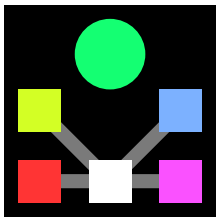

Handbook of Network Analysis*

The KONECT Project

KONECT.cc

Jérôme Kunegis
University of Namur, Belgium
naXys – Namur Center for Complex Systems

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Abstract

This is the handbook for the KONECT project, a scientific project to archive network datasets, compute systematic network theoretic statistics about them, visualize their properties, and provide corresponding data and Free Software tools to programmers, researchers and teachers in fields related to network analysis and graph mining, by Jérôme Kunegis at the Namur Center for Complex Systems (naXys) at the University of Namur, Belgium. The name *KONECT* stands for *Koblenz Network Collection*, as parts of the KONECT project were initiated for the PhD thesis of Jérôme Kunegis at the University of Koblenz–Landau in Germany (2011). This handbook documents all methods, definitions and conventions used in the project, and serves as a general handbook of network mining, with an emphasis on spectral graph theoretical methods, i.e., such methods that are based on the use of specific characteristic matrices of graphs. KONECT datasets and code is used in academia as the basis for research on real-world datasets, in education as the basis for teaching, and in particular it serves as the basis for research projects with the goal to study large numbers of network datasets. KONECT borrows datasets from many sources in and outside academia, and lends itself datasets to other network dataset collection projects.

*This handbook is continuously updated; an up-to-date version can always be found at https://github.com/kunegis/konect-handbook/raw/master/konect-handbook.pdf.

1 Introduction

Everything is a network – whenever we look at the interactions between things, a network is formed implicitly. In the areas of data mining, machine learning, information retrieval, etc., networks are modeled as *graphs*. Many, if not most problem types can be applied to graphs: clustering, classification, prediction, pattern recognition, and others. Networks arise in almost all areas of research, commerce and daily life in the form of social networks, road networks, communication networks, trust networks, hyperlink networks, chemical interaction networks, neural networks, collaboration networks and lexical networks. The content of text documents is routinely modeled as document–word networks, taste as person–item networks and trust as person–person networks. In recent years, whole database systems have appeared specializing in storing networks. In fact, a majority of research projects in the areas of web mining, web science and related areas uses datasets that can be understood as networks. Unfortunately, results from the literature can often not be compared easily because they use different datasets. What is more, different network datasets have slightly different properties, such as allowing multiple or only single edges between two nodes. In order to provide a unified view on such network datasets, and to allow the application of network analysis methods across disciplines, the KONECT project defines a comprehensive network taxonomy and provides a consistent access to network datasets. To validate this approach on real-world data from the Web, KONECT also provides a large number (1,000+) of network datasets of different types and different application areas.

KONECT, the Koblenz Network Collection, contains over 1,000 network datasets as of 2017. In addition to these datasets, KONECT consists of Matlab code to generate statistics and plots about them, which are shown on the KONECT website[1]. KONECT contains networks of all sizes, from small classical datasets from the social sciences such as Kenneth Read's Highland Tribes network with 16 vertices and 58 edges (HT), to the Twitter social network with 52 million nodes and 1.9 billion edges (TF). Figure 1 shows a scatter plot of all networks by the number of nodes and the average degree in the network. Each network in KONECT is represented by a unique two- or three-character code which we write in a sans-serif font, and is indicated in parentheses as used previously in this paragraph. The full list of codes is given online.[2]

1.1 Software and Software Packages

The KONECT project consists of several components, whose interactions is summarized in Figure 2. Various parts of the KONECT project are available at Github, including this Handbook.[3][4][5][6][7]

1.2 History of KONECT

The roots of KONECT lie in the research of Jérôme Kunegis at the Technical University of Berlin, within the DAI Laboratory. The first networks were bipartite rating graphs, collected to support

[1] http://konect.cc/
[2] http://konect.cc/networks/
[3] github.com/kunegis/konect-analysis
[4] github.com/kunegis/konect-toolbox
[5] github.com/kunegis/konect-handbook
[6] github.com/kunegis/konect-extr
[7] github.com/kunegis/konect-www

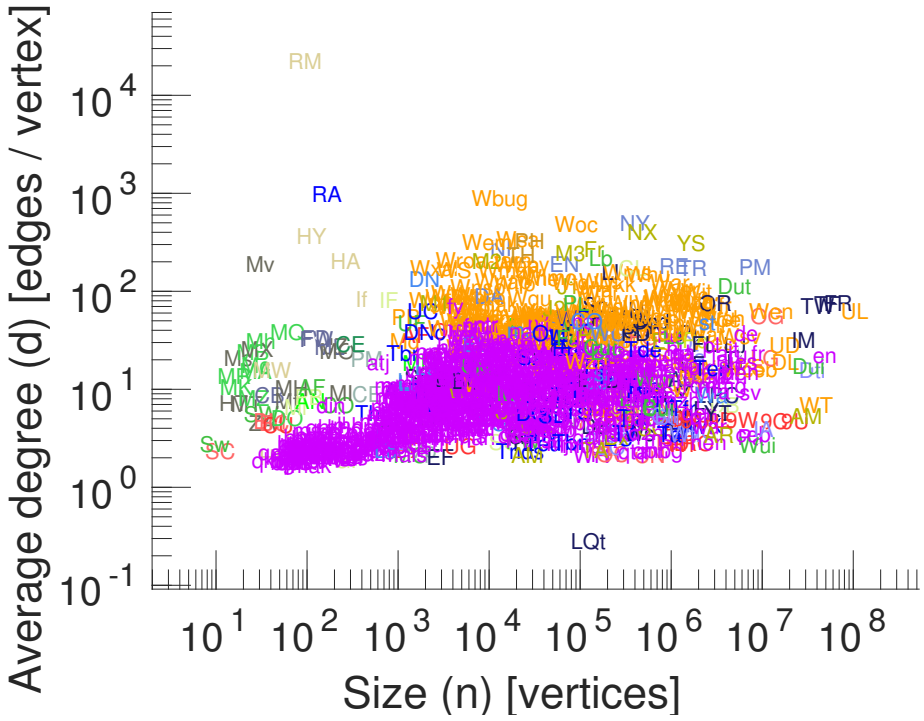


Figure 1: All networks in KONECT arranged by the size (the number of nodes) and the average number of neighbors of all nodes. Each network is represented by a two- or three-character code. The color of each code corresponds to the network category as given in Table 3.

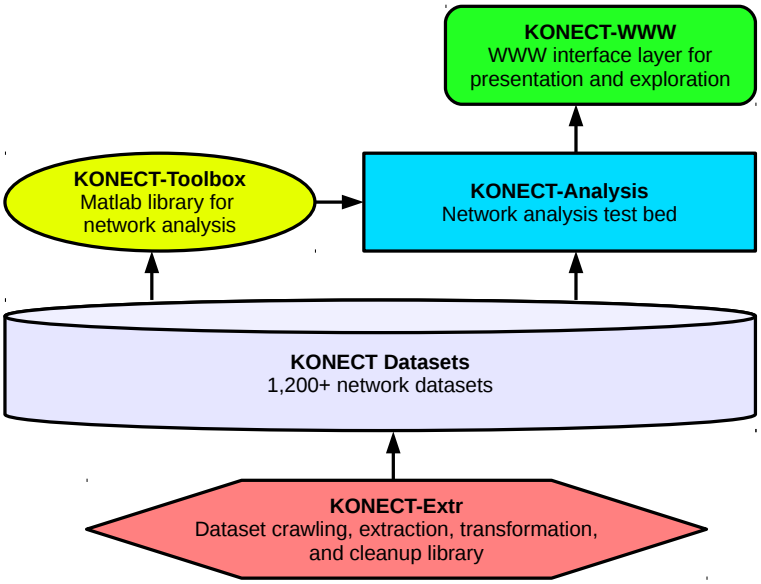


Figure 2: Overview of KONECT's components.

Jérôme Kunegis' work on collaborative filtering (Kunegis & Schmidt 2007, Kunegis & Şahin Albayrak 2007). The earliest networks include MovieLens, Jester, BookCrossing, and the Netflix Prize dataset.

What later became known as the KONECT project's collection of networks properly started out in December 2008, as evaluation for Jérôme Kunegis' ICML 2009 paper *Learning Spectral Graph Transformations for Link Prediction* (Kunegis & Lommatzsch 2009), codenamed *Spectral Transformation*. It then consisted of a collection of network datasets and spectral link prediction methods, i.e., code based on the decomposition of various characteristic graph matrices. The first unipartite networks appeared at this time, one of the earliest being the trust network of Advogato. The first dataset crawled specifically for KONECT was the Slashdot Zoo, for which crawling began in 2008, with the corresponding paper published a year later (Kunegis et al. 2009).

Later, more datasets were added and the codebase was called the *Graph Store*. This was at a time when the word *store* was in fashion, in part due to the emergence of *app stores*. In that phase, the project was used for the experiments of several papers in the area of collaborative filtering and recommender systems.

When Jérôme moved from TU Berlin to the University of Koblenz–Landau in Koblenz (Germany) the project was renamed *Web Store*, in line with Koblenz' Institute for Web Science and Technologies (WeST). The name *KONECT – Koblenz Network Collection* was adopted sometime in 2011. Jérôme wrote his PhD thesis at the University of Koblenz–Landau in the same year. In that phase, most of the mathematical notation of KONECT was settled. In particular, Jérôme Kunegis' PhD thesis contains a mathematical glossary that contains many of the symbol in the present document. That PhD thesis was also the first time that a list of all networks was generated, before the website went online. The first KONECT website was created in 2011 at `konect.uni-koblenz.de`. In this phase of the project, extraction of datasets and maintenance of the KONECT website was performed in collaboration with Daniel Dünker, Holger Heinz, and Martina Sekulla.

Code for dataset extraction and the Matlab Toolbox were first published on the KONECT website as downloadable tarballs. All published code was licensed under the GNU General Public License (GPL) version 3 from the beginning. A short overview paper of the KONECT system was published in 2013 at the International World Wide Web Conference (WWW), as part of the Web Observatory Workshop (Kunegis 2013)[8], and remains the primary publication to cite when using KONECT. In 2015 and 2016, various parts of the KONECT project were placed on GitHub, again under the GNU General Public License version 3, including this handbook.

From 2017 on, the KONECT project continued to be developed at the University of Namur (Belgium), with web hosting provided by the Institute for Web Science and Technologies (WeST) at the University of Koblenz–Landau in Koblenz (Germany). In that phase, the number of networks, statistics, and plots was yet again increased, and a more powerful computation server was acquired. The domain name `konect.cc` was adopted in October 2017, with hosting now switched to the University of Namur. A tutorial on the use of large network dataset collections, largely based on KONECT, was given in the same year at the Conference on Information and Knowledge Management (CIKM 2017) under the name *Network Analysis in the Age of Large Network Dataset Collections – Challenges, Solutions and Applications*.[9]

The Stu build tool[10] was developed in parallel with KONECT. KONECT used `make(1)` in the

[8] The Google Scholar citation page for that publication gives a list of papers using KONECT, see https://scholar.google.com/scholar?cites=7174338004474749050

[9] http://xn.unamur.be/network-collection-tutorial-cikm2017/

[10] https://github.com/kunegis/stu

first years of its operation. The Stu build system was initiated in 2014, and KONECT was gradually switched to it, serving as its main use case. Stu was essentially completed in August 2017 with publication of version 2.5.

1.3 Status of this Handbook

This handbook is constantly updated. The most up to date version can always be found on GitHub.[11] Approximately every year, a version of the handbook is published on arXiv, as a new version of the existing arXiv entry.[12] The handbook is explicitly *not* peer reviewed, as we update it as needed. Since the handbook serves as the central place in which we add new definitions, the reader will notice that the prose may vary greatly from section to section. Also, some notation used may be very establish and will not change, but other choices in notation are not set in stone, and we reserve the possibility to change our notation to suit our needs. However, any change in notation will be reflected in KONECT source code and on the KONECT world wide web site. While it is perfectly acceptable to cite this handbook, we urge readers not to refer to any equation or section number as these *will* change constantly.

What will *not* change, by design, are all internal names. For instance, the clustering coefficient will always be denoted as `clusco`. Thus, when citing statistics, decompositions or other items defined in this handbook, please use internal names; these are always given in a non-proportional font. Throughout the handbook, we use margin notes to give the internal names of various parameters, ⟨`name`⟩ as shown here on the side of this text. In KONECT, almost everything has an *internal name*, i.e., a systematic name that is used throughout code, and which is stable. Internal names are never changed. Therefore, some internal names may be slightly inaccurate or inconsistent. For purposes of backward compatibility, this is preferred over changing names. For instance, KONECT's Enron email network is called `enron`, the edge weights of networks with negative edges is called `signed`, the set of all unipartite networks is called `SQUARE`, and the category of online social networks is called `Social`. Internal names are case-sensitive.

Certain parts of the handbook are based on previous work, and certain parts have been published as papers by the authors. These works are always cited in the relevant sections, and should be cited in preference when using the corresponding methods.

1.4 Structure of this Handbook

This handbook serves both as a compendium of mathematical definitions used in the KONECT project, as well as documentation of the software used in KONECT.

- Section 2 – **Networks** gives the basic definition of a network as used in KONECT – as networks exists in many different variants such as directed, bipartite, signed, etc., the exact definitions are often overlooked, but can be crucial for understanding individual relations. The taxonomy of networks given in this section serves for the rest of the project. This section also attempts to illustrate the breadth of the concept of *network*, bridging many different disciplines.

- Section 3 – **Graph Theory** gives the graph-theoretical definitions underlying all topics of network analysis. Again, the devil is in the details and the graph theory literature is often split

[11] https://github.com/kunegis/konect-handbook/raw/master/konect-handbook.pdf
[12] http://arxiv.org/abs/1402.5500

between multiple incompatible definitions for commonly used terms. This sections presents the terminology and definitions as used in KONECT, which represents a compromise that has been proven useful in practice.

- Section 4 – **Statistics** is devoted to network statistics, i.e., numerical measures that characterize a network as a whole. These are central to network analysis, and most common network statistics are covered. The section is structured by the type of analysis underlying the different statistics, roughly order from simple to complex. Each subsection serves as an introduction to the underlying topic, although not all subsections (yet) contain enough exposition to server as a general introduction to these topics.
- Section 5 – **Matrices and Decompositions** reviews characteristic matrices used to analyse graphs, with a focus on their decompositions. These are crucial in various types of analyses, include pairwise node measures such as distances and similarities, as well as node-based measures such as centralities. Due to the focus of the KONECT project on such decompositions, this section quite detailed and complete, although the emphasis is mainly on the eigenvalue and singular value decompositions, and matrix to which they can be applied.
- Section 6 – **Plots** reviews common ways to visualize properties of a network. Some of those, such as the degree distribution, are ubiquitous in the literature. Since KONECT plots are extensively used by the authors in published research, the plots cover quite distinct areas of network analysis.
- Section 7 – **Other Definitions** covers other definitions used in KONECT, including node features, i.e., numerical measures that characterize individual nodes in a network, and error measures. These are often used as measures of centrality or importance, etc.
- Section 8 – **The KONECT Toolbox** describes the GNU Octave and Matlab toolbox that is part of the KONECT project.
- Section 9 – **File Formats** finally documents the different file formats used by KONECT.

2 Networks

Datasets in KONECT represent networks, i.e., a set of nodes connected by links. Networks can be classified by their format (directed/undirected/bipartite), by their edge weight types and multiplicities, by the presence of metadata such as timestamps and node labels, and by the types of objects represented by nodes and links. The full list of networks is given online.[13]

2.1 Format

The format of a network is always one of the following. The network formats are summarized in Table 1.

- In **undirected networks** (U), edges are undirected. That is, there is no difference between the edge from u to v and the edge from v to u; both are the edge $\{u, v\}$. An example of an undirected network is the social network of Facebook (Ow), in which there is no difference between the statements "A is a friend of B" and "B is a friend of A." `sym`

[13] http://konect.cc/networks/

Table 1: The network formats allowed in KONECT. Each network dataset is exactly of one type.

#	Icon	Type	Edge partition	Edge types	Internal name
1	U	Undirected	Unipartite	Undirected	`sym`
2	D	Directed	Unipartite	Directed	`asym`
3	B	Bipartite	Bipartite	Undirected	`bip`

- In a **directed network** (D), the links are directed. That is, there is a difference between the edge (u, v) and the edge (u, v). Directed networks are sometimes also called *digraphs* (for *directed graphs*), and their edges *arcs*. An example of a directed social network is the follower network of Twitter (TF), in which the fact that user A follows user B does not imply that user B follows user A. `asym`

- **Bipartite networks** (B) include two types of nodes, and all edges connect one node type with the other. An example of a bipartite network is a rating graph, consisting of the node types *user* and *movie*, and each rating connects a user and a movie (M3). Bipartite networks are always undirected in KONECT. Datasets that can be represented as hypergraphs can also be represented without loss as bipartite networks, explaining why KONECT does not have a *hypergraph* format, and also why bipartite networks are common. `bip`

2.2 Weights

The edge weight and multiplicity types of networks are represented by one of the following types. The types of edge weights and multiplicities are summarized in Table 2.

- An **unweighted network** ($-$) has edges that are unweighted, and only a single edge is allowed between any two nodes. `unweighted`

- In a **network with multiple edges** ($=$), two nodes can be connected by any number of edges, and all edges are unweighted. This type of network is also called a multigraph. `positive`

- In a **positive network** ($+$), edges are annotated with positive weights, and only a single edge is allowed between any node pair. The weight of zero is identified with the lack of an edge and thus, we require that each edge has a weight strictly larger than zero, except when the special tag `#zeroweight` is defined. `posweighted`

- In a **signed network** ($\pm$), both positive and negative edges are allowed. Positive and negative edges are represented by positive and negative edge weights. Many networks of this type have only the weights ± 1, but in the general case we allow any nonzero weight; this distinction is not made. `signed`

- **Networks with multiple signed edges** ($\overset{\pm}{=}$) allow multiple edges between two nodes, which may have the same values as edges in a signed network. `multisigned`

- **Rating networks** ($*$) have arbitrary real edge weights. They differ from positive and signed networks in that the edge weights are interpreted as an interval scale, and thus the value zero has no special meaning. Adding a constant to all edge weights does not change the `weighted`

Table 2: The edge weight and multiplicity types allowed in KONECT. Each network dataset is exactly of one type. Note that due to historical reasons, networks with multiple unweighted edges have the internal name `positive`, while positively weighted networks have the internal `posweighted`. For signed networks and positive edge weights, weights of zero are only allowed when the tag `#zeroweight` is set.

