiv W(t) + \frac{P'(t)}{\mu(t)} = Z^*(t) - n(t),$$

$$Y'(t) \equiv Y(t) - F(t) - \mu(t)n(t),$$

$$U'(C'(t), t) \equiv U(C^*(t), t),$$

$$B'(Z'(t), t) \equiv B(Z^*(t), t).$$

we see that the form of (16')–(20) remains unchanged with the primed variables and functions substituted for the unprimed ones. Thus without loss of generality we can assume that there are no minimum levels of consumption or legacy [or more precisely that any such minimums are deducted from the future earnings stream $Y(t)$ leaving a net future earnings stream].

We have shown that minimum levels of consumption and legacy are unessential to our problem by incorporating them in the future earnings stream. We may next reasonably question whether or not the future earnings stream is essential to the problem. The following theorem shows it is not.¹¹

Theorem 3. If security prices have a log-normal distribution and there exists a riskless security and if at any time $t = \tau$, any investor relinquishes his right to his future earnings $Y(t)$ for $t > \tau$ and receives instead a certain addition to wealth $b(\tau)$, where

$$\begin{aligned} b(t) &= \int_t^{\bar{T}} Y(s) \exp[-r(s-t) - \int_t^s \mu(\tau) d\tau] ds \\ &= \int_t^{\bar{T}} Y(s) G(s)/G(t) \exp[-r(s-t) - \{H(t) - H(s)\}] ds, \end{aligned} \quad (25)$$

which is independent of his preferences and the return on risky securities and depends only upon the risk-free rate of return and the life insurance rates, then the investor's optimal consumption rate, legacy, risky investment and expected future utility will remain unchanged. Furthermore, if we define the investor's new wealth positions at $t = \tau$ to be his 'adjusted wealth',

$$X(\tau) = W(\tau) + b(\tau),$$

then following optimal policies at any $t > \tau$, the investor's wealth will be

$$X(t) = W(t) + b(t). \quad (26)$$

Proof. Define

$$\hat{P}(t) \equiv P^*(t) - \mu(t)b(t), \quad (27)$$

and

$$I(X, t) = I(W + b(t), t) \equiv J(W, t), \quad (28)$$

so that

$$J_W = I_X, \quad J_{WW} = I_{XX}, \quad J_t = I_t + I_X b'(t), \quad (29)$$

where

$$b'(t) = -Y(t) + [r + \mu(t)]b(t). \quad (30)$$

¹¹Merton (1971) obtains the same result for non-stochastic lifetime and wage income (p. 394) and a similar result for a stochastic wage income and non-stochastic lifetime (p. 397).

Now defining $\hat{x}X = w^*W$ and $\hat{C} = C^*$ we substitute (25)–(30) into (16'), (17), (18') and (19) to find

$$0 = \lambda B\left(X + \frac{\hat{P}}{\mu}, t\right) + U(\hat{C}, t) + I_t - \lambda I \\ + I_X[Xr - \hat{C} - \hat{P}] - \frac{1}{2} \frac{I_X^2}{I_{XX}} \left(\frac{\alpha - r}{\sigma}\right)^2, \quad (16'')$$

$$0 = U_C(\hat{C}, t) - I_X, \quad (17'')$$

$$0 = I_X(\alpha - r) + I_{XX}\sigma^2\hat{x}X, \quad (18'')$$

$$0 = \frac{\lambda}{\mu} B_Z\left(X + \frac{\hat{P}}{\mu}, t\right) - I_X \quad (19'')$$

subject to

$$I(X(T), T) = B(Z(T), T).$$

Now an investor with total wealth X and a risky investment of $\hat{x}X$ must have a riskless investment \hat{x}_nX , such that

$$\hat{x}_n = 1 - \hat{x}. \quad (20'')$$

We recognize (16'')–(20'') as the equations satisfied by the optimal policies of an investor with wealth $X(t)$ and no future earnings. Thus we conclude that $\hat{C} = C^*$, $\hat{x}X = w^*W$, $\hat{Z} = X + (\hat{P}/\mu) = W + (P^*/\mu) = Z^*$ are the optimal policies for the investor with no future earnings and further that $I(X, \tau) = J(W, \tau)$ for any $t = \tau$, such that $X(\tau) = W(\tau) + b(\tau)$.

To complete the proof we must show that if the investor without a future earnings stream chooses $\hat{C} = C^*$, $\hat{Z} = Z^*$ and $\hat{x}X = w^*W$ for all $t > \tau$, then $X(t) = W(t) + b(t)$. The asset dynamics equations are

$$dW = -C^* dt - P^* dt + Y dt + w^*W\alpha dt + (1 - w^*)Wr dt \\ + w^*W\sigma dq, \quad (31)$$

and

$$dX = -\hat{C} dt - \hat{P} dt + \hat{x}X\alpha dt + (1 - \hat{x})Xr dt + \hat{x}X\sigma dq. \quad (32)$$

Under our assumptions,

$$dX = dW - Y(t) dt + (r + \mu)b dt \\ = dW + b'(t) dt. \quad (33)$$

Using Ito's Lemma to integrate (33), we find that, for $t > \tau$,

$$\begin{aligned} X(t) &= X(\tau) + W(t) + W(\tau) - b(t) - b(\tau) \\ &= W(t) + b(t). \end{aligned}$$

Q.E.D.