#	Icon	Type	Multiple edges	Edge weight range	Edge weight scale	Internal name
1	$-$	Unweighted	No	$\{1\}$	–	`unweighted`
2	$=$	Multiple unweighted	Yes	$\{1\}$	–	`positive`
3	$+$	Positive weights	No	$(0, \infty)$	Ratio scale	`posweighted`
4	$\pm$	Signed	No	$(-\infty, +\infty)$	Ratio scale	`signed`
5	$\overset{+}{=}$	Multiple signed	Yes	$(-\infty, +\infty)$	Ratio scale	`multisigned`
6	$*$	Rating	No	$(-\infty, +\infty)$	Interval scale	`weighted`
7	${}_*{}^*$	Multiple ratings	Yes	$(-\infty, +\infty)$	Interval scale	`multiweighted`
8	$\rightleftharpoons$	Dynamic	Yes	$\{1\}$	–	`dynamic`
9		Multiple positive weights	Yes	$(0, \infty)$	Ratio scale	`multiposweighted`

semantics of a rating network. Ratings can be discrete, such as the one-to-five star ratings, or continuous, such as a rating given in percent. This type of network allows only a single edge between two nodes.

- **Networks with multiple ratings** (${}_*{}^*$) have edges annotated with rating values, and allow multiple edges between two nodes. `multiweighted`
- **Dynamic networks** ($\rightleftharpoons$) are networks in which edges can appear and disappear. They are always temporal. Individual edges are not weighted. `dynamic`

2.3 Temporal Networks

Furthermore, networks can have one more property:

- **Temporal networks** (🕓) include a timestamp for each edge, and thus the network can be reconstructed for any moment in the past. By default, the timestamp is in Unix time, and gives one timestamp to each edge, denoting when that edge was added, or when the event represented by the edge took place. It is unspecified whether timestamps include or not leap seconds. In practice, no dataset has precise enough information for that to matter. When the tag `#unspecifiedtime` is set, the timestamps are not in Unix time, but rather in an unspecified monotonous time scale. The fact that a network is temporal is not saved in any special location, but is determined by the fact that timestamps are present in the TSV file, as will be described later.

2.4 Categories

Finally, the network categories classify networks by the type of data they represent. An overview of the categories is given in Table 3.

Table 3: The network categories in KONECT. Each category is assigned a color, which is used in plots, for instance in Figure 1. The property icons are defined in Table 2. U: Undirected network, D: Directed network, B: Bipartite network.

	Internal name	Vertices	Edges	Properties	Count
●	`Affiliation`	Actors, groups	Membership	B − =	17
●	`Animal`	Animals	Tie	U D − +	9
●	`Authorship`	Authors, works	Authorship	B − =	809
●	`Citation`	Documents	Citation	D −	7
●	`Coauthorship`	Authors	Coauthorship	U − =	7
●	`Cocitation`	Authors	Cocitation	U =	2
●	`Communication`	Persons	Message	U D − =	42
●	`Computer`	Computers	Connection	U D − =	14
●	`Feature`	Items, features	Property	B − = +	17
●	`HumanContact`	Persons	Real-life contact	U − = +	7
●	`HumanSocial`	Persons	Real-life tie	U D − + ± $\stackrel{\pm}{=}$	12
●	`Hyperlink`	Web page	Hyperlink	U D B − = ⇌	197
●	`Infrastructure`	Location	Connection	U D − = +	23
●	`Interaction`	Persons, items	Interaction	U D B − = $\stackrel{\pm}{=}$	26
●	`Lexical`	Words	Lexical relationship	U D − =	5
●	`Metabolic`	Metabolites	Interaction	U D − =	8
●	`Misc`	Various	Various	U D B − = +	15
●	`Neural`	Neurons	Connection	D +	1
●	`OnlineContact`	Users	Online interaction	U D − = ± $\stackrel{\pm}{=}$ ⇌	15
●	`Rating`	Users, items	Rating	B − = $*$ $*^*$	21
●	`Social`	Persons	Online tie	U D − + ± $*$ ⇌	50
●	`Software`	Software Component	Dependency	D − =	3
●	`Text`	Documents, words	Occurrence	B =	10
●	`Trophic`	Species	Carbon exchange	U D − +	4
	Total				**1321**

Affiliation networks are bipartite networks denoting the membership of actors in groups. Groups can be defined as narrowly as individual online communities in which users have been active (FG) or as broadly as countries (CN). The actors are mainly persons, but can also be other actors such as musical groups. Note that in all affiliation networks we consider, each actor can be in more than one group, as otherwise the network cannot be connected. `Affiliation`

Animal networks are networks of contacts between animals. They are the animal equivalent to human social networks. Note that datasets of websites such as Dogster (Sd) are *not* included here but in the `Social` (online social network) category, since the networks are generated by humans. `Animal`

Authorship networks are unweighted bipartite networks consisting of links between authors and their works. In some authorship networks such as that of scientific literature (Pa), works have typically only few authors, whereas works in other authorship networks may have many authors, as in Wikipedia articles (en). `Authorship`

Citation networks consist of documents that reference each other. The primary example are scientific publications, but the category also allow patents and other types of documents that reference each other. The category does not include hyperlink networks, i.e., web pages that reference each other. Those are in the category `Hyperlink`. `Citation`

Co-authorship networks are unipartite networks connecting authors who have written works together, for instance academic literature, but also other types of works such as music or movies. These are also often called collaboration networks. `Coauthorship`

Co-citation networks are unipartite networks of documents, connected by an edge if they are both cited by a same other document. As a general rule, co-citation networks are the self-join of citation networks.

Communication networks contain edges that represent individual messages between persons. Communication networks are directed and allow multiple edges. Examples of communication networks are those of emails (EN) and those of Facebook messages (Ow). Note that in some instances, edge directions are not known and KONECT can only provide an undirected network. `Communication`

Computer networks are networks of connected computers. Nodes in them are computers, and edges are connections. When speaking about *networks* in a computer science context, one often means only computer networks. An example is the internet topology network (TO). `Computer`

Feature networks are bipartite, and denote any kind of feature assigned to entities. Feature networks are unweighted and have edges that are not annotated with edge creation times. Examples are songs and their genres (GE). `Feature`

Human contact networks are unipartite networks of actual contact between persons, i.e., talking with each other, spending time together, or at least being physically close. Usually, these datasets are collected by giving out RFID tags to people with chips that record which other people are in the vicinity. Determining when an actual contact has happened (as opposed to for instance to persons standing back to back) is a nontrivial research problem. An example is the Reality Mining dataset (RM). `HumanContact`

Human social networks are real-world social networks between humans. The ties are offline, and not from an online social network. Also, the ties represent a state, as opposed to human contact networks, in which each edge represents an event. `HumanSocial`

Hyperlink networks are the networks of pages on the World Wide Web or on another system of linked knowldge, connected by hyperlinks or similar types of links. These are in general directed. Since any type of information can be represented on the World Wide Web, hyperlink networks are often simultaneously in another category. For instance, user pages linked to each other represent a hyperlink network and a social network at the same time. In such cases, only the more specific (non-hyperlink) category is used in KONECT. `Hyperlink`

Infrastructure networks are networks of physical infrastructure. Examples are road networks (RO), airline connection networks (OF), and power grids (UG). `Infrastructure`

Interaction networks are bipartite networks consisting of people and items or between people and other people, where each edge represents an interaction. In interaction networks, we always allow multiple edges between the same person–item pair, and an interaction is always to be understood as an event. Interaction networks can be online or offline. Examples are people writing in forums (UF), commenting on movies (Fc), listening to songs (Ls) and sports results. `Interaction`

Lexical networks consist of words from natural languages and the relationships between them. Relationships can be semantic (i.e, related to the meaning of words) such as the synonym relationship (WO), associative such as when two words are associated with each other by people in experiments (EA), or denote co-occurrence, i.e., the fact that two words co-occur in text (SB). `Lexical`

Metabolic networks model metabolic pathways, in which nodes a chemical substances and edges are often directed and represents chemical interactions. A subset are protein–protein interaction networks (PPI), in which nodes are proteins, and which are usually undirected. `Metabolic`

Miscellaneous networks are any networks that do not fit into one of the other categories. `Misc`

Neural networks are networks reprensentating the structure of the brain. Nodes are neurons are higher-level groupings of the brain, while edges are connections between them. The field concerned with the network analysis of such structures is called *network neuroscience*.

Online contact networks consist of people and interactions between them. Contact networks are unipartite and allow multiple edges, i.e., there can always be multiple interactions between the same two persons. They can be both directed or undirected. `OnlineContact`

Physical networks represent physically existing network structures in the broadest sense. This category covers such diverse data as physical computer networks (TO), transport networks (OF) and biological food networks (FD). `Physical`

Rating networks consist of assessments given to items by users, weighted by a rating value. Rating networks are bipartite. Networks in which users can rate other users are not included here, but in the Social category instead. If only a single type of rating is possible, for instance the "favorite" relationship, then rating networks are unweighted. Examples of items that are rated are movies (M3), songs (YS), jokes (J1), and even sexual escorts (SX). `Rating`

Online social networks represent ties between persons in online social networking platforms. Certain social networks allow negative edges, which denote enmity, distrust or dislike. Examples are Facebook friendships (OI), the Twitter follower relationship (TF), and friends and foes on Slashdot (SZ). Note that some social networks can be argued to be rating networks, for instance the user–user rating network of a dating site (LI). These networks are all included in the Social category. Online social networks may be undirected (such as on Facebook), or directed (such as on Twitter). For historical reasons, the internal name of this category is Social, even though it includes only online social networks. `Social`

Software networks are networks of interacting software component. Node can be software packages connected by their dependencies, source files connected by includes, and classes connected by imports. `Software`

Text networks consist of text documents containing words. They are bipartite and their nodes are documents and words. Each edge represents the occurrence of a word in a document. Document types are for instance newspaper articles (TR) and Wikipedia articles (EX). `Text`

Trophic networks consist of biological species connected by edges denoting which pairs of species are subject to exchange of substances such as carbon or nitrogen. In simple cases, these networks can be described as "who eats whom", but the category also includes the exchanges of more specific chemical species. The term *food chain* may describe such relationships, but note that in the general case, a trophic network is not a chain, i.e., it is not linear. Trophic networks are directed. Nodes may be individual species, may may also be broader or narrower classes of organisms. `Trophic`

Note that the category system of KONECT is in flux. As networks are added to the collection, large categories are split into smaller ones.

2.5 What Is and Is Not a Network

While the KONECT motto asserts that *everything is a network*, this does not imply that everything is a complex network. Thus, certain networks have a structure that is trivial in a way that render many network analysis methods moot. In KONECT, we do not include such networtks. This includes networks without a giant connected component, in which most nodes are not reachable from each other, and trees, in which there is only a single path between any two nodes. Note that bipartite relationships extracted from n-to-1 relationships are therefore excluded, as they lead to a disjoint network. For instance, a bipartite person–city network containing *was-born-in* edges would not be included, as each city would form its own component disconnected from the rest of the network. On the other hand, a band–country network where edges denote the country of origin of individual band members is included, as members of a single band can have different countries of origin. In fact the Countries network (CN) is of this form. Another example is a bipartite song–genre network, which would only be included in KONECT when songs can have multiple genres. As an example of the lack of complex structure when only a single genre is allowed, the degree distribution in such a song–genre network is skewed because all song nodes have degree one, the diameter cannot be computed since the network is disconnected, and each connected component trivially has a diameter of two or less.

Certain other types of structures are equivalent to networks. For instance, hypergraphs, in which each hyperedge may include any number of nodes, can be presented equivalently as bipartite graphs,

Table 4: The list of groups in KONECT. Groups represent sets of networks that have similar metadata, based on their format, weights and other attributes.

Internal name	Definition
ALL	All networks
SYM	Undirected, unipartite
ASYM	Directed, unipartite
BIP	Bipartite
SQUARE	Unipartite
NEGATIVE	Networks where edges can be interpreted to have negative edges
NONUNWEIGHTED	Networks where edges can be interpreted to have a weight or multiplicity
ASYMNEGATIVE	Directed networks where edges that can be negative
SQUARENEGATIVE	Unipartite networks where edges can be negative
TIME	Temporal networks
TIME_NEGATIVE	Temporal networks where edges can be negative
MULTI	Networks with multiple edges

up to certain small differences described in Section 3. The same holds e.g. for partially-ordered sets and directed graphs. In KONECT, we choose the "network" aspect to model those.

2.6 Groups

Types of networks are separated into groups, as given in Table 4. Each groups represents a combination of metadata nased on network format, weights and other attributes. The importance of groups is that for most network analysis methods, we can specify to which groups of networks they apply.

2.7 Tags

The following tags can be given to networks. They are declared in the `tags` field in the `meta.*` file for each network.

- `#acyclic`: The network is acyclic. Can only be set for directed networks. If this is not set, a directed network must contain at least two pairs of reciprocal edges of the form (u, v) and (v, u). If the network does not contain reciprocal edges, but has cycles, the tag `#nonreciprocal` is used.
- `#aggregatetime`: The smallest value in the timestamp column stands for any earlier time; all timestamp having this smallest value should not be considered when performing time-based methods and plots. In most cases, this value is zero, but it can just as well be -1 or the most negative integer of the integer type used when creating the dataset.
- `#clique`: All possible edges are present, i.e., the graph is complete. This is *not* the opposite of `#incomplete`. Which edges are taken into account depends on the type of graph, i.e., whether the graph is bipartite, directed, allows loops. This does not take into account edge weights and multiplicities, and in fact this tag only really makes sense when those are present.

- `#incomplete`: The network is incomplete, i.e., the present dataset represents a subset of the actual data. This is mostly due to the fact that the data was crawled, or aggregated in another way that cannot guarantee that all nodes and links were seen. In such graphs, it is not specified *which* nodes and edges are missing. This implies that strictly speaking, certain statistics or plots like the degree distribution are not meaningful, since they may depend on which parts are missing. In practice, many datasets fall into this category, and they do not necessarily all have this tag.

- `#join`: The network is the join (in the database-theoretical sense) of more fundamental networks. For instance, a co-authorship network is a join of the authorship network with itself. In general, all networks that can be described as being a *co-X network*, where X is some other network, are of this type. Networks that have this tag may have skewed properties, such as skewed degree distributions. In KONECT, we recommend to *not* generate the join of a given network, but to publish the underlying network(s) itself. However, many networks are only known as their join, and thus are included in KONECT. (Note that in many cases, the self join has been used to make a unipartite network out of a bipartite one, in order to apply network analysis methods that otherwise do not apply to bipartite networks.)

- `#kcore`: The network contains only nodes with a certain minimal degree k. In other words, the nodes with degree less than a certain number k were removed from the dataset. This changes a network drastically, and is called the "k-core" of a network. This is sometimes done to get a less sparse network in applications that do not perform well on sparse networks. This tag implies the `#incomplete` tag.

- `#lcc`: The dataset actually contains only the largest connected component of the actual network. Implies `#incomplete`. This tag is not used when the network is connected for other reasons.

- `#loop`: The network may contain loops, i.e., egdes connecting a vertex to itself. This tag is only allowed for unipartite networks. When this tag is not present, loops are not allowed, and the presence of loops will be considered an error by analysis code.

- `#lowmultiplicity`: Set in networks with multiple edges in which the actual maximal edge multiplicity is very low. Used to be able to use the maximal multiplicity as a sanity check. Indicates a dataset error, as edge multiplicities usually have a power law-like distribution, and thus very high edge multiplicities are usually present.

- `#missingmultiplicity`: This tag is used when the underlying network had inherent multiple edges, but these are not present in the dataset. For instance, any email network that does not contain multiple edges is tagged with this.

- `#missingorientation`: This tag is used for undirected networks which are based on an underlying directed network. For instance, in a citation network, we may only know that the documents A and B are linked, but not which one cites the other. In such a case, the network in KONECT is undirected, although the underlying network is actually directed.

- `#nonreciprocal`: For directed networks only. The network does not contain reciprocal edges. This is only used when the network is non-reciprocal, but does contain directed cycles. If the network is acyclic, `#acyclic` is used.

- `#unspecifiedtime`: The timestamps do not represent Unix time, but something else. The only assumption that can be made is that timestamps are monotonously increasing, i.e., that larger values denote later times. These could be simply year numbers, number of days since a starting point, or something else. In networks having this tag, the number of unique timestamps may be very low, making temporal methods unsuited. For instance, a network that is known from only two snapshots may have the timestamp values 1 and 2.
- `#path`: The edges represent paths have have been navigated in some form. Thus, a purely network-analysis approach will fail to take into account the paths and only consider the adjacencies.
- `#regenerate`: The network can be regenerated periodically and may be updated when a more recent dataset becomes available.
- `#skew`: The graph is directed, and can be interpreted such that an inverted edge is the same than a negative edge. This applies for instance to sports results network, where a directed edge means A won against B, but could be equivalently expressed as a negative edge from B to A. It also applies to networks denoting dominance behavior, in particular between animals.
- `#tournament`: The graph is directed and for each pair of nodes $\{u, v\}$, either the directed edge $u \to v$ or the directed edge $v \to u$ exists, but not both. It is an error for a non-directed graph to have this tag. If `#tournament` is defined, then `#nonreciprocal` must also be defined. Also, the graph must not contain loops, and thus `#loop` must not be defined.
- `#trianglefree`: The network is triangle-free, i.e., the network does not contain any triangles. Can only be used on unipartite networks. By default, it is an error for a unipartite network to not contain triangles. This tag allows a network to be triangle-free.
- `#zeroweight`: Must be set if it is allowed for edge weights to be zero. Only used for networks with positive edge weights and signed/multisigned networks.

3 Graph Theory

The areas of graph theory and network analysis are young, and many concepts within them notoriously lack a single established notation. The notation chosen in KONECT represents a compromise between familiarity with the most common conventions, and the need to use an unambigous choice of letters and symbols. This section gives an overview of the basic definitions used within KONECT, including in the rest of this handbook. A general mathematical reference for basic definitions used in graph theory is given by a book by Bollobás (1998).

To give an example of the inconsistency across different discipline, the degree of a node (the number of neighbors it has) is usually denoted d in mathematics, but k in certain other fields, in particular physics and network science. In KONECT we have chosen to always denote it d. Similar choice have been made for other symbols. For the sake of completeness, we give the definition as used in KONECT, then mention other commonly used symbols; the rest of the handbook and KONECT as a whole then uses the symbol defined initially.

3.1 Graphs

Graphs will be denoted as $G = (V, E)$, in which V is the set of vertices, and E is the set of edges. Without loss of generality, we assume that the vertices V are consecutive natural numbers starting at one, i.e.,

$$V = \{1, 2, 3, \ldots, |V|\}. \tag{1}$$

Edges $e \in E$ will be denoted as sets of two vertices, i.e., $e = \{u, v\}$. We say that two vertices are adjacent if they are connected by an edge; this will be written as $u \leftrightarrow v$, and is equivalent to $\{u, v\} \in E$. For directed networks, $u \to v$ will denote the existence of a directed edge from u to v, and $u \rightleftarrows v$ will denote that two directed edges of opposite orientation exist between u and v. We say that an edge is incident to a vertex if the edge touches the vertex. Strictly speaking, an edge e is incident to node u when $u \in e$, but this notation may be confusing and we avoid it.

We also allow loops, i.e., edges of the form $\{u, u\} = \{u\}$. Loops appear for instance in email networks, where it is possible to send an email to oneself, and therefore an edge may connect a vertex with itself. Loops are also called *self loops* in the literature, but in KONECT, we use the simple term *loop*, as there are no other loops to consider – cycles of multiple nodes and edges are called *cycles*. Most networks do not contain loops, and therefore networks that allow loops are annotated in KONECT with the `#loop` tag, as described in Section 9. Loops are special in networks because they often need special treatment: For instance, when defining the degree of a node, do we count a loop once (defining the degree as the number of edges that are incident to a node), twice (defininining the degree as the number of half-edges attached to it) or none at all (counting the number other nodes that are neighbours).

Most of the time, we work with only one given graph, and therefore it is unambigous which node and edge set are meant by V and E. When ambiguity is possible, we will use the notation $V[G]$ and $E[G]$ to denote the vertex and edge sets of a graph G. This notation using brackets may occasionally be extended to other graph characteristics.

In directed networks, edges are pairs instead of sets, i.e., $e = (u, v)$. In this case, we have $(u, v) \neq (v, u)$ whenever $u \neq v$, as both edges connecting two nodes can exist independently of each other. In directed networks, edges are sometimes called *arcs*, in which case the term *edge* os often reserved for undirected edges; in KONECT, we use the term *edge* or *directed edge* for them, and the term *edge* does not imply undirectedness. Note that the term *directed* may apply to both an individual edge as well as to a whole graph, but in KONECT, graphs never contain both directed and undirected edges. For directed graphs, we use the following terminology: A directed graph is *symmetric* if all edges are reciprocated, i.e., if, for every directed edge (u, v), there is another directed edge (j, i). A symmetric directed graph is equivalent to an undirected graph, but we must take care of a subtelty: In a symmetric directed graph, the degree of each node is twice of that in the corresponding undirected graph. Thus, we cannot just identify both with each other, and must be careful whether we are talking about an undirected graph, or a symmetric directed graph. Directed graphs that are not symmetric will be called asymmetric.

In bipartite graphs, we can partition the set of nodes V into two disjoint sets V_1 and V_2, which we will call the left and right set respectively. Although the assignment of a bipartite network's two node types to left and right sides is mathematically arbitrary, it is chosen in KONECT such that the left nodes are *active* and the right nodes are *passive*, as such a distinction can often be made, and may provide useful hints to the users of a dataset. For instance, a rating graph with users and items will always have users on the left since they are active in the sense that it is they who give

the ratings. Such a distinction is sensible in most networks (Opsahl 2012), as certain patterns can be observed. The degree distribution, for instance, is usually more regular for the *passive* than for the *active* nodes. The number of left and right nodes are denoted $n_1 = |V_1|$ and $n_2 = |V_2|$. As a general rule, a certain number of quantities used in graph theory can be applied to the left and to the right node left separately, in which case we will use the indexes of one and two consistently.

Additionally, bipartite networks can be interpreted as a way to represent hypergraphs. A hypergraph $G = (V, E)$ is similar to a graph, except that the edges in E are sets that may contain any number of nodes, and not only two as in ordinary graphs. These sets in E are called hyperedges. Equivalently, a hypergraph $G = (V, E)$ can be represented as bipartite graph $G = (V \cup E, F)$, where $\{v, e\} \in F$ whenever $v \in V$, $e \in E$ and $v \in e$. In other words, nodes of the hypergraph are kept as left nodes of the bipartite graph, hyperedges become right nodes, and the edges of the bipartite graph correspond to the inclusion relation between them. Since this equivalence is isomorphic, KONECT does not include a format *hypergraph*, and represents all such networks as bipartite ones. Note that for this isomorphism to be precise, one must allow empty and singleton hyperedges.

Networks with multiple edges are written as $G = (V, E)$ just as other networks, with E being a multiset. The degree of nodes in such networks takes into account multiple edges. Thus, the degree does not equal the number of adjacent nodes but the number of incident edges. When E is a multiset, it can contain the edge $\{u, v\}$ multiple times. Mathematically, we may write $\{u, v\}_1$, $\{u, v\}_2$, etc. to distinguish multiple such edges. Note however that we will be lax with this notation. In expressions valid for all types of networks, we will use sums such as $\sum_{\{u,v\}\in E}$ and understand that the sum is over all edges, taking into account multiplicities.

In positively weighted networks, we define w as the weight function, returning the edge weight when given an edge. In such networks, the weights are not taken into account when computing the degree.

In a signed network, each edge is assigned a signed weight such as $+1$ or -1 (Zaslavsky 1982). In such networks, we define w to be the signed weight function. In the general case, we allow arbitrary nonzero real numbers, representing degrees of positive and negative edges. Signed relationships have been considered in both phychology (Heider 1946) and anthropology (Hage & Harary 1983).

In rating networks, we define r to be the rating function, returning the rating value when given an edge. Note that rating values are interpreted to be invariant under shifts, i.e., adding a real constant to all ratings in the network must not change the semantics of the network. Thus, we will often make use of the mean rating defined as

$$\mu = \frac{1}{|E|} \sum_{e \in E} r(e). \tag{2}$$

For consistency, we also define the edge weight function w for unweighted and rating networks:

$$w(e) = \begin{cases} 1 & \text{when } G \text{ is unweighted} \\ r(e) - \mu & \text{when } G \text{ is a rating network} \end{cases} \tag{3}$$

We also define a weighting function for node pairs, also denoted w. This function takes into account both the weight of edges and edge multiplicities. It is defined as $w(u, v) = 0$ when the

nodes u and v are not connected and if they are connected as

$$w(u,v) = \begin{cases} 1 & \text{when } G \text{ is } - \\ |\{k \mid \{u,v\}_k \in E\}| & \text{when } G \text{ is } = \\ w(\{u,v\}) & \text{when } G \text{ is } + \\ w(\{u,v\}) & \text{when } G \text{ is } \pm \\ r(\{u,v\}) - \mu & \text{when } G \text{ is } * \\ \sum_{\{u,v\}_{k \in E}} [r(\{u,v\}_k) - \mu] & \text{when } G \text{ is } {}_*{}^* \end{cases} \tag{4}$$

Dynamic networks are special in that they have a set of events (edge addition and removal) instead of a set of edges. In most cases, we will model dynamic networks as unweighted networks $G = (V, E)$ representing their state at the latest known timepoint. For analyses that are performed over time, we consider the graph at different time points, with the graph always being an unweighted graph.

In an unweighted graph $G = (V, E)$, the degree of a vertex is the number of neighbors of that node

$$d(u) = \{v \in V \mid \{u,v\} \in E\}. \tag{5}$$

In networks with multiple edges, the degree takes into account multiple edges, and thus to be precise, it equals the number of incident edges and not the number of adjacent vertices.

$$d(u) = \{\{u,v\}_k \in E \mid v \in V\} \tag{6}$$

In directed graphs, the sum is over all of u's neighbors, regardless of the edge orientation. Note that the sum of the degrees of all nodes always equals twice the number of edges, i.e.,

$$\sum_{v \in V} d(u) = 2|E|. \tag{7}$$

In a directed graph we define the outdegree d^+ of a node as the number of outgoing edges, and the indegree d^- as the number of ingoing edges.

$$d^+(u) = \{v \in V \mid (u,v) \in E\} \tag{8}$$

$$d^-(u) = \{v \in V \mid (v,u) \in E\} \tag{9}$$

Within source code in KONECT, the outdegree and indegree are also often denoted $d_1(u)$ and $d_2(u)$, respectively – the indices 1 and 2 used for out- and indegree respectively can be thought as referring to rows and columns of a matrix, and are consistent with the usage in bipartite graphs, where the nodes in the first set of nodes correspond to rows of matrices, and nodes in the second set to columns.

The sum of all outdegrees, and likewise the sum of all indegrees always equals the number of nodes in the network.

$$\sum_{u \in V} d^+(u) = \sum_{u \in V} d^-(u) = |E| \tag{10}$$

Thus, the sum of all outdegrees always equals the sum of all indegrees, and therefore the average outdegree always equals the average indegree.

We also define the weight of a node, also denoted by the symbol w, as the sum of the absolute weights of incident edges

$$w(u) = \sum_{\{u,v\} \in E} |w(\{u,v\})|. \tag{11}$$

The weight of a node coincides with the degree of a node in unweighted networks and networks with multiple edges. The weight of a node may also be called its strength (Opsahl et al. 2010).

For directed graphs, we can distinguish the outdegree weight and the indegree weight:

$$w_1(u) = \sum_{(u,v) \in E} |w((u,v))| \tag{12}$$

$$w_2(u) = \sum_{(v,u) \in E} |w((v,u))| \tag{13}$$

3.2 Graph Transformations

Sometimes, it is necessary to construct a graph out of another graph. In the following, we briefly review such constructions.

Let $G = (V, E, w)$ be any weighted, signed or rating graph, regardless of edge multiplicities. Then, $\bar{G}$ will denote the corresponding unweighted graph, i.e.,

$$\bar{G} = (V, E). \tag{14}$$

Note that the graph $\bar{G}$ may still contain multiple edges.

Let $G = (V, E, w)$ be any graph with multiple edges. We define the corresponding unweighted simple graphs as

$$\bar{\bar{G}} = (V, \bar{\bar{E}}), \tag{15}$$

where $\bar{\bar{E}}$ is the set underlying the multiset E. For simple graphs, we define $\bar{\bar{G}} = G$.

Let $G = (V, E, w)$ be a signed or rating network. Then, $|G|$ will denote the corresponding unsigned graph defined by

$$\begin{aligned} |G| &= (V, E, w') \\ w'(e) &= |w(e)|. \end{aligned} \tag{16}$$

Let $G = (V, E, w)$ be any network with weight function w. The negative network to G is then defined as

$$\begin{aligned} -G &= (V, E, w') \\ w'(e) &= -w(e). \end{aligned} \tag{17}$$

This construction is possible for all types of networks. For unweighted and positively weighted networks, it leads to signed networks.

3.3 Algebraic Graph Theory

A very useful representation of graph is using matrices. In fact, a subfield of graph theory, algebraic graph theory, is devoted to this representation (Godsil & Royle 2001, Brouwer & Haemers 2012, Mieghem 2011). When a graph is represented as a matrix, operations on graphs can often be expressed as simple algebraic expressions. For instance, the number of common friends of two people in a social network can be expressed as the square of a matrix. This section gives brief definition of the most important graph matrices. More properties of these matrices, in particular their decompositions, as well as other such matrices, are given in Section 5.

An unweighted graph $G = (V, E)$ can be represented by a $|V|$-by-$|V|$ matrix containing the values 0 and 1, denoting whether a certain edges between two nodes is present. This matrix is called the adjacency matrix of G and will be denoted $\mathbf{A}$. Remember that we assume that the vertices are the natural numbers $1, 2, \ldots, |V|$. Then the entry $\mathbf{A}_{uv}$ is one when $\{u, v\} \in E$ and zero when not. This makes $\mathbf{A}$ square and symmetric for undirected graphs, generally asymmetric (but still square) for directed graphs.

For a bipartite graph $G = (V_1 \cup V_2, E)$, the adjacency matrix has the form

$$\mathbf{A} = \begin{bmatrix} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \end{bmatrix}. \tag{18}$$

The matrix $\mathbf{B}$ is a $|V_1|$-by-$|V_2|$ matrix, and thus generally rectangular. $\mathbf{B}$ will be called the biadjacency matrix.

In weighted networks, the adjacency matrix takes into account edge weights. In networks with multiple edges, the adjacency matrix takes into account edge multiplicities. Thus, the general definition of the adjacency matrix is given by

$$\mathbf{A}_{uv} = w(u, v). \tag{19}$$

The degree matrix $\mathbf{D}$ is a diagonal $|V|$-by-$|V|$ matrix containing the absolute weights of all nodes, i.e.,

$$\mathbf{D}_{uu} = |w(u)|. \tag{20}$$

Note that we define the degree matrix explicitly to contain node weights instead of degrees, to be consistent with the definition of $\mathbf{A}$.

For directed graphs, we can define the diagonal degree matrix specifically for outdegrees and indegrees as follows:

$$[\mathbf{D}_1]_{uu} = |w_1(u)| \tag{21}$$

$$[\mathbf{D}_2]_{uu} = |w_2(u)| \tag{22}$$

Note that while the outdegree and indegree is denoted d^+ and d^-, we avoid superscripts for matrices, as they can be confused with powers or matrix inverses, etc.

The normalized adjacency matrix $\mathbf{N}$ is a $|V|$-by-$|V|$ matrix given by

$$\mathbf{N} = \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}. \tag{23}$$

Finally the Laplacian matrix $\mathbf{L}$ is an $|V|$-by-$|V|$ matrix defined as

$$\mathbf{L} = \mathbf{D} - \mathbf{A}. \tag{24}$$

The term *Laplacian matrix* is also used in a wider sense to refer to similar matrices, which, like $\mathbf{L}$, are positive-semidefinite. An alternative convention exists in which the Laplacian matrix is defined as $\mathbf{A} - \mathbf{D}$, effectively negating all eigenvalues. In KONECT, we use the $\mathbf{D} - \mathbf{A}$ convention for all Laplacian-type matrices. The Laplacian is also called the *Laplace matrix*.

Note that in most cases, we work on just a single graph, and it is implicit that the characteristic matrices apply to this graph. In a few cases, we may need to consider the characteristic matrices of multiple graphs. In these cases, we will write

$$\mathbf{A}[G], \mathbf{D}[G], \mathbf{L}[G], \dots$$

to denote the characteristic matrices of the graph G.

Another matrix that appears in relation to graph analysis is the incidence matrix. Unlike almost all other matrices used in KONECT and in network analysis in general, the incidence matrix is not indexed by nodes of the graph, but by edges. The incidence matrix $\mathbf{E} \in \{0,1\}^{|V|\times|E|}$ is a nodes-by-edges matrix, and it contains the value one when a node is incident to an edge, hence the name:

$$\mathbf{E}_{u\{u,v\}} = 1 \text{ for edges } \{u,v\} \in E$$
$$\mathbf{E}_{u\{v,w\}} = 0 \text{ when } u \neq v \text{ and } u \neq w$$

To be precise, the exact definition of $\mathbf{E}$ requires one to define a numbering for the edges, just as a number of the nodes is necessary to define the adjacency matrix.

The incidence matrix $\mathbf{E}$ is highly sparse: Only a proportion $2/m$ of all entries are nonzero. Since the incidence matrix represents the same information as the adjacency matrix, it is not necessary to represent it programmatically, except when additional edge attributes are used, which is not the case in KONECT. Rather, the matrix is of theoretical interest. For instance, we can derive the equality $\mathbf{E}\mathbf{E}^{\mathrm{T}} = \mathbf{D} + \mathbf{A}$, which corresponds to the *signless Laplacian matrix*, as will be defined in Section 5.1.7.

A variant is the signed incidence matrix $\mathbf{E}^{\pm}$, which has the same size and values as $\mathbf{E}$, with the exception that in each column, a single value of 1 is replaced by -1. The choice of which of the two nonzero values in each column to negate is arbitrary, and corresponds to chosing an orientation for each edge. This matrix appears for instance in the equality $\mathbf{E}^{\pm}[\mathbf{E}^{\pm}]^{\mathrm{T}} = \mathbf{D} - \mathbf{A} = \mathbf{L}$, which proves that the matrix $\mathbf{L}$ is positive-semidefinite. Note that the equality holds independently of the chosen edge orientations.

Both variants of the incidence matrix may also be denoted $\mathbf{B}/\mathbf{R}$ (e.g. in Mieghem 2011) and $\mathbf{M}$ (e.g. in Brouwer & Haemers 2012) in other literature.

Finally, the line matrix $\mathbf{F} \in \{0,1\}^{|E|\times|E|}$ of a graph is the edge-by-edge matrix given by

$$\mathbf{F} = [\mathbf{E}^{\pm}]^{\mathrm{T}}\mathbf{E}^{\pm}. \tag{25}$$

This matrix denotes which pairs of edges are incident, i.e., share an endpoint. This matrix is sometimes also denoted $\mathbf{G}$. It shares its spectrum with the signless Laplacian matrix $\mathbf{K} = \mathbf{D} + \mathbf{A}$, and thus, as will be shown in Section 5.1.7, it is positive-semidefinite, and has eigenvalue zero if and only if the graph contains a connected component that is bipartite.

The line matrix $\mathbf{F}$ is also the adjacency matrix of the original graph's line graph.