The function $b(t)$ represents the investors net 'human capital' at time t . It is the expected present value of future (wage) earnings found by discounting back to time t . The instantaneous rate of discount is the sum of the risk-free rate and the cost rate of insurance, i.e., to discount from $t + \Delta t$ to t the discount factor is

$$\{1 - [r + \mu(t + \Delta t)]\Delta t\} \approx \exp \{-[r + \mu(t)]\Delta t\}.$$

Thus to discount from $t + T$ to t the discount factor is

$$\prod_{i=1}^n \{1 - [r + \mu(t + i\Delta t)]\Delta t\} \rightarrow \exp \{-\int_t^{t+T} [r + \mu(\tau)] d\tau\},$$

as

$$\Delta t \rightarrow 0,$$

while

$$n\Delta t = T.$$

Hence the more costly insurance is, i.e., the greater is η , the less that future earnings are worth. It is interesting to note that the investor's human capital does not directly depend on the force of mortality, $\lambda(t)$ [except that $\mu(t) \geq \lambda(t)$].

In a sense Theorem 3 is a separation theorem in that the certainty equivalent of future net earnings $b(t)$ is independent of risky market opportunities and preferences. Hence $X(t) = W(t) + b(t)$ represents the 'objectively' obtained certainty equivalent of any investors present wealth and future earnings, i.e., independent of preferences any investors would trade his future net wages for $b(t)$. Furthermore, we may interpret

$$\mu(t)b(t)$$

as the premium necessary to cover $b(t)$ with insurance and

$$\hat{P}(t) = P^*(t) - \mu(t)b(t)$$

as the excess premium paid to insure beyond (or below) $b(t)$. It is commonly thought that insuring against the risk of losing $b(t)$ requires that $\hat{P} = 0$, but in general this will not be the case.

4. Solutions for utility function with constant relative risk aversion

Let

$$U(C, t) = \{h(t)/\gamma\}C^\gamma, \quad \gamma < 1, \quad h > 0, \quad C > 0, \quad (34)$$

be the investor's utility for consumption.¹² By substituting (34) into (17), we find that

$$C^* = (h/J_w)^{1/(1-\gamma)}. \quad (35)$$

Similarly, let¹³

$$B(Z, t) = \{m(t)/\gamma\}Z^\gamma, \quad \gamma < 1, \quad m > 0, \quad Z > 0, \quad (36)$$

be the investor's utility for bequest. Substituting (36) into (19) yields

$$Z^* = W + \frac{P^*}{\mu} = \left(\frac{\lambda m}{\mu J_w} \right)^{1/(1-\gamma)}. \quad (37)$$

We now use (34)–(37) in (16') and $\delta \equiv 1 - \gamma$ to find that

$$0 = k(t) \frac{\delta}{\gamma} J_w^{-\gamma/\delta} - \lambda J + J_t - \frac{1}{2} \frac{J_w^2}{J_{ww}} \left(\frac{\alpha - r}{\sigma} \right)^2 + J_w[(r + \mu)W + Y], \quad (38)$$

where

$$k(t) = [\{\lambda(t)/\mu(t)\}^{\gamma/\delta} \{\lambda(t)m(t)\}^{1/\delta} + h(t)^{1/\delta}], \quad (39)$$

subject to

$$J(W, T) = B(Z, T) = \{\bar{m}/\gamma\}Z^\gamma,$$

where the overbar denotes evaluation at T , e.g., $Z = Z(T)$. A solution to (38) is

$$J(W, t) = \{a(t)/\gamma\}(W + b(t))^\gamma, \quad (40)$$

where $b(t)$ is given by (25), and

$$a(t) = \left\{ \int_t^T k(s) \frac{G(s)}{G(t)} \exp \left[\frac{\gamma}{\delta} (v + r)(s - t) + \frac{\gamma}{\delta} [H(t) - H(s)] \right] ds \right\}^\delta, \quad (41)$$

¹²When $\gamma = 0$, $U(C, t) = h(t) \ln C$.

¹³It is mathematically necessary that the exponent γ be the same for both consumption and legacy in order to get solutions in closed form.

where

$$v = (\alpha - r)^2 / 2\delta\sigma^2.$$

From (18'), (25), (35), (37), (40) and (41) the optimal consumption, life insurance and portfolio rules can be explicitly written as

$$C^*(W, t) = \left(\frac{h(t)}{a(t)}\right)^{1/\delta} [W + b(t)], \quad (42)$$

$$Z^*(W, t) \equiv W + \frac{P^*(W, t)}{\mu(t)} = \left(\frac{m(t)\lambda(t)}{a(t)\mu(t)}\right)^{1/\delta} [W + b(t)], \quad (43)$$

and

$$w^*(W, t)W = \frac{\alpha - r}{\delta\sigma^2} [W + b(t)]. \quad (44)$$

By using L'Hopital's rule in (41), we find that $\bar{a} = \bar{m}$. Thus from (43) we find that $Z^* = W(T)$, i.e., the investor will purchase no insurance at T (because the cost rises without limit). Furthermore,

$$J(W, T) = \{\bar{m}/\gamma\} Z^{*\gamma} = B(Z^*, T),$$

satisfying the boundary conditions.