4 Statistics

A network statistic is a numerical value that characterizes a network. Examples of network statistics are the number of nodes and the number of edges in a network, but also more complex measures such as the diameter and the clustering coefficient. Statistics are the basis of most network analysis methods; they can be used to compare networks, classify networks, detect anomalies in networks and for many other tasks. Network statistics are also used to map a network's structure to a simple numerical space, in which many standard statistical methods can be applied. Thus, network statistics are essential for the analysis of almost all network types. All statistics described in KONECT are real numbers. Graph statistics are also called graph invariants and grapg metrics. In KONECT we use the term *statistic* exclusively.

This section gives the definitions for the statistics supported by KONECT, and briefly reviews their uses. All network statistics can be computed using the KONECT Toolbox using the function `konect_statistic()`. Each statistic has an internal name that must be passed as the first argument to `konect_statistic()`. The internal names are given in the margin in this section. Additionally, the KONECT Toolbox includes functions named `konect_statistic_<NAME>()` which compute a single statistic `<NAME>`.

The values of selected statistics are shown for the KONECT networks on the website[14].

4.1 Basic Network Statistics

Some statistics are simple to define, trivial to compute, and are reported universally in studies about networks. These include the number of nodes, the number of edges, and statistics derived from them such as the average number of neighbors a node has.

The size of a network is the number of nodes it contains, and is almost universally denoted n. The size of a graph is sometimes also called the order of the graph.

$$n = |V| \qquad \texttt{size}$$

In a bipartite graph, the size can be decomposed as $n = n_1 + n_2$ with $n_1 = |V_1|$ and $n_2 = |V_2|$. The size of a network is not necessarily a very meaningful number. For instance, adding a node without edges to a network will increase the size of the network, but will not change anything in the network. In the case of an online social network, this would correspond to creating a user account and not connecting it to any other users – this adds an inactive user, which are often not taken into account. Therefore, a more representative measure of the *size* of a network is actually given by the number of edges, giving the volume of a network.

The volume of a network equals the number of edges and is defined as

$$m = |E|. \qquad \texttt{volume}$$

Note that in mathematical contexts, the number of edges may be called the *size* of the graph, in which case the number of nodes is called the *order*. In this text, we will consistently use *size* for the number of nodes and *volume* for the number of edges.

The volume can be expressed in terms of the adjacency or biadjacency matrix of the underlying

[14] konect.cc/statistics

unweighted graph as

$$m = \begin{cases} \frac{1}{2}\|\mathbf{A}[\bar{G}]\|_{\mathrm{F}}^2 & \text{when } G \text{ is undirected} \\ \|\mathbf{A}[\bar{G}]\|_{\mathrm{F}}^2 & \text{when } G \text{ is directed} \\ \|\mathbf{B}[\bar{G}]\|_{\mathrm{F}}^2 & \text{when } G \text{ is bipartite} \end{cases} \tag{26}$$

The number of edges in network is often considered a better measure of the *size* of a network than the number vertices, since a vertex unconnected to any other vertices may often be ignored. On the practical side, the volume is also a much better indicator of the amount of memory needed to represent a network.

We will also make use of the number of edges without counting multiple edges. We will call this the unique volume of the graph.

$$\bar{\bar{m}} = m[\bar{\bar{G}}] \qquad \texttt{uniquevolume}$$

The weight w of a network is defined as the sum of absolute edge weights. For unweighted networks, the weight equals the volume. For rating networks, remember that the weight is defined as the sum over ratings from which the overall mean rating has been subtracted, in accordance with the definition of the adjacency matrix for these networks.

$$w = \sum_{e \in E} |w(e)| \qquad \texttt{weight}$$

The number of loops is also a statistic, and is denoted l. `loops`

The average degree is defined as

$$d = \frac{1}{|V|} \sum_{u \in V} d(u) = \frac{2m}{n}. \qquad \texttt{avgdegree}$$

The average degree is sometimes called the *density* or *node density*. We avoid the term *density* in KONECT as it is sometimes used for the fill, which denotes the probability that an edge exists. The average degree is also called the expected degree. In bipartite networks, we additionally define the left and right average degree

$$d_1 = \frac{1}{|V_1|} \sum_{u \in V_1} d(u) = \frac{m}{n_1} \tag{27}$$

$$d_2 = \frac{1}{|V_2|} \sum_{u \in V_2} d(u) = \frac{m}{n_2} \tag{28}$$

Note that in directed networks, the average outdegree equals the average indegree, and both are equal to m/n.

The fill of a network is the proportion of edges to the total number of possible edges. The fill is used as a basic parameter in the Erdős–Rényi random graph model (Erdős & Rényi 1959), where it denotes the probability that an edge is present between two randomly chosen nodes, and is usually called p, which is the notation we also use in KONECT.

$$p = \begin{cases} 2m/[n(n-1)] & \text{when } G \text{ is undirected without loop} \\ 2m/[n(n+1)] & \text{when } G \text{ is undirected with loops} \\ m/[n(n-1)] & \text{when } G \text{ is directed without loops} \\ m/n^2 & \text{when } G \text{ is directed with loops} \\ m/(n_1 n_2) & \text{when } G \text{ is bipartite} \end{cases} \qquad \texttt{fill}$$

In the undirected case, the expression is explained by the fact that the total number of possible edges is $n(n-1)/2$ excluding loops. The fill is sometimes also called the density or edge density of the network, in particular in a mathematical context, or the connectance of the network[15].

The maximum degree equals the highest degree value attained by any node.

$$d_{\max} = \max_{u \in V} d(u) \qquad \texttt{maxdegree}$$

This is can be applied to directed and bipartite networks in the obvious way:

$$\begin{aligned} d^+_{\max} &= \max_{u \in V} d^+(u) \\ d^-_{\max} &= \max_{u \in V} d^-(u) \\ d_{1\,\max} &= \max_{u \in V_1} d(u) \\ d_{2\,\max} &= \max_{v \in V_2} d(u) \end{aligned}$$

The maximum degree can be divided by the average degree to normalize it.

$$d_{\mathrm{MR}} = \frac{d_{\max}}{d} \qquad \texttt{relmaxdegree}$$

In a directed network, the reciprocity equals the proportion of edges for which an edge in the opposite direction exists, i.e., that are reciprocated (Garlaschelli & Loffredo 2004).

$$y = \frac{1}{m} |\{(u,v) \in E \mid (v,u) \in E\}| \qquad \texttt{reciprocity}$$

The reciprocity has also been noted r (e.g. Szell et al. 2010). The reciprocity can give an idea of the type of network. For instance, citation networks only contain only few pairs of papers that mutually cite each other. On the other hand, an email network will contain many pairs of people who have sent emails to each other. Thus, citation networks typically have low reciprocity, and communnication networks have high reciprocity.

In a graph with multiple edges, we may consider the average multiplicity, i.e., the average number of edges connecting two nodes that are connected by at least one edge.

$$\tilde{m} = \frac{m}{\bar{\bar{m}}} \qquad \texttt{avgmult}$$

4.2 Subgraph Count Statistics

The fundamental building block of a network are the edges. Thus, the number of edges is a basic statistic of any network. To understand the structure of a network, it is however not enough to analyse edges individually. Instead, larger patterns such as triangles must be considered. These patterns can be counted, and give rise to

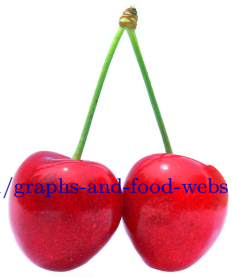
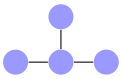


Figure 3: A 2-star is a graph consisting of three nodes, two of which are connected. 2-stars are occasionally called *cherries* due to their resemblance to the fruit.

[15] Used for instance in this blog entry: proopnarine.wordpress.com/2010/02/11/graphs-and-food-webs

Table 5: Subgraph patterns that occur in networks. Each pattern can be counted, giving rise to a count statistic.

Pattern	Name(s)	Statistic	Internal name
	Node, 0-star, 0-path, 1-clique	n	`size`
	Edge, 1-star, 1-path, 2-clique	m	`volume`
	Loop	l	`loops`
	Wedge, 2-star, 2-path	s	`twostars`
	Triangle, 3-cycle, 3-clique	t	`triangles`
	Claw, 3-star	z	`threestars`
	Square, 4-cycle	q	`squares`
	Cross, 4-star	x	`fourstars`
	k-Star k-Path k-Cycle k-Clique	S_k P_k C_k K_k	

count statistics, i.e., statistics that count the number of ocurrences of specific patterns.

Table 5 gives a list of fundamental patterns in networks, and their corresponding count statistics.

A star is defined as a graph in which a central node is connected to all other nodes, and no other edges are present. Specifically, a k-star is defined as a star in which the central node is connected to k other nodes. Thus, a 2-star consists of a node connected to two other nodes, or equivalently two incident edges, or a path of length 2. The specific name for 2-stars is *wedges*. The number of wedges can be defined as

$$s = \sum_{u \in V} \binom{d(u)}{2} = \sum_{u \in V} \frac{1}{2} d(u)(d(u) - 1), \qquad \texttt{twostars}$$

where $d(u)$ is the degree of node u. Wedges have many different names: 2-stars, 2-paths, hairpins (e.g. Gleich & Owen 2012) and cherries.

Three-stars are defined analogously to two-stars, and their count denoted z. Three-stars are also called *claws* and *tripins* (e.g. Gleich & Owen 2012).

$$z = \sum_{u \in V} \binom{d(u)}{3} = \sum_{u \in V} \frac{1}{6} d(u)(d(u)-1)(d(u)-2) \qquad \texttt{threestars}$$

In the general case, the number of k-stars is defined as

$$S_k = \sum_{u \in V} \binom{d(u)}{k} \tag{29}$$

The number of triangles defined in the following way is independent of the orientation of edges when the graph is directed. Loops in the graph, as well as edge multiplicities, are ignored.

$$t = |\{\{u, v, w\} \mid u \leftrightarrow v \leftrightarrow w \leftrightarrow u\}| \,/\, 6 \qquad \texttt{triangles}$$

A square is a cycle of length four, and the number of squares in a graph is denoted q.

$$q = |\{u, v, w, x \mid u \leftrightarrow v \leftrightarrow w \leftrightarrow x \leftrightarrow u\}| \,/\, 8 \qquad \texttt{squares}$$

The factor 8 ensures that squares are counted regardless of their edge labeling.

Multiple edges are ignored in these count statistics, and edges in patterns are not allowed to overlap.

Triangles and squares are both cycles – which we can generalize to k-cycles, sequences of k distinct vertices that are cyclically linked by edges. We denote the number of k-cycles by C_k. For small k, we note the following equivalences:

$$\begin{aligned} C_1 &= 0 \\ C_2 &= m \\ C_3 &= t \\ C_4 &= q \end{aligned}$$

for graphs without loops. Cycles of length three and four have special notation: $C_3 = t$ and $C_4 = q$ and are called triangles and squares.

A cycle cannot the same node twice. Due to this combinatorial restriction, C_k is quite complex to compute for large k. Therefore, we may use *tours* instead, defined as cyclical lists of connected vertices in which we allow several vertices to overlap. The number of k-tours will be denoted T_k. For computational conveniance, we will define labeled tours, where two tours are not equal when they are identical up to shifts or inversions. We note the following equalities:

$$\begin{aligned} T_1 &= 0 \\ T_2 &= 2m \\ T_3 &= 6t \\ T_4 &= 8q + 4s + 2m \qquad \texttt{tour4} \end{aligned}$$

Again, these are true when the graph is loopless. The last equality shows that trying to divide the tour count by $2k$ to count them up to shifts and inversions is a bad idea, since it cannot be implemented by dividing the present definition by $2k$.

As mentioned before, counting cycles is a complex problem. Counting tours is however much easier. The number of tours of length k can be expressed as the trace of a power of the graph's adjacency matrix, and thus also as a moment of the adjacency matrix's spectrum when $k > 2$.

$$T_k = \mathrm{Tr}(\mathbf{A}^k) = \sum_i \lambda_i[\mathbf{A}]^k \tag{30}$$

This remains true when the graph includes loops.

4.3 Connectivity Statistics

Connectivity statistics measure to what extent a network is connected. Two nodes are said to be connected when they are either directly connected through an edge, or indirectly through a path of several edges. A connected component is a set of vertices all of which are connected, and unconnected to the other nodes in the network. The largest connected component in a network is usually very large and called the giant connected component. When it contains all nodes, the network is connected.

The size of the largest connected component is denoted N.

$$N = \max_{F \subseteq \mathcal{C}} |F| \qquad \texttt{coco}$$
$$\mathcal{C} = \{C \subseteq V \mid \forall u, v \in C : \exists w_1, w_2, \ldots \in V : u \leftrightarrow w_1 \leftrightarrow w_2 \leftrightarrow \cdots \leftrightarrow v\}$$

In bipartite networks, the number of left and right nodes in the largest connected components are denoted N_1 and N_2, with $N_1 + N_2 = N$.

The relative size of the largest connected component equals the size of the largest connected component divided by the size of the network

$$N_{\mathrm{rel}} = \frac{N}{n}. \qquad \texttt{cocorel}$$

We also use an inverted variant of the relative size of the largest connected component, which makes it easier to plot the values on a logarithmic scale.

$$N_{\mathrm{inv}} = 1 - \frac{N}{n} \qquad \texttt{cocorelinv}$$

In directed networks, we additionally define the size of the largest strongly connected component N_{s}. A strongly connected component is a set of vertices in a directed graph such that any node is reachable from any other node using a path following only directed edges in the forward direction. We always have $N_{\mathrm{s}} \leq N$. `cocos`

4.4 Distance Statistics

The distance between two nodes in a network is defined as the number of edges needed to reach one node from another, and serves as the basis for a class of network statistics. For instance, the well-known characterisation of networks as having the *small-world* property by Watts and Strogatz (1998) implies that distances between nodes in networks are typically short.

A path in a network is a sequence of incident edges, or equivalently, a sequence of nodes $P = (u_0, u_2, \ldots, u_k)$, such that $(u_i, u_{i+1}) \in E$ for all $i \in \{0, \ldots, k-1\}$. The number k is called the length

of the path, and will also be denoted $l(P)$. A further restriction can be set on the visited nodes, definining that each node can only be visited at most once. If the distinction is made, the term *path* is usually reserved for sequences of non-repeating nodes, and general sequence of adjacent nodes are then called *walks*. We will not make this distinction here.

Paths in networks can be used to model browsing behavior of people in hyperlink networks, navigation in transport networks, and other types of movement-like activities in a network. When considering navigation and browsing, an important problem is the search for shortest paths. Since the length of a path determines the number of steps needed to reach one node from another, it can be used as a measure of distance between nodes of a network. The distance defined in this way may also be called the shortest-path distance to distinguish it from other distance measures between nodes of a network.

$$d(u,v) = \begin{cases} \min_{P=(u,\ldots,v)} l(P) & \text{when } u \text{ and } v \text{ are connected} \\ \infty & \text{when } u \text{ and } v \text{ are not connected} \end{cases} \tag{31}$$

In the case that a network is not connected, the distance is defined as infinite. In practice, only the largest connected component of a network may be used, making it unnecessary to deal with infinite values. The distribution of all $|V|^2$ values $d(u,v)$ for all $u, v \in V$ is called the distance distribution, and it too characterizes the network.

The eccentricity of a node can then be defined as the maximal distance from that node to any other node, defining a measure of *non-centrality*:

$$\epsilon(u) = \max_{v \in V} d(u,v) \tag{32}$$

The diameter δ of a graph equals the longest shortest path in the network (Newman 2003*b*). It can be equivalently defined as the largest eccentricity of all nodes.

$$\delta = \max_{u \in V} \epsilon(u) = \max_{u,v \in V} d(u,v) \qquad \texttt{diam}$$

Note that the diameter is undefined (or infinite) in unconnected networks, and thus in numbers reported for actual networks in KONECT we consider always the diameter of the network's largest connected component. Du to the high runtime complexity of computing the diameter, it may be estimated by various methods, in which case it is noted noted $\tilde{\delta}$.

A statistic related to the diameter is the radius, defined as the smallest eccentricity

$$r = \min_{u \in V} \epsilon(u) = \min_{u \in V} \max_{v \in V} d(u,v) \qquad \texttt{radius}$$

The diameter is bounded from below by the radius, and from above by twice the radius.

$$r \leq \delta \leq 2r \tag{33}$$

The first inequality follows directly from the definitions of r and δ as the minimal and maximal eccentricity. The second inequality follows from the fact that between any two nodes, the path joining them cannot be longer that the path joining them going through a node with minimal eccentricity, which has length of at most $2r$.

The radius and the diameter are not very expressive statistics: Adding or removing an edge will, in many cases, not change their values. Thus, a better statistic that reflects the typical distances in a network in given by the mean and average distance.

The mean path length δ_{m} in a network is defined as as the mean distance over all node pairs, including the distance between a node and itself (Newman 2003*b*):

$$\delta_{\mathrm{m}} = \frac{1}{n^2} \sum_{u \in V} \sum_{v \in V} d(u, v) \qquad \texttt{meandist}$$

The mean path length defined in this way is undefined when a graph is disconnected. Also, the average inverse distance has been used, or equivalently, the inverse of the harmonic mean of distances (Latora & Marchiori 2001). The sum of path lengths has been used on chemistry, where it is referred to as the Wiener index (Rouvray 2002). `mediandist`

Likewise, the median path length δ_{M} is the median length of shortest paths in the network. In KONECT, both the median and mean path lengths are computed taking into account node pairs of the form (u, u).

Both the mean and median path length can be called the *characteristic path length* of the network.

A related statistic is the 90-percentile effective diameter $\delta_{0.9}$, which equals the number of edges needed on average to reach 90% of all other nodes.

4.5 Degree Distribution Statistics

The distribution of degree values $d(u)$ over all nodes u is often taken to characterize a network. Thus, a certain number of network statistics are based solely on this distribution, regardless of overall network structure. The simplest network statistics that depend only on the degree distribution are the number of nodes n, the number of edges m and the average degree $d = 2m/n$, as described previously. More elaborative statistics however reveal information about the detailed shape of the degree distribution.

The power law exponent is a number that characterizes the degrees of the nodes in the network. In many circumstances, networks are modeled to follow a degree distribution power law, i.e., the number of nodes with degree n is taken to be proportional to the power $n^{-\gamma}$, for a constant γ larger than one (Barabási & Albert 1999). This constant γ is called the power law exponent. Given a network, its degree distribution can be used to estimate a value γ. There are multiple ways of estimating γ, and thus a network does not have a single definite value of it. In KONECT, we estimate γ using the robust method given in (Newman 2006, Eq. 5)

$$\gamma = 1 + n \left(\sum_{u \in V} \ln \frac{d(u)}{d_{\min}} \right)^{-1}, \qquad \texttt{power}$$

in which $d_{\min}$ is the minimal degree. This calculation of γ takes into account the entire distribution. Another way to calculate the power law exponent takes only into account the tail, and it is based on code by Aaron Clauset.[16] We call that variant γ_{t}. `power2`

The Gini coefficient is a measure of inequality from economics, typically applied to distributions of wealth or income. In KONECT, we apply it to the degree distribution, as described in (Kunegis & Preusse 2012). The Gini coefficient can either be defined in terms of the Lorenz curve, a type of plot that visualizes the inequality of a distribution, or using the following expression. Let

[16] http://tuvalu.santafe.edu/ aaronc/powerlaws/

$d_1 \leq d_2 \leq \cdots \leq d_n$ be the sorted list of degrees in the network. Then, the Gini coefficient is defined as

$$G = \frac{2\sum_{i=1}^{n} i d_i}{n \sum_{i-1}^{n} d_i} - \frac{n+1}{n}. \qquad \texttt{gini}$$

The Gini coefficient takes values between zero and one, with zero denoting total equality between degrees, and one denoting the dominance of a single node.

The relative edge distribution entropy is a measure of the equality of the degree distribution, and equals one when all degrees are equal, and attains the limit value of zero when all edges attach to a single node (Kunegis & Preusse 2012). It is defined as

$$H_{\text{er}} = \frac{1}{\ln n} \sum_{u \in V} -\frac{d(u)}{2m} \ln \frac{d(u)}{2m}. \qquad \texttt{dentropyn}$$

Another statistic for measuring the inequality in the degree distribution is associated with the Lorenz curve (see Section 6.4). Even though it often does appear indirectly in discussions of skewed distributions, it does not have a common name. In KONECT, we call it the balanced inequality ratio. It is given by the intersection point of the Lorenz curve with the antidiagonal given by $y = 1 - x$ (Kunegis & Preusse 2012). By construction, this point equals $(1 - P, P)$ for some $0 < P < 1$, where the value P corresponds exactly to the number "25%" in the statement "25% of all users account for 75% of all friendship links on Facebook". By construction, we can expect P to be smaller when G is larger, but there is no precise relationship between the two. `own`

Another statistic, which has been called the "hubiness" [sic], is given by the variance of the degree distribution (e.g. by Ferrer-i-Cancho 2013, Mosleh & Heydari 2017).

Statistics such as the power law exponent γ, the Gini coefficient G, the relative edge distribution entropy H_{er}, as well as the ratio P all correlate with each other, as they measure the inequality, skewness or "hubiness" of the degree distribution.

The analysis of degrees can be generalized to pairs of nodes: What is the distribution of degrees for pairs of connected edges? In some networks, high-degree nodes are connected to other high-degree nodes, while low-degree nodes are connected to low-degree nodes. This property is called the degree assortativity (Newman 2003*a*), or, very often, simply the assortativity. Inversely, in a network with dissortativity, high-degree nodes are typically connected to low-degree and vice versa. The amount of assortativity can be measured by the Pearson correlation ρ between the degree of connected nodes.

$$\rho = \frac{\sum_{\{u,v\} \in E} (d(u) - d)(d(v) - d)}{\sum_{\{u,v\} \in E} (d(u) - d)^2} \qquad \texttt{assortativity}$$

The assortativity is undefined whenever the Pearson correlation is undefined, for instance, if all nodes have the same degree (i.e., when the graph is regular), and when the graph does not contain any edges.

In directed networks, we define the in-out assortativity as the Pearson correlation coefficient between in-degree and the out-degree of nodes, taking the logarithm of one plus the degree. `inoutassort`

4.6 Preferential Attachment Statistics

The term *preferential attachment* refers to the observation that in networks that grow over time, the probability that an edge is added to a node with d neighbors is proportional to d. This linear

relationship lies at the heart of Barabási and Albert's *scale-free* network model (Barabási & Albert 1999), and has been used in a vast number of subsequent work to model networks, online and offline. The scale-free network model results in a distribution of degrees, i.e., number of neighbors of individual nodes, that follows a power law with negative exponent. In other words, the number of nodes with degree d is proportional to $d^{-\gamma}$ in these networks, for a constant $\gamma > 1$.

In basic preferential attachment, the probability that an edge attached to a vertex u is proportional to its degree $d(u)$. An extension of this basic model uses a probability that is a power of the degree, i.e., $d(u)^{\beta}$. The exponent β is a positive number, and can be measured empirically from a dataset (Kunegis et al. 2013). The value of β then determines the type of preferential attachment:

1. **Constant case** $\beta = 0$. This case is equivalent to a constant probability of attachment, and thus this graph growth model results in networks in which each edge is equally likely and independent from other edges. This is the Erdős–Rényi model of random graphs (Erdős & Rényi 1959).

2. **Sublinear case** $0 < \beta < 1$. In this case, the preferential attachment function is sublinear. This model gives rise to a stretched exponential degree distribution (Dereich & Mörters 2009), whose exact expression is complex and given in (Dorogovtsev & Mendes 2002, Eq. 94).

3. **Linear case** $\beta = 1$. This is the scale-free network model of Barabási and Albert (Barabási & Albert 1999), in which attachment is proportional to the degree. This gives a power law degree distribution.

4. **Superlinear case** $\beta > 1$. In this case, a single node will acquire 100% of all edges asymptotically (Rudas et al. 2007). Networks with this behavior will however display power law degree distributions in the pre-asymptotic regime (Krapivsky & Krioukov 2008).

The following minimization problem gives an estimate for the exponent β (Kunegis et al. 2013). `prefatt`

$$\min_{\alpha,\beta} \sum_{u \in V} \left(\alpha + \beta \ln[1 + d_1(u)] - \ln[\lambda + d_2(u)]\right)^2 \tag{34}$$

The resulting value of β is the estimated preferential attachment exponent.

To measure the error of the fit, the root-mean-square logarithmic error ϵ can be defined in the following way:

$$\epsilon = \exp\left\{\sqrt{\frac{1}{|V|} \sum_{u \in V} \left(\alpha + \beta \ln[1 + d_1(u)] - \ln[\lambda + d_2(u)]\right)^2}\right\}$$

This gives the average factor by which the actual new number of edges differs from the predicted value, computed logarithmically. The value of ϵ is larger or equal to one by construction.

4.7 Clustering Statistics

The term *clustering* refers to the observation that in almost all networks, nodes tend to form small groups within which many edges are present, and such that only few edges connected different clusters with each other. In a social network for instance, people form groups in which almost every

member known the other members. Clustering thus forms one of the primary characteristics of real-world networks, and thus many statistics for measuring it have been defined. As an example, the well-known characterisation of networks as having the *small-world* property by Watts and Strogatz (1998) uses the clustering coefficient as one network statistic, and states that it will be particularly small. The main method for measuring clustering numerically is the clustering coefficient, of which there exist several variants. As a general rule, the clustering coefficient measures to what extent edges in a network tend to form triangles. Since it is based on triangles, it can only be applied to unipartite networks, because bipartite networks do not contain triangles.

The number of triangles t itself as defined in Section 4.2 is however not a statistic that can be used to measure the clustering in a network, since it correlates with the size and volume of the network. Instead, the clustering coefficients in all its variants can be understood as a count of triangles, normalized in different ways in order to compare several networks with it.

The local clustering coefficient $c(u)$ of a node u is defined as the probability that two randomly chosen (but distinct) neighbors of u are connected.

$$c(u) = \begin{cases} \frac{\{v,w \in V | u \leftrightarrow v \leftrightarrow w \leftrightarrow u\}}{\{v,w \in V | u \leftrightarrow v \neq w \leftrightarrow u\}} & \text{when } d(u) > 1 \\ 0 & \text{when } d(u) \leq 1 \end{cases} \tag{35}$$

The global clustering of a network can be computed in two ways. The first way defines it as the probability that two incident edges are completed by a third edge to form a triangle (Newman et al. 2002). This is also called the transitivity ratio, or simply the transitivity.

$$c = \frac{|\{u, v, w \in V \mid u \leftrightarrow v \leftrightarrow w \leftrightarrow u\}|}{|\{u, v, w \in V \mid u \leftrightarrow v \neq w \leftrightarrow u\}|} = \frac{3t}{s} \qquad \texttt{clusco}$$

This variant of the global clustering coefficient has values between zero and one, with a value of one denoting that all possible triangles are formed (i.e., the network consists of disconnected cliques), and zero when it is triangle free. Note that the clustering coefficient is trivially zero for bipartite graphs. This clustering coefficient is however not defined when each node has degree zero or one, i.e., when the graph is a disjoint union of edges and unconnected nodes. This is however not a problem in practice.

The second variant variant of the clustering coefficient uses the average of the local clustering coefficients. This second variant was historically the first to be defined. In was defined in 1998 (Watts & Strogatz 1998) and precedes the first variant by four years.

$$c_2 = \frac{1}{|V|} \sum_{u \in V} c(u) \qquad \texttt{clusco2}$$

This second variant of the global clustering coefficient is zero when a graph is triangle-free, and one when the graph is a disjoint union of cliques of size at least three. This variant of the global clustering coefficient is defined for all graphs, except for the empty graph, i.e., the graph with zero nodes. A slightly different definition of the second variant computes the average only over nodes with a degree of at least two, as seen for instance in (Bansal et al. 2008).

Because of the arbitrary decision to define $c(u)$ as zero when the degree of c is zero or one, we recommend to use the first variant of the clustering coefficient. In the following, the extensions to the clustering coefficient we present are all based on the first variant, c.

For directed graphs, the directed clustering $c^{\pm}$ can be considered. It is defined in analogy to c, `cluscoasym`

but only considering directed 2-paths closed by a directed edge that has the same orientation as the path.

For signed graphs, we may define the clustering coefficient to take into account the sign of edges. The signed clustering coefficient is based on balance theory (Kunegis et al. 2009). In a signed network, edges can be positive or negative. For instance in a signed social network, positive edges represent friendship, while negative edges represent enmity. In such networks, balance theory stipulates than triangles tend to be balanced, i.e., that three people are either all friends, or two of them are friends with each other, and enemies with the third. On the other hand, a triangle with two positive and one negative edge, or a triangle with three negative edges is unbalanced. In other words, we can define the sign of a triangle as the product of the three edge signs, which then leads to the stipulation that triangles tend to have positive weight. To extend the clustering coefficient to signed networks, we thus distinguis between balanced and unbalanced triangles, in a way that positive triangles contribute positively to the signed clustering coefficient, and negative triangles contribute negatively to it. For a triangle $\{u, v, w\}$, let $\sigma(u, v, w) = w(u, v)w(v, w)w(w, u)$ be the sign of the triangle, then the following definition captures the idea:

$$c_{\mathrm{s}} = \frac{\sum_{u,v,w \in V} \sigma(u, v, w)}{|\{u, v, w \in V \mid u \leftrightarrow v \neq w \leftrightarrow u\}|} \tag{36}$$

Here, the sum is over all triangles $\{u, v, w\}$, but can also be taken over all triples of vertices, since $w(u, v) = 0$ when $\{u, v\}$ is not an edge.

The signed clustering coefficient is bounded by the clustering coefficient:

$$|c_{\mathrm{s}}| \leq c \tag{37}$$

The relative signed clustering coefficient can then be defined as

$$c_{\mathrm{r}} = \frac{c_{\mathrm{s}}}{c} = \frac{\sum_{u,v,w \in V} \sigma(u, v, w)}{|\{u, v, w \in V \mid u \leftrightarrow v \leftrightarrow w \leftrightarrow u\}|} \tag{38}$$

which also equals the proportion of all triangles that are balanced, minus the proportion of edges that are unbalanced.

4.8 Algebraic Statistics

Algebraic statistics are based on a network's characteristic matrices. They are motivated by the broader field of spectral graph theory, which characterizes graphs using the spectra of these matrices (Chung 1997). This section only describes algebraic statistics that do not fall under another category, such as bipartivity statistics, or signed graph statistics.

In the following we will denote by $\lambda_k[\mathbf{X}]$ the k^{th} dominant eigenvalue of the matrix $\mathbf{X}$. For the adjacency matrix $\mathbf{A}$, the dominant eigenvalues are the largest absolute ones; for the Laplacian $\mathbf{L}$ they are the smallest ones.

Also, the matrix $\mathbf{L}$ will only be considered for the network's largest connected component.

The spectral norm of a network equals the spectral norm (i.e., the largest absolute eigenvalue) of the network's adjacency matrix

$$\alpha = \|\mathbf{A}\|_2 = |\lambda_1[\mathbf{A}]|. \qquad \texttt{snorm}$$

The spectral norm can be understood as an alternative measure of the size of a network.

For directed graphs, we consider the largest singular value of the graph's (generally) asymmetric adjacency matrix. This is called the operator 2-norm or the Ky Fan 1-norm. We denote it ν. `opnorm`

Also for directed graphs only, the largest absolute eigenvalue of the (generally asymmetric) adjacency matrix $\mathbf{A}$ can be considered. We denote it π and call it the cyclic eigenvalue in KONECT. It is zero if and only if the graph is acyclic. `maxdiag`

The algebraic connectivity equals the second smallest nonzero eigenvalue of $\mathbf{L}$ (Fiedler 1973)

$$a = \lambda_2[\mathbf{L}]. \qquad \texttt{alcon}$$

The algebraic connectivity is zero when the network is disconnected – this is one reason why we restrict the matrix $\mathbf{L}$ to each network's giant connected component. The algebraic connectivity is larger the better the network's largest connected component is connected.

The largest absolute eigenvalue of $\mathbf{W}$ is a network statistic. `seidelnorm`

4.9 Bipartivity Statistics

Some unipartite networks are almost bipartite. Almost-bipartite networks include networks of sexual contact (Liljeros et al. 2001) and ratings in online dating sites (Brožovský & Petříček 2007, Kunegis, Gröner & Gottron 2012). Other, more subtle cases, involve online social networks. For instance, the follower graph of the microblogging service Twitter is by construction unipartite, but has been observed to reflect, to a large extent, the usage of Twitter as a news service (Kwak et al. 2010). This is reflected in the fact that it is possible to indentify two kinds of users: Those who primarily get followed and those who primarily follow. Thus, the Twitter follower graph is almost bipartite. Other social networks do not necessarily have a near-bipartite structure, but the question might be interesting to ask to what extent a network is bipartite. To answer this question, measures of bipartivity have been developed.

Instead of defining measures of bipartivity, we will instead consider measures of non-bipartivity, as these can be defined in a way that they equal zero when the graph is bipartite. Given an (a priori) unipartite graph, a measure of non-bipartivity characterizes the extent to which it fails to be bipartite. These measures are defined for all networks, but are trivially zero for bipartite networks. For non-bipartite networks, they are larger than zero.

A first measure of bipartivity consists in counting the minimum number of *frustrated edges* (Holme et al. 2003). Given a bipartition of vertices $V = V_1 \cup V_2$, a frustrated edge is an edge connecting two nodes in V_1 or two nodes in V_2. Let f be the minimal number of frustrated edges in any bipartition of V, or, put differently, the minimum number of edges that have to be removed from the graph to make it bipartite. Then, a measure of non-bipartivity is given by

$$F = \frac{f}{|E|}. \qquad \texttt{frustration}$$

This statistic is always in the range $[0, 1/2]$. It attains the value zero if and only if G is bipartite.

The minimal number of frustrated edges f can be approximated by algebraic graph theory. First, we represent a bipartition $V = V_1 \cup V_2$ by its characteristic vector $\mathbf{x} \in \mathbb{R}^{|V|}$ defined as

$$\mathbf{x}_u = \begin{cases} +1/2 & \text{when } u \in V_1 \\ -1/2 & \text{when } u \in V_2 \end{cases}$$

Note that the number of edges connecting the sets V_1 and V_2 is then given by

$$\{\{u,v\} \mid u \in V_1, v \in V_2\} = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{K}[\bar{G}]\mathbf{x} = \frac{1}{2}\sum_{(u,v)\in E}(\mathbf{x}_u + \mathbf{x}_v)^2,$$

where $\mathbf{K}[\bar{G}] = \mathbf{D}[\bar{G}] + \mathbf{A}[\bar{G}]$ is the signless Laplacian matrix of the underlying unweighted graph (see Section 5.1.7). Thus, the minimal number of frustrated edges f is given by

$$f = \min_{\mathbf{x}\in\{\pm 1/2\}^{|V|}} \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{K}[\bar{G}]\mathbf{x}.$$

By relaxing the condition $\mathbf{x} \in \{\pm 1/2\}^{|V|}$, we can express f in function of $\mathbf{K}[\bar{G}]$'s minimal eigenvalue, using the fact that the norm of all vectors $\mathbf{x} \in \{\pm 1/2\}^{|V|}$ equals $\sqrt{|V|/4}$, and the property that the minimal eigenvalue of a matrix equals its minimal Rayleigh quotient.

$$\frac{2f}{|V|/4} \approx \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^{\mathrm{T}}\mathbf{K}[\bar{G}]\mathbf{x}}{\|\mathbf{x}\|^2} = \lambda_{\min}[\mathbf{K}[\bar{G}]]$$

We can thus approximate the previous measure of non-bipartivity by

$$b_{\mathrm{K}} = \frac{|V|}{8|E[\bar{G}]|}\lambda_{\min}[\mathbf{K}[\bar{G}]] \qquad \texttt{anticonflict}$$

The eigenvalue itself,

$$\chi = \lambda_{\min}[\mathbf{K}[\bar{G}]], \qquad \texttt{nonbipal}$$

as the algebraic non-bipartivity of the graph.

The eigenvalue $\lambda_{\min}[\mathbf{K}[\bar{G}]]$ can also be interpreted as the algebraic conflict in G interpreted as a signed graph in which all edges have negative weight.