The stochastic process generating wealth can be derived by substituting (42)–(44) in (3'),

$$\begin{aligned} dW = X(t) \left\{ \left[\frac{(\alpha - r)^2}{\delta\sigma^2} - \left(\frac{h}{a}\right)^{1/\delta} - \mu \left(\frac{m\lambda}{a\mu}\right)^{1/\delta} \right] dt + \frac{\alpha - r}{\delta\sigma} dq \right\} \\ + [W(r + \mu) + Y] dt, \end{aligned} \quad (45)$$

where

$$X(t) = W(t) + b(t).$$

Now by Ito's Lemma

$$\begin{aligned} dX &= dW + b'(t) dt \\ &= dW - Y dt + (r + \mu)b dt. \end{aligned} \quad (46)$$

Thus $X(t)$ is the solution to

$$\frac{dX}{X} = \left[\frac{(\alpha - r)^2}{\delta\sigma^2} + r + \mu - \frac{k(t)}{a(t)^{1/\delta}} \right] dt + \frac{\alpha - r}{\delta\sigma} dq. \quad (47)$$

We may use Ito's Lemma again to integrate (47), finding that

$$X(t) = X(0) \left[\frac{a(t)}{a(0)} \right]^{1/\delta} \exp \left\{ \left(v + \frac{r}{\delta} \right) t + \frac{1}{\delta} [H(0) - H(t)] + \frac{\alpha - r}{\delta \sigma} \int_0^t dq \right\}, \quad (48)$$

and hence the 'adjusted wealth' $X(t)$ is log-normally distributed. Thus,

$$W(t) = X(t) - b(t)$$

is, as Merton found, a 'displaced' or 'three-parameter' log-normally distributed random variable. By Ito's Lemma (48) is a solution to (47) with probability one and $W(t)$ is continuous with probability one, so that

$$\begin{aligned} \bar{W} &= \lim_{t \rightarrow \bar{T}} W(t) \\ &= X(0) \left[\frac{\bar{m}}{a(0)} \right]^{1/\delta} \exp \left[\left(v + \frac{r}{\delta} \right) \bar{T} + \frac{1}{\delta} H(0) + \frac{\alpha - r}{\delta \sigma} \int_0^{\bar{T}} dq \right]. \end{aligned} \quad (49)$$

Thus $\bar{W} = 0$ if and only if $\bar{m} = 0$, i.e., it is optimal to be penniless at \bar{T} if and only if the investor has no bequest motive at \bar{T} . Substituting (48) into (42), we find with probability one that

$$\begin{aligned} C^*(W, t) & \\ &= \left[\frac{h(t)}{a(0)} \right]^{1/\delta} X(0) \exp \left[\left(v + \frac{r}{\delta} \right) t + \frac{1}{\delta} [H(0) - H(t)] + \frac{\alpha - r}{\delta \sigma} \int_0^t dq \right], \end{aligned} \quad (50)$$

so that $\bar{C} = 0$ if and only if $\bar{h} = 0$. Hence, if the investor has no legacy motive at \bar{T} , he will control his wealth to make it approach zero while at the same time keeping his consumption rate non-negative and finite.

It seems reasonable, and we shall do so, to assume that the investor would not want to at any time have a legacy greater than $X(t)$, i.e., assume $Z^*(W, t) \leq W + b(t)$.¹⁴ From (43) we have then that $\{m(t)\lambda(t)/a(t)\mu(t)\} \leq 1$ so that at any time t for W large enough the investor will certainly be a seller of insurance, while for W small enough he will purchase insurance. It seems quite reasonable

¹⁴It is of course possible that the investor's preferences for legacy are strong enough so that he wishes to insure to a level beyond the present value of his total wealth (capital plus human capital). However, this case is not what is commonly thought of as insuring against the risk of loss of future income.

and intuitive that, neglecting tax effects, the wealthy will find little need for life insurance. Moreover, for large enough $b(t)$ the investor will purchase insurance no matter what his relative likings for consumption and legacy are. At the same time we see from (44) that as wealth increases, the *fraction* of wealth w^* placed in the risky security should decline [provided $b(t) > 0$], while the dollar amount w^*W increases absolutely. This is again quite reasonable in the following sense. The individual with small wealth W relative to future prospects $b(t)$, depends upon his wages both for his consumption and for his life insurance premium, which he must pay to protect against the loss of his main asset – his future. The wealthy individual with W relatively large to b , depends upon his current wealth to earn most of his funds for consumption and to supply the bulk of his estate. In the same way the ‘poor’ man must protect his assets, the wealthy individual protects his by, at any given time, placing a greater proportion of his wealth in the riskless security, the more wealth he has at that time.

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