A further measure of bipartivity exploits the fact that the adjacency matrix $\mathbf{A}$ of a bipartite graph has eigenvalues symmetric around zero, i.e., all eigenvalues of a bipartite graph come in pairs $\pm\lambda$. Thus, the ratio of the smallest and largest eigenvalues can be used as a measure of non-bipartivity

$$b_{\mathrm{A}} = 1 - \left|\frac{\lambda_{\min}[\mathbf{A}[\bar{G}]]}{\lambda_{\max}[\mathbf{A}[\bar{G}]]}\right|, \qquad \texttt{nonbip}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalue of the given matrix, and $\bar{G}$ is the unweighted graph underlying G. Since the largest eigenvalue always has a larger absolute value than the smallest eigenvalue (due to the Perron–Frobenius theorem, and from the nonnegativity of $\mathbf{A}[\bar{G}]$), it follows that this measure of non-bipartivity is always in the interval $[0, 1)$, with zero denoting a bipartite network. The value one is excluded in non-empty loopless graphs. This can be seen by considering the trace of the graph's adjacency matrix, which equals the sum of its diagonal values (and therefore equals the number of loops), and at the same time equals the sum of eigenvalues of the adjacency matrix.

Another spectral measure of non-bipartivity is based on considering the smallest eigenvalue of the matrix $\mathbf{N}[\bar{G}]$. This eigenvalue is -1 exactly when G is bipartite. Thus, this value minus one is a measure of non-bipartivity. Equivalently, it equals two minus the largest eigenvalue of the normalized Laplacian matrix $\mathbf{Z}$.

$$b_{\mathrm{N}} = \lambda_{\min}[\mathbf{N}[\bar{G}]] + 1 = 2 - \lambda_{\max}[\mathbf{Z}[\bar{G}]] \qquad \texttt{nonbipn}$$

4.10 Signed Network Statistics

In networks that allow negative edges such as signed networks and rating networks, we may be interested in the proportion of edges that are actually negative. We call this the *negativity* of the network.

$$\zeta = \frac{|\{e \in E \mid w(e) < 0\}|}{m} \qquad \texttt{negativity}$$

The negativity is denoted q in (Facchetti et al. 2011).

In directed signed networks, we can additionally compute the dyadic conflict, i.e., the propostion of node pairs connected by two oppositely oriented edges of different, compared to the total number of pairs of nodes connected by two edges of opposite orientation.

$$\eta = \frac{|\{u, v \mid u \rightleftarrows v, w(u, v) = -w(v, u)\}|}{|\{u, v \mid u \rightleftarrows v\}|} \qquad \texttt{dconflict}$$

Furthermore, the triadic conflict can be defined as the proportion of triangles that are in conflict, i.e., that are unbalanced.

$$\tau = \frac{|\{u, v, w \mid w(u, v)w(v, w)w(w, u) < 0\}|}{|\{u, v, w \mid u \sim v \sim w \sim u\}|} \qquad \texttt{tconflict}$$

This is also known as the triangle index. It is also related to the relative signed clustering coefficient by

$$\tau = 2c_{\mathrm{r}} - 1. \tag{39}$$

The smallest eigenvalue of the adjacency matrix $\mathbf{L}$ can be larger than zero. (In other networks, it is always zero.) The algebraic conflict equals this smallest eigenvalue

$$\xi = \lambda_1[\mathbf{L}]. \qquad \texttt{conflict}$$

The algebraic conflict measures the amount of conflict in the network, i.e., the tendency of the network to contain cycles with an odd number of negatively weighted edges. It is applied in KONECT only to the network's largest connected component. A normalized version of the algebraic conflict is an approximation to the number of edges whose sign must be switched for the graph to become balanced. We call it the spectral signed frustration ϕ, defined as

$$\phi = \frac{n\xi}{8m} = \frac{n\lambda_1[\mathbf{L}]}{8m}. \qquad \texttt{fconflict}$$

The normalization factor can be derived in analogous way as for the corresponding measure of bipartivity that is based on the signless Laplacian matrix.

4.11 Miscellaneous

This section lists miscellaneous statistics that do not fit any of the other sections

4.11.1 Controllability

A less-known way to assess the structure of a network consists in measuring how well it can be controlled. For instance, assume that we want to influence opinions in a social network, but are only able to directly influence k persons in the network. Assuming that opinions will spread through the network, how big has k to be in order for us to be able to influence all nodes in a network, in a way that any arbitrary opinion can be given to any node? A solution to this problem is given in (Liu et al. 2011), in which such *driver nodes* are identified and, surprisingly, they are not necessarily the nodes with highest degree. In fact, the authors of that article state that driver nodes tend to *avoid the hubs* of the network.

The resulting computational model uses differential equations to model diffusion and can be reduced to finding a maximal matching in the bipartite double cover of the network (Liu et al. 2011). The maximal matching in a bipartite graph can be computed efficiently because of Kőnig's theorem. It states that perfect matchings and minimal vertex covers have equal size in bipartite graphs, and thus the corresponding integer program formulations are equivalent to their relaxations.

The number of driver nodes C needed to control a graph $G = (V, E)$ equals $|V|$ minus the size of the maximum matching in the bipartite cover of the network, which equals the maximal directed 2-matching in the network. The bipartite double cover of a network $G = (V, E)$ is constructed by mapping each vertex $u \in V$ to two vertices $(u, 1)$ and $(u, 2)$, and mapping each edge $\{u, v\}$ to the edges $\{(u, 1), (v, 2)\}$ and $\{(u, 2), (v, 1)\}$. Equivalently this corresponds to the size of the maximal directed 2-matching. A 2-matching is a set of edges such that each vertex is incident to at most two edges. A directed 2-matching is a set of directed edges, such that each vertex is incident to at most one ingoing and one outgoing edge. Here, we interpret an undirected graph as a directed graph where each edge corresponds to two directed edges:

$$
\begin{aligned}
C = |V| - \max |M| & & & \texttt{controllability} \\
\text{s.t.} \quad & |\{(i,j) \in M \mid i = u, \{i,j\} \in E\}| \leq 1 & \text{for all } u \in V, & \\
& |\{(i,j) \in M \mid j = u, \{i,j\} \in E\}| \leq 1 & \text{for all } u \in V. &
\end{aligned}
$$

The result is the number C of vertices needed to control a given network. A maximal matching in a bipartite graph can be found in runtime $O(|V|^{1/2}|E|)$ (Liu et al. 2011), and thus can be computed efficiently even for large networks.

We also define the relative controllability as

$$
C_{\mathrm{r}} = \frac{C}{n} \qquad \texttt{controllabilityr}
$$

This statistic has been used by the authors of KONECT in (Kunegis, Sizov, Schwagereit & Fay 2012).

5 Matrices and Decompositions

In order to analyse graphs, algebraic graph theory is a common approach. In algebraic graph theory, a graph with n vertices is represented by an $n \times n$ matrix called the adjacency matrix, from which other matrices can be derived. The defined matrices can then be decomposed to reveal properties of the graph. In this section, we review characteristic graph matrices, their decompositions, and

their uses. Since most decompositions are based on a specific matrix, this section also serves as a survey of characteristic graph matrices.

As a general rule, matrices are noted with bold uppercase letters. The letters shown in this section for each characteristic matrix are used systematically within KONECT, but not necessarily in the literature. Among established matrix names, there is the adjacency matrix ($\mathbf{A}$) and the Laplacian ($\mathbf{L}$), but we must note that even the adjacency matrix is not always called $\mathbf{A}$ in the literature. Each matrix resp. decomposition has an internal name in KONECT, which is given in the margin. Using the KONECT Toolbox, the corresponding matrices can be generated using the `konect_matrix()` function, and the corresponding (rank reduced) matrix decomposition can be computed with the function `konect_decomposition()`.

The first subsection contains decompositions that apply to simple graphs. Subsequent subsections contain matrices that apply to specific types of graphs: signed, bipartite, and directed graphs. Almost all decompositions given in the first subsection can also be applied to non-simple graphs, in which cases the KONECT implementation will ignore the extra information. Exceptions to this are when a decomposition given in a subsequent subsection shares an internal name with one given in the first subsection. In that case, KONECT will do the obvious generalization when a signed or directed graph is given.

Table 6 gives a tabular overview. A more detailed overview is given in Table 7, including the internal names of corresponding decompositions and eigenvalue statistics.

5.1 Decompositions of Undirected Graphs

This section covers the case of unipartite unsigned undirected graphs. Weighted graphs and graphs with multiple edges are included, as their treatment is usually simple. We define nine basic matrices in this section, which can be thought of spanning the two dimensions of matrix type (adjacency/Laplacian/signless Laplacian) and normalization type (none, symmetric normalization, stochastic). These matrices have the names $\mathbf{A}$, $\mathbf{L}$, $\mathbf{K}$, $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{C}$, $\mathbf{P}$, $\mathbf{S}$ and $\mathbf{J}$. An overview is given in Table 8 These matrices serve as the "basic nine" matrices in KONECT, and derivatives with similar names are used for other types of graphs. Of these, $\mathbf{A}$, $\mathbf{L}$, $\mathbf{N}$ and $\mathbf{P}$ are ubiquitous in network analysis; $\mathbf{K}$ is used from time to time; $\mathbf{Z}$, $\mathbf{C}$, $\mathbf{S}$, and $\mathbf{J}$ can be derived by shifts of eigenvalues from $\mathbf{N}$ and $\mathbf{P}$, and thus don't usually need to be considered separately. $\mathbf{N}$ and $\mathbf{P}$ share eigenvalues, but not eigenvectors. Only the names $\mathbf{A}$ and $\mathbf{L}$ can be expected to be understood without explanation – the other names are specific to KONECT, although the notation $\mathbf{N}$, $\mathbf{K}$ and $\mathbf{P}$ are used in the literature from time to time.

In KONECT, these decompositions can be applied to directed graphs as well, in which case edge directions are ignored. A directed graph that contains reciprocal edges (the majority) will thus result in an undirected graph with multiple parallel edges, even if the original directed graph did not contain parallel directed edges. As a result, one must be careful: Expressions involving the number of edges m in this chapter refer to the number of unique edges, which gives a different result for directed and undirected graphs. In the same way, the matrix $\mathbf{A}$ in this section refers to the symmetric adjacency matrix; in the case of directed graphs, this thus equals the matrix normally written as $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$.

5.1.1 Symmetric Adjacency Matrix (A)

The symmetric adjacency matrix $\mathbf{A}$ is the most basic graph characteristic matrix. It is a symmetric $n \times n$ matrix defined as $\mathbf{A}_{uv} = 1$ when the nodes u and v are connected, and $\mathbf{A}_{uv} = 0$ when u and

Table 6: Matrices that characterize networks. All shown cases include weighted and signed graphs. All matrices shown here are written with a diacritic or a subscript. When used without, $\bar{\mathbf{X}}$ is meant for undirected graphs, and $\vec{\mathbf{X}}$ for undirected graphs; hence the need to use diacritics when covering both types of networks.

	Name	Plain	Normalized	Stochastic
Undirected	Adjacency matrix	$\bar{\mathbf{A}}$	$\bar{\mathbf{N}}$	$\bar{\mathbf{P}}$
	Laplacian	$\bar{\mathbf{L}}$	$\bar{\mathbf{Z}}$	$\bar{\mathbf{S}}$
	Signless Laplacian	$\bar{\mathbf{K}}$	$\bar{\mathbf{C}}$	$\bar{\mathbf{J}}$
	Repulsive adjacency matrix[1]	$\hat{\mathbf{A}}$	$\hat{\mathbf{N}}$	$\hat{\mathbf{P}}$
	Repulsive Laplacian†[2]	$\hat{\mathbf{L}}$	$\hat{\mathbf{Z}}$	$\hat{\mathbf{S}}$
	Repulsive signless Laplacian†[2]	$\hat{\mathbf{K}}$	$\hat{\mathbf{C}}$	$\hat{\mathbf{J}}$
Directed	Symmetric adjacency matrix	$\bar{\mathbf{A}}$	$\bar{\mathbf{N}}$	$\bar{\mathbf{P}}$
	Symmetric Laplacian	$\bar{\mathbf{L}}$	$\bar{\mathbf{Z}}$	$\bar{\mathbf{S}}$
	Symmetric signless Laplacian	$\bar{\mathbf{K}}$	$\bar{\mathbf{C}}$	$\bar{\mathbf{J}}$
	Hermitian adjacency matrix	$\check{\mathbf{A}} = \mathbf{H}$	$\check{\mathbf{N}}$	
	Hermitian Laplacian	$\check{\mathbf{L}} = \mathbf{T}$	?	
	Hermitian signless Laplacian			
	Skew-Hermitian adjacency matrix	$\acute{\mathbf{A}} = i\mathbf{Y}$	$\acute{\mathbf{N}}$	
	Skew-Hermitian Laplacian	$\acute{\mathbf{L}}$	?	
	Skew-Hermitian signless Laplacian			
	Magnetic adjacency matrix	$\mathbf{A}_\theta$	?	
	Magnetic Laplacian	$\mathbf{L}_\theta$		
	Magnetic signless Laplacian	$\mathbf{K}_\theta$		
	Unidirectional adjacency matrix†	$\vec{\mathbf{A}}$	$\vec{\mathbf{N}}$	$\vec{\mathbf{P}}$
	Unidirectional Laplacian†	$\vec{\mathbf{L}}$	$\vec{\mathbf{Z}}$	$\vec{\mathbf{S}}$
	Unidirectional signless Laplacian†	$\vec{\mathbf{K}}$	$\vec{\mathbf{C}}$	$\vec{\mathbf{J}}$
Bipartite	Biadjacency matrix*	$\grave{\mathbf{A}} = \mathbf{B}$	$\grave{\mathbf{N}} = \mathbf{M}$	$\times$

* Non-square. As a general rule, bipartite graphs are a special case of unipartite graphs and thus all variants apply; we only show the two cases here in which the matrix has the bipartite form $[\mathbf{0X}; \mathbf{X}^{\mathrm{T}}\mathbf{0}]$.

† Non-normal. These matrices allow both the singular value decomposition, as well as considering their eigenvalues.

[1] Accent usually omitted because the matrix is the natural extension of the non-accented matrix

[2] Identical to the non-accented matrix for unsigned networks

Table 7: The nine base matrices of KONECT for undirected, unipartite graphs, including unweighted, weighted, and signed cases. The transpose of the stochastic matrices are not shown; they are left-stochastic and are analogous to the shown right-stochastic variants. Question marks indicate aspects we have not yet investigated.

	Expression	Int.	Name	Prop.	Eigenvalues
5.1.1	$\mathbf{A}$	`sym`	Adjacency m.	Sym	$\ldots, \alpha$ `snorm`
5.1.2	$\mathbf{N} = \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}$	`sym-n`	Norm. adjacency m.	Sym	$-1+?, \ldots, 1-?, 1-?$
5.1.3	$\mathbf{P} = \mathbf{D}^{-1}\mathbf{A}$	`stoch1`	Stoch. adjacency m.	Right-stoch[1]	$\lambda[\mathbf{N}]$
5.1.4	$\mathbf{L} = \mathbf{D} - \mathbf{A}$	`lap`	Laplacian	Sym	$0 \leq \xi$ `conflict`, a `alcon`, $\ldots, ?$
5.1.5	$\mathbf{Z} = \mathbf{I} - \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}$	$*$ $\mathbf{N}$	Norm. Laplacian	Sym	$1 - \lambda[\mathbf{N}]$
5.1.6	$\mathbf{S} = \mathbf{I} - \mathbf{D}^{-1}\mathbf{A}$	$*$ $\mathbf{P}$	Stoch. Laplacian	Row sum $= 0$[1]	$1 - \lambda[\mathbf{N}]$
5.1.7	$\mathbf{K} = \mathbf{D} + \mathbf{A}$	`lapq`	Signless Laplacian	Sym	$0 \leq \chi$ `nonbipal`, $?, \ldots, ?$
5.1.7	$\mathbf{C} = \mathbf{I} + \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}$	$*$ $\mathbf{N}$	Norm. signless Laplacian	Sym	$1 + \lambda[\mathbf{N}]$
5.1.7	$\mathbf{J} = \mathbf{I} + \mathbf{D}^{-1}\mathbf{A}$	$*$ $\mathbf{P}$	Stoch. signless Laplacian	Row sum $= 2$[1]	$1 + \lambda[\mathbf{N}]$
5.2.3	$\hat{\mathbf{A}} = \mathbf{A}$	$*$ $\mathbf{A}$	Repulsive adjacency m.	Sym	$\lambda[\mathbf{A}]$
5.2.3	$\hat{\mathbf{N}} = \mathbf{N}$	$*$ $\mathbf{N}$	Repulsive norm. adjacency m.	Sym	$\lambda[\mathbf{N}]$
5.2.3	$\hat{\mathbf{P}} = \mathbf{P}$	$*$ $\mathbf{P}$	Repulsive stoch. adjacency m.	Right-stoch[1]	$\lambda[\mathbf{N}]$
5.2.3	$\hat{\mathbf{L}} = \hat{\mathbf{D}} - \mathbf{A}$	?	Repulsive Laplacian	Sym	?
5.2.3	$\hat{\mathbf{Z}} = \mathbf{D}^{-\frac{1}{2}}(\hat{\mathbf{D}} - \mathbf{A})\mathbf{D}^{-\frac{1}{2}}$	?	Repulsive norm. Laplacian	Sym	?
5.2.3	$\hat{\mathbf{S}} = \mathbf{D}^{-1}(\hat{\mathbf{D}} - \mathbf{A})$	$*$ $\hat{\mathbf{Z}}$	Repulsive stoch. Laplacian		$\lambda[\hat{\mathbf{Z}}]$
5.2.3	$\hat{\mathbf{K}} = \hat{\mathbf{D}} + \mathbf{A}$	?	Repulsive signless Laplacian	Sym	?
5.2.3	$\hat{\mathbf{C}} = \mathbf{D}^{-\frac{1}{2}}(\hat{\mathbf{D}} + \mathbf{A})\mathbf{D}^{-\frac{1}{2}}$	?	Repulsive norm. signless Laplacian	Sym	?
5.2.3	$\hat{\mathbf{J}} = \mathbf{D}^{-1}(\hat{\mathbf{D}} + \mathbf{A})$	$*$ $\hat{\mathbf{C}}$	Repulsive stoch. signless Laplacian		$\lambda[\hat{\mathbf{C}}]$
5.1.1	$\bar{\mathbf{A}} = \vec{\mathbf{A}} + \vec{\mathbf{A}}^{\mathrm{T}}$	`sym`	Symmetric adjacency m.	Sym	$\ldots, \alpha$ `snorm`
5.1.2	$\bar{\mathbf{N}} = \mathbf{D}^{-\frac{1}{2}}(\vec{\mathbf{A}} + \vec{\mathbf{A}}^{\mathrm{T}})\mathbf{D}^{-\frac{1}{2}}$	`sym-n`	Symmetric norm. adjacency m.	Sym	$-1+?, \ldots, 1-?, 1-?$
5.1.3	$\bar{\mathbf{P}} = \mathbf{D}^{-1}(\vec{\mathbf{A}} + \vec{\mathbf{A}}^{\mathrm{T}})$	`stoch1`	Symmetric stoch. adjacency m.	Right-stoch[1]	$\lambda[\bar{\mathbf{N}}]$
5.1.4	$\bar{\mathbf{L}} = \mathbf{D} - (\vec{\mathbf{A}} + \vec{\mathbf{A}}^{\mathrm{T}})$	`lap`	Symmetric Laplacian	Sym	$0 \leq \xi$ `conflict`, a `alcon`, $\ldots, ?$
5.1.5	$\bar{\mathbf{Z}} = \mathbf{I} - \mathbf{D}^{-\frac{1}{2}}(\vec{\mathbf{A}} + \vec{\mathbf{A}}^{\mathrm{T}})\mathbf{D}^{-\frac{1}{2}}$	$*$ $\bar{\mathbf{N}}$	Symmetric norm. Laplacian	Sym	$1 - \lambda[\bar{\mathbf{N}}]$
5.1.6	$\bar{\mathbf{S}} = \mathbf{I} - \mathbf{D}^{-1}(\vec{\mathbf{A}} + \vec{\mathbf{A}}^{\mathrm{T}})$	$*$ $\bar{\mathbf{P}}$	Symmetric stoch. Laplacian	Row sum $= 0$[1]	$1 - \lambda[\bar{\mathbf{N}}]$
5.1.7	$\bar{\mathbf{K}} = \mathbf{D} + \vec{\mathbf{A}} + \vec{\mathbf{A}}^{\mathrm{T}}$	`lapq`	Symmetric signless Laplacian	Sym	$0 \leq \chi$ `nonbipal`, $?, \ldots, ?$
5.1.7	$\bar{\mathbf{C}} = \mathbf{I} + \mathbf{D}^{-\frac{1}{2}}(\vec{\mathbf{A}} + \vec{\mathbf{A}}^{\mathrm{T}})\mathbf{D}^{-\frac{1}{2}}$	$*$ $\bar{\mathbf{N}}$	Symmetric norm. signless Laplacian	Sym	$1 + \lambda[\bar{\mathbf{N}}]$
5.1.7	$\bar{\mathbf{J}} = \mathbf{I} + \mathbf{D}^{-1}(\vec{\mathbf{A}} + \vec{\mathbf{A}}^{\mathrm{T}})$	$*$ $\bar{\mathbf{P}}$	Symmetric stoch. signless Laplacian	Row sum $= 2$[1]	$1 + \lambda[\bar{\mathbf{N}}]$
5.4.6	$\acute{\mathbf{A}} = i\mathbf{Y} = i(\vec{\mathbf{A}} - \vec{\mathbf{A}}^{\mathrm{T}})$	`skewi`	Skew-Hermitian Adjacency m.	Skew-Herm.	?
5.4.7	$\acute{\mathbf{N}} = i\mathbf{D}^{-\frac{1}{2}}(\vec{\mathbf{A}} - \vec{\mathbf{A}}^{\mathrm{T}})\mathbf{D}^{-\frac{1}{2}}$	`skewin`	Skew-Hermitian norm. adjacency m.	Skew-Herm., ?	?
?	$\acute{\mathbf{P}} = i\mathbf{D}^{-1}(\vec{\mathbf{A}} - \vec{\mathbf{A}}^{\mathrm{T}})$	?	Skew-Hermitian stoch. adjacency m.	?	$\lambda[\acute{\mathbf{N}}]$
5.4.3, 5.4.1	$\vec{\mathbf{A}}$	`diag;svd`	Unidirectional adjacency m.		$\ldots, \pi$ `maxdiag`; ?
5.4.2, 5.4.4	$\vec{\mathbf{N}} = \mathbf{D}_1^{-\frac{1}{2}}\vec{\mathbf{A}}\mathbf{D}_2^{-\frac{1}{2}}$	`diag-n;svd-n`	Unidirectional norm. adjacency m.		?; ?
?;?	$\vec{\mathbf{P}} = \mathbf{D}_1^{-1}\vec{\mathbf{A}}$	`stoch1;?`	Unidirectional stoch. adjacency m.	Right-stoch[1]	?
5.4.12;?	$\vec{\mathbf{L}} = \mathbf{D}_1 - \vec{\mathbf{A}}$	`lapdiag2;?`	Unidirectional Laplacian		?

$*$ These decompositions can be derived by a transformation of eigenvalues of the given decomposition, and thus no internal name is used, and the decompositions are not explicitly computed in KONECT.

[1] Only valid for unsigned graphs

When two decomposition names are given, the matrix is non-normal and the first represents eigenvectors and -values and the second is the singular value decomposition.

Table 8: The basic nine matrices of KONECT.

	Plain	Normalized	Stochastic
Adjacency matrix	**A**	**N**	**P**
Laplacian	**L**	**Z**	**S**
Signless Laplacian	**K**	**C**	**J**

v are not connected.

The eigenvalue decomposition of the matrix $\mathbf{A}$ for undirected graphs is widely used to analyse graphs:

$$\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}} \qquad \texttt{sym}$$

$\mathbf{\Lambda}$ is an $n \times n$ real diagonal matrix containing the eigenvalues of $\mathbf{A}$, i.e., $\mathbf{\Lambda}_{ii} = \lambda_i[\mathbf{A}]$. $\mathbf{U}$ is an $n \times n$ orthogonal matrix having the corresponding eigenvectors as columns.

The eigenvalues of $\mathbf{A}$ are widely used to characterize graphs, and often, the expression *eigenvalue of a graph* is used without further qualification to refer to the eigenvalues of $\mathbf{A}$. An example is given in (Lovász 2007).

The largest absolute eigenvalue of $\mathbf{A}$ is the networks spectral norm α, i.e.,

$$\alpha = \max_i |\mathbf{\Lambda}_{ii}| = \|\mathbf{A}\|_2 .$$

The sum of all eigenvalues λ_i equal the trace of $\mathbf{A}$, i.e., the sum of its diagonal elements. The sum of the eigenvalues of $\mathbf{A}$ thus equals the number of loops in the graphs. In particular, when a graph has no loops, then the sum of the eigenvalues of its adjacency matrix is zero.

Higher moments the eigenvalues of $\mathbf{A}$ give the number of tours in the graph. Remember that a tour of length k is defined as a sequence of k connected nodes, such that the first and the last node are connected, such that two tours are considered as distinct when they have a different starting node or orientation. The sum of k^{th} powers of the eigenvalues of $\mathbf{A}$ then equals the number of k-tours T_k. We thus have in a loopless graph, that the traces of powers of $\mathbf{A}$ are related to the number of edges m, the number of triangles t, the number of squares q and the number of wedges s by:

$$\begin{aligned}
\mathrm{Tr}(\mathbf{A}) &= 0 \\
\mathrm{Tr}(\mathbf{A}^2) &= 2m \\
\mathrm{Tr}(\mathbf{A}^3) &= 6t \\
\mathrm{Tr}(\mathbf{A}^4) &= 8q + 4s + 2m
\end{aligned}$$

The traces of $\mathbf{A}$ can also be expressed as sums of powers (moments) of the eigenvalues of $\mathbf{A}$:

$$\mathrm{Tr}(\mathbf{A}^k) = \sum_{i=1}^{n} \lambda_i^k$$

The spectrum of $\mathbf{A}$ can also be characterized in terms of graph bipartivity. When the graph is bipartite, then all eigenvalues come in pairs $\{\pm\lambda\}$, i.e., they are distributed around zero symmetrically. When the graph is not bipartite, then their distribution is not symmetric. It follows that when the graph is bipartite, the smallest and largest eigenvalues have the same absolute value.

The sum of the absolute values of $\mathbf{A}$ is called the graph energy (Gutman 2001).

5.1.2 Normalized Adjacency Matrix (N)

The normalized adjacency matrix $\mathbf{N}$ of an undirected graph is defined as

$$\mathbf{N} = \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}, \qquad \texttt{sym-n}$$

where we remind the reader that the diagonal matrix $\mathbf{D}$ contains the node degrees, i.e., $\mathbf{D}_{uu} = d(u)$. The matrix $\mathbf{N}$ is symmetric and its eigenvalue decomposition can be considered:

$$\mathbf{N} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}} \tag{40}$$

The eigenvalues λ_i of $\mathbf{N}$ can be used to characterize the graph, in analogy with those of the nonnormalized adjacency matrix. The spectrum of $\mathbf{N}$ is also called the weighted spectral distribution (Fay 2010). All eigenvalues of $\mathbf{N}$ are contained in the range $[-1, +1]$. When the graph is unsigned, the largest eigenvalue is one. In addition, the eigenvalue one has multiplicity one if the graph is connected and unsigned. It follows that for general unsigned graphs, the multiplicity of the eigenvalue one equals the number of connected components of the graph.

Minus one is the smallest eigenvalue of $\mathbf{N}$ if and only iff the graph is bipartite. As with the nonnormalized adjacency matrix, the eigenvalues of $\mathbf{N}$ are distributed symmetrically around zero if and only if the graph is bipartite.

When the graph is connected, the eigenvector corresponding to eigenvalue one has entries proportional to the square root of node degrees, i.e.,

$$\mathbf{U}_{u1} = \sqrt{\frac{d(u)}{2m}}. \tag{41}$$

Note that this equivalence only holds for undirected graphs. For directed graphs, there is no such equivalence.

5.1.3 Stochastic Adjacency Matrix (P)

The consideration of random walks on a graph leads to the definition of the stochastic adjacency matrix $\mathbf{P}$. Imagine a random walker on the nodes of a graph, who can walk from node to node by following edges. If, at each edge, the probability that the random walker will go to each neighboring node with equal probability, then the random walk can be described be the transition probability matrix defined as

$$\mathbf{P} = \mathbf{D}^{-1}\mathbf{A}. \qquad \texttt{stoch1}$$

This matrix is called the stochastic adjacency matrix. It is asymmetric, even when the graph is undirected, except when the graph is regular, i.e., when all degrees are the same. Thus, its eigenvalue decomposition is not always defined, and in any case may not involve orthogonal matrices.

The sum of rows of $\mathbf{P}$ sum to one, making this matrix right-stochastic. For directed graphs we may distinguish the right-stochastic (or row-stochastic) matrix $\mathbf{P} = \mathbf{D}^{-1}\mathbf{A}$ and the left-stochastic (or column-stochastic) matrix $\mathbf{A}\mathbf{D}^{-1}$. Note the subtle terminology here: $\mathbf{D}^{-1}\mathbf{A}$ is left-normalized but right-stochastic.

The matrix $\mathbf{P}$ is related to the normalized adjacency matrix $\mathbf{N}$ by

$$\mathbf{P} = \mathbf{D}^{-\frac{1}{2}}\mathbf{N}\mathbf{D}^{\frac{1}{2}} \tag{42}$$

and therefore both matrices have the same set of eigenvalues. Thus, the eigenvalues of $\mathbf{P}$ are all real, even though $\mathbf{P}$ is asymmetric, and they are contained in the range $[-1, +1]$. Also, the relationship between $\mathbf{P}$ and $\mathbf{N}$ implies that that eigenvectors of $\mathbf{P}$ are related to those of $\mathbf{N}$ by factors of the diagonal elements of $\mathbf{D}^{\frac{1}{2}}$, i.e., the square roots of node degrees. Since $\mathbf{P}$ is asymmetric, its left eigenvectors differ from its right eigenvectors. When the graph is undirected, the left eigenvector corresponding to the eigenvalue one has entries proportional to the degree of nodes, while the right eigenvector corresponding to the eigenvalue one is the constant vector. This is consistent with the fact that for a random walk on an undirected graph, the stationary distribution of nodes is proportional to the node degrees.

The alternative matrix $\mathbf{A}\mathbf{D}^{-1}$ can also be considered. It is left-stochastic, and can be derived by considering random walks that tranverse edges in a backward direction. `stoch2`

The matrix $\mathbf{P}$ is the state transition matrix of a random walk on the graph, and thus its largest eigenvector is one if the graph is (strongly) connected. The matrix $\mathbf{P}$ is also related to the PageRank matrix $\mathbf{G}$ ("Google matrix"), which equals

$$\mathbf{G} = (1 - \alpha)\mathbf{P} + \alpha\mathbf{1} \tag{43}$$

where $0 < \alpha < 1$ is a damping factor (the teleportation probability), and $\mathbf{1}$ is the matrix containing all ones. The left eigenvalues of the PageRank matrix give the PageRank values, and thus we see that (ignoring the teleportation term), the PageRank of nodes in an undirected network equals the degrees of the nodes.

The matrix $\mathbf{P}$ is also related to random walks with restarts on the graph, i.e., random walks that have a certain probability $0 < \alpha < 1$ to return to an initial node at each step, instead of taking an edge at random. For any two nodes u and v, the number

$$\left[\alpha(\mathbf{I} - (1 - \alpha)\mathbf{P}^{\mathrm{T}})^{-1}\right]_{uv} \tag{44}$$

gives the asymptotic probability that a random walk with restart starting at node u finds itself at node v.

The matrix $\mathbf{P}$ is further related to the mean first-passage time (MFPT) on the network (Zhang et al. 2011).

5.1.4 Laplacian Matrix (L)

The Laplacian matrix of an undirected graph is defined as

$$\mathbf{L} = \mathbf{D} - \mathbf{A}, \tag{lap}$$

i.e., the diagonal degree matrix from which we subtract the adjacency matrix. The Laplacian matrix is also called the Kirchhoff matrix. The Laplacian matrix is the discrete analogue of the Laplace operator ubiquitous in physics and other areas employing multivariate calculus. This operator is also called the Laplacian, and denoted Δ or ∇^2. In that context, the Laplacian matrix is also called the discrete Laplace operator. Due to this equivalence, the Laplacian matrix $\mathbf{L}$ has many uses in network analysis. The Laplacian matrix is also denoted with other letters, such as for instance $\mathbf{Q}$ as used by Mieghem (2011). It may also be called the admittance matrix.

We consider the eigenvalue decomposition of the Laplacian:

$$\mathbf{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}} \tag{45}$$

The Laplacian matrix of positive-semidefinite, i.e., all eigenvalues are nonnegative. When the graph is unsigned, the smallest eigenvalue is zero and its multiplicity equals the number of connected components in the graph.

The second-smallest eigenvalue is called the algebraic connectivity of the graph, and is denoted $a = \lambda_2[\mathbf{L}]$ (Fiedler 1973). If the graph is unconnected, that value is zero, i.e., an unconnected graph has an algebraic connectivity of zero.

When the graph is connected, the eigenvector corresponding to eigenvalue zero is a constant vector, i.e., a vector with all entries equal. The eigenvector corresponding the the second-smallest eigenvalue is called the Fiedler vector, and can be used to cluster nodes in the graph. Together with further eigenvectors, it can be used to draw graphs (see Kunegis et al. 2010).

The Laplacian matrix $\mathbf{L}$ can be expressed in terms of the signed incidence matrix $\mathbf{E}^{\pm}$:

$$\mathbf{L} = \mathbf{E}^{\pm}[\mathbf{E}^{\pm}]^{\mathrm{T}} \tag{46}$$

From this equality follows that $\mathbf{L}$ is positive-semidefinite. Note that while the matrix $\mathbf{E}^{\pm}$ is called the signed incidence matrix, it is not related to signed graphs. In fact, the equality only holds for unsigned, unweighted graphs.

The Laplacian matrix can be used to compute the *effective resistance* in an electrical network (Klein & Randić 1993). If a given graph is interpreted as an electrical network in which each edge is a resistor, then the graph as a whole acts as a resistor between any two nodes u and v. The value of the resistance between two nodes u and v can be expressed using the Moore–Penrose inverse $\mathbf{\Gamma} = \mathbf{L}^{+}$ of the Laplacian $\mathbf{L}$:

$$r(u, v) = \mathbf{\Gamma}_{uu} + \mathbf{\Gamma}_{vv} - \mathbf{\Gamma}_{uv} - \mathbf{\Gamma}_{vu} \tag{47}$$

$$= (x^u - x^v)^{\mathrm{T}} \mathbf{\Gamma} (x^u - x^v) \tag{48}$$

Here, x^u is a vector of size $|V|$ with $(x^u)_u = 1$ and $(x^u)_v = 0$ whenever $u \neq v$. The Moore–Penrose pseudoinverse can be expressed using the eigenvalue decomposition $\mathbf{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}}$ as

$$\mathbf{\Gamma} = \mathbf{L}^{+} = \mathbf{U}\mathbf{\Lambda}^{+}\mathbf{U}^{\mathrm{T}}, \tag{49}$$

in which the Moore–Penrose pseudodoinverse of the diagonal matrix $\mathbf{\Lambda}$ is also diagonal and given by $(\mathbf{\Lambda}^{+})_{ii} = (\mathbf{\Lambda}_{ii})^{-1}$ when $\mathbf{\Lambda}_{ii} \neq 0$, and $(\mathbf{L}^{+})_{ii} = 0$ otherwise.

5.1.5 Normalized Laplacian Matrix (Z)

The Laplacian matrix too, can be normalized. It turns out that the normalized Laplacian and the normalized adjacency matrix are tighly related to each other: They share the same set of eigenvectors, and their eigenvalues are reflections of each other.

The normalized Laplacian matrix of an undirected graph is defined as

$$\mathbf{Z} = \mathbf{D}^{-\frac{1}{2}}\mathbf{L}\mathbf{D}^{-\frac{1}{2}}. \tag{50}$$

As opposed to $\mathbf{A}$, $\mathbf{L}$ and $\mathbf{N}$, there is no standardized notation of the normalized Laplacian. The notation $\mathbf{Z}$ is specific to KONECT, and was chosen as the letter Z resembles a turned letter N, and the matrices represented by those letters share eigenvectors and have flipped eigenvalues. The matrix $\mathbf{Z}$ is also called the *random walk Laplacian*.

The normalized Laplacian is related to the normalized adjacency matrix by

$$\mathbf{Z} = \mathbf{I} - \mathbf{N} = \mathbf{I} - \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}, \tag{51}$$

as can be derived directly from their definitions. It follows that $\mathbf{Z}$ and $\mathbf{N}$ have the same set of eigenvectors, and that their eigenvalues are related by the transformation $1-\lambda$. Thus, the properties of $\mathbf{Z}$ can be derived from those of $\mathbf{N}$. For instance, all eigenvalues of $\mathbf{Z}$ are contained in the range $[0, 2]$, and the multiplicity of the eigenvalue zero equals the number of connected components (when the graph is unsigned). If the undirected graph is connected, the eigenvector of eigenvalue zero contains entries proportional to the square root of the node degrees.

In KONECT, the decomposition of the normalized Laplacian is not included, since it can be derived from that of the normalized adjacency matrix.

5.1.6 Stochastic Laplacian Matrix (S)

A further variant of the Laplacian exists, based on the stochastic adjacency matrix:

$$\mathbf{S} = \mathbf{I} - \mathbf{P} = \mathbf{I} - \mathbf{D}^{-1}\mathbf{A} = \mathbf{I} - \mathbf{D}^{-\frac{1}{2}}\mathbf{N}\mathbf{D}^{\frac{1}{2}} = \mathbf{D}^{-\frac{1}{2}}\mathbf{Z}\mathbf{D}^{\frac{1}{2}} = \mathbf{D}^{-1}\mathbf{L} \tag{52}$$

This matrix shares much properties with $\mathbf{P}$ and thus with $\mathbf{N}$ and $\mathbf{Z}$. The eigenvalues of $\mathbf{S}$ are contained in the interval $[0, 2]$. The eigenvalue zero has a multiplicity equal to the number of connected components of the graph, and when the graph is connected its corresponding right eigenvector is the constant vector, while its corresponding left eigenvector is proportional to the node degrees. For connected graphs, the largest eigenvalue of $\mathbf{S}$ is two if and only if the graph is bipartite. In the general case, the eigenvalue two has a multiplicity equal to the number of connected components that are bipartite.

5.1.7 Signless Laplacian (K)

The signless Laplacian of a graph is defined as the Laplacian of the corresponding graph in which all edges are interpreted as negative (Cvetković et al. 2007). It thus equals

$$\mathbf{K} = \mathbf{D} + \mathbf{A}. \qquad \texttt{lapq}$$

The signless Laplacian is also denoted $\mathbf{Q}$. It corresponds to the ordinary Laplacian $\mathbf{L}$ of the graph with inverted edge weights, i.e., $\mathbf{K}[G] = \mathbf{L}[-G]$.

This matrix is positive-semidefinite, and its smallest eigenvalue is zero if and only if at least one connected component of the graph is bipartite. Thus, $\mathbf{K}$ is used in measures of bipartivity (see e.g. Kunegis 2015). We call this smallest eigenvalue the algebraic non-bipartivity $\chi = \lambda_{\min}[\mathbf{K}]$.

In unsigned graphs, the multiplicity of the eigenvalue zero equals the number of connected components that are bipartite. In signed graphs, the multiplicity of the eigenvalue zero equals the number of connected components that are balanced after all their edges have been negated (i.e., switch between positive and negative).

In connected graphs (that are not necessarily bipartite), the smallest eigenvalue of $\mathbf{K}$ is a measure of the non-bipartivity of the graph.

The signless Laplacian exists in the following two normalized variants:

$$\mathbf{C} = \mathbf{I} + \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}} \tag{53}$$

$$\mathbf{J} = \mathbf{I} + \mathbf{D}^{-1}\mathbf{A} \tag{54}$$

5.1.8 Seidel Adjacency Matrix (W)

The Seidel adjacency matrix $\mathbf{W}$ is a symmetric $n \times n$ matrix given by

$$\mathbf{W}_{uv} = \begin{cases} 0 & \text{when } u = v \\ -1 & \text{when } u \neq v \text{ and } u \leftrightarrow v \\ +1 & \text{when } u \neq v \text{ and } u \not\leftrightarrow v \end{cases} \qquad \texttt{seidel}$$

The notation $\mathbf{W}$ is not standardized. This type of matrix originates in (van Lint & Seidel 1966); see (Seidel 1968) for another reference. This matrix is also called the $(-1, 1, 0)$-adjacency matrix. In that context, the ordinary adjacency matrix $\mathbf{A}$ may be called the $(0, 1)$-adjacency matrix. The matrix is also written as $\mathbf{A}^*$ (e.g. in Zhou 2008) and $\mathbf{S}$ (e.g. in Ghorbani 2013). Symmetric matrices such as $\mathbf{W}$ which have a zero diagonal and ± 1 off the diagonal are also studied outside of graph theory, see e.g. Szöllősi & Östergård (2018).

The matrix can also be expressed as

$$\mathbf{W} = \mathbf{1} - \mathbf{I} - 2\mathbf{A}, \tag{55}$$

where $\mathbf{1}$ is the matrix completely filled with ones, and $\mathbf{A}$ is the symmetric adjacency matrix of the graph.

The matrix is not sparse, but due to its simple definition, its eigenvalue decomposition can be computed using sparse matrix methods. In the general case, the matrix has both positive and negative eigenvalues. The largest absolute eigenvalue of $\mathbf{W}$ is a network statistic. The sum of the absolute values of all eigenvalues of $\mathbf{W}$ has been called the Seidel energy of the graph (Ghorbani 2013), in analogy with the sum of absolute eigenvalues of $\mathbf{A}$, called the energy.

The Seidel adjacency matrix has a particular significance for regular graphs (Zhou 2008). In particular, the number of eigenvalues of $\mathbf{A}$ and $\mathbf{W}$ whose eigenvector is not orthogonal to the all-ones vector has been considered. This number is one for regular graphs, and larger otherwise.

Relations between the spectrum of $\mathbf{W}$ and the spectrum of $\mathbf{A}$ are given by Zhou (2008).

5.2 Decompositions of Signed Graphs

In this subsection, we review matrices and decompositions that apply to signed graphs.

In KONECT, in cases where the matrices are generalizations of a matrix and decomposition for unsigned graphs, they share the internal name of the decomposition with the unsigned case. On matrix at least is however specific to signed graphs.

5.2.1 Signed Adjacency Matrix

In undirected signed graphs, the adjacency matrix contains the values ± 1. In general, it can be used in the same fashion as the adjacency matrix of an unsigned graph, with the property that when edge weights get multiplied with each other, the multipliciation follows the multiplication rules for signed multiplication, i.e.,

$$\begin{aligned} +1 \times +1 &= +1 \\ +1 \times -1 &= -1 \\ -1 \times +1 &= -1 \\ -1 \times -1 &= +1 \end{aligned}$$

As an example of this, powers of the adjacency matrix of a signed graph will not contain counts of paths, but counts of paths weighted by the sign of each path, i.e., denoting whether each path contains an even or odd number of edges. As another example, the trace of the cube of $\mathbf{A}$ will equal six times the number of balanced triangles minus the number of unbalanced triangles.

5.2.2 Signed Laplacian Matrix

In a signed graph, the Laplacian matrix is defined as in an unsigned graph as

$$\mathbf{L} = \mathbf{D} - \mathbf{A}.$$

Note that the values in $\mathbf{A}$ are signed, but those in $\mathbf{D}$ are not. Thus, the properties of $\mathbf{L}$ differ from the unsigned case. In particular, it is no longer true that the smallest eigenvalue of $\mathbf{L}$ is always zero. While $\mathbf{L}$ is still positive-semidefinite in the signed case, the smallest eigenvalue of $\mathbf{L}$ may be larger than zero, rendering $\mathbf{L}$ positive-definite. The smallest eigenvalue of $\mathbf{L}$ is signed graphs is called the algebraic conflict ξ. In connected signed graphs, it is zero if and only if the graph is balanced, i.e., when the nodes can be divided into two groups such that all positive edges connect nodes within the same group, and all negative edges connect nodes of different groups. Equivalently, ξ is larger than zero if and only if each connected component contains at least one cycle with an odd number of negative edges. In signed graphs consisting of multiple connected components, the smallest eigenvalue of $\mathbf{L}$ is zero if and only if at least one connected component is balanced.

If a graph G is balanced, then the eigenvalue decomposition of $\mathbf{L}[G]$ can be derived from that of underlying unsigned graph (Kunegis et al. 2010). If G is a balanced signed graph, and $|G|$ is underlying unsigned graph, then the eigenvalue decompositions of both graphs' Laplacian matrices can be considered:

$$\mathbf{L}[G] = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}}$$
$$\mathbf{L}[|G|] = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathrm{T}}$$

In that case, both Laplacian matrices have the same eigenvalues, and $\mathbf{U}$ and $\mathbf{V}$ differ only by negation of rows corresponding to nodes in one of the two clusters.

In signed graphs, the equivalent resistance can be defined by defining that an edge carrying a negative resistance value acts like the corresponding positive resistance in series with a component that negates potentials (Kunegis et al. 2008). The result gives a *signed resistance distance kernel* based on the signed Laplacian:

$$r(u, v) = \mathbf{\Gamma}_{uu} + \mathbf{\Gamma}_{vv} - \mathbf{\Gamma}_{uv} - \mathbf{\Gamma}_{vu} \tag{56}$$

As in the unsigned case, the matrix $\mathbf{\Gamma} = \mathbf{L}^{+}$ is the Moore-Penrose pseudoinverse of $\mathbf{L}$. See also Section 5.8.1 in the author's Phd thesis (Kunegis 2011). Note that if all connected components in the graph are unbalanced, then the Moore–Penrose pseudoinverse reduces to the ordinary matrix inverse, i.e., $\mathbf{L}^{+} = \mathbf{L}^{-1}$, due to the fact that $\mathbf{L}$ does not have eigenvalues of zero for such graphs.

5.2.3 Repulsive Matrices

The Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ as defined earlier is based on the the degree matrix $\mathbf{D}$, whose diagonal elements equal the sum of absolute weights of all edges incident to each node. Taking the absolute value in that definition is justified for many applications related to signed graphs, and

results in a positive-semidefinite matrix, whose smallest eigenvalue is zero if and only if there is at least one balanced component in the signed network. Alternatively, using the sum of edge weights instead results in a different matrix, which may have negative eigenvalues:

$$\hat{\mathbf{L}} = \hat{\mathbf{D}} - \mathbf{A}, \tag{57}$$

where the diagonal matrix $\hat{\mathbf{D}}$ is defined as

$$[\hat{\mathbf{D}}]_{uu} = \sum_{v \sim u} \mathbf{A}_{uv}.$$

$\hat{\mathbf{L}}$ has also been called simply the *signed Laplacian matrix*, but should not be confused with the matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ as defined in Section 5.2.2. The matrix $\hat{\mathbf{L}}$ appears in various contexts; see e.g. the examples given by Bronski & DeVille (2014). We call it the *repulsive Laplacian* due to its applications; the name is not standard. The matrix $\hat{\mathbf{D}}$ as defined above will be called the *repulsive degree matrix*.

The matrix $\hat{\mathbf{D}}$ may have zero entries on the diagonal even when no node has degree zero – this happens when a node has an equal number of positive and negative edges. As a result, $\hat{\mathbf{D}}$ must be considered non-invertible. Thus, variants of the repulsive Laplacian are defined using the ordinary degree matrix for normalization.

Other matrices that can be labeled *repulsive* are given in the following. When applied to ajacency matrices the hat may be regarded as a no-op. Thus, $\hat{\mathbf{A}} = \mathbf{A}$, $\hat{\mathbf{N}} = \mathbf{N}$, and $\hat{\mathbf{P}} = \mathbf{P}$. The repulsive Laplacian matrices are

$$\hat{\mathbf{L}} = \hat{\mathbf{D}} - \mathbf{A} \tag{58}$$

$$\hat{\mathbf{Z}} = \mathbf{D}^{-\frac{1}{2}}(\hat{\mathbf{D}} - \mathbf{A})\mathbf{D}^{-\frac{1}{2}} \tag{59}$$

$$\hat{\mathbf{S}} = \mathbf{D}^{-1}\hat{\mathbf{D}} - \mathbf{P} = \mathbf{D}^{-1}(\hat{\mathbf{D}} - \mathbf{A}) \tag{60}$$

$$\hat{\mathbf{K}} = \hat{\mathbf{D}} + \mathbf{A} \tag{61}$$

$$\hat{\mathbf{C}} = \mathbf{D}^{-\frac{1}{2}}(\hat{\mathbf{D}} + \mathbf{A})\mathbf{D}^{-\frac{1}{2}} \tag{62}$$

$$\hat{\mathbf{J}} = \mathbf{D}^{-1}\hat{\mathbf{D}} + \mathbf{P} = \mathbf{D}^{-1}(\hat{\mathbf{D}} + \mathbf{A}) \tag{63}$$

The use of the repulsive matrices is rare, and we have not seen uses of any beyond $\hat{\mathbf{L}}$ in the wild.

5.3 Decompositions of Bipartite Graphs

Bipartite graphs can be considered a subset of unipartite graphs, and thus the methods described previously apply to them. However, this treatment ignores the special structure, and instead, methods specific to bipartite graphs can be used. Indeed, bipartite graphs have adjacency matrices of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{0} \end{bmatrix}, \tag{64}$$

where $\mathbf{B}$ is called the biadjacency matrix of the graph. This form can be exploited to reduce the eigenvalue decomposition of $\mathbf{A}$ to the equivalent singular value decomposition of $\mathbf{B}$ (Mirzal &

Furukawa 2010). Given the singular value decomposition $\mathbf{B} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}$, the eigenvalue decomposition of $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} \bar{\mathbf{U}} & \bar{\mathbf{U}} \\ \bar{\mathbf{V}} & -\bar{\mathbf{V}} \end{bmatrix} \begin{bmatrix} +\mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & -\mathbf{\Sigma} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{U}} & \bar{\mathbf{U}} \\ \bar{\mathbf{V}} & -\bar{\mathbf{V}} \end{bmatrix}^{\mathrm{T}} \tag{65}$$

with $\bar{\mathbf{U}} = \mathbf{U}/\sqrt{2}$ and $\bar{\mathbf{V}} = \mathbf{V}/\sqrt{2}$. In this decomposition, each singular value σ corresponds to the eigenvalue pair $\{\pm\sigma\}$. Odd powers of $\mathbf{A}$ then have the form

$$\mathbf{A}^{2k+1} = \begin{bmatrix} \mathbf{0} & (\mathbf{B}\mathbf{B}^{\mathrm{T}})^k\mathbf{B} \\ (\mathbf{B}^{\mathrm{T}}\mathbf{B})^k\mathbf{B}^{\mathrm{T}} & \mathbf{0} \end{bmatrix},$$

where the alternating power $(\mathbf{B}\mathbf{B}^{\mathrm{T}})^k\mathbf{B}$ can be explained by the fact that in the bipartite network, a path will follow edges from one vertex set to the other in alternating directions, corresponding to the alternating transpositions of $\mathbf{B}$.

The same is true in the normalized case: The eigenvalue decomposition of the normalized adjacency matrix $\mathbf{N}$ can be computed using the singular value decomposition of the normalized biadjacency matrix $\mathbf{M}$.

However, this technique cannot be extended to other matrices such as the Laplacian $\mathbf{L}$, since that matrix contains nonzero entries in its "block diagonal" entries. I.e., there is no corresponding expression relating the decomposition of the bipartite Laplacian matrix $\mathbf{L} = [\mathbf{D}_1\ -\mathbf{B}; -\mathbf{B}^{\mathrm{T}}\ \mathbf{D}_2]$ and the decomposition of the biadjacency matrix $\mathbf{B}$.

5.4 Decompositions of Directed Graphs

In directed graphs, the adjacency matrix is itself asymmetric, and there is no special half-adjacency matrix. Since the adjacency matrix is symmetric, decompositions are more complex. For instance, the adjacency is not normal in the general case, and therefore there is no simply defined eigenvalue decomposition anymore.

For directed graphs, there is an ambiguity about the meaning of the matrix $\mathbf{A}$: It may refer to the (generally asymmetric) adjacency matrix, or to its symmetrized variants. As a general rule in KONECT, in contexts in which all graphs (directed and undirected) are considered, $\mathbf{A}$ is always symmetric. To avoid ambiguities in this section, we use $\vec{\mathbf{A}}$ for the (generally asymmetric) adjacency matrix, and $\bar{\mathbf{A}} = \vec{\mathbf{A}} + \mathbf{A}^{\mathrm{T}}$ for the symmetric variant. When a bare $\mathbf{A}$ is used, it is equal to $\vec{\mathbf{A}}$.

5.4.1 Singular Value Decomposition

The singular value decomposition is defined for any matrix, including those that are not symmetric, and even those that are not quadratic. Thus, it can be applied to the adjacency matrix of directed graphs.

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}} \qquad \texttt{svd}$$

The matrices $\mathbf{U}$ and $\mathbf{V}$ are each orthogonal, but they are not equal. They contain the left and right singular vectors as columns. The matrix $\mathbf{\Sigma}$ is diagonal, and contains the singular values, which are all nonnegative.

This decomposition corresponds to the eigenvalue decomposition of the directed graph's bipartite double cover. It also corresponds to HITS (Hyperlink-Induced Topic Search, Kleinberg 1999).

The largest singular value, i.e., Σ_{11}, equals the graphs operator 2-norm ν.

5.4.2 Normalized Adjacency Matrix

The adjacency matrix can be normalized for directed network, in the same way as for undirected networks.

$$\mathbf{N} = \mathbf{D}_1^{-\frac{1}{2}} \mathbf{A} \mathbf{D}_2^{-\frac{1}{2}} \tag{66}$$

Here, $\mathbf{D}_1^{-\frac{1}{2}}$ and $\mathbf{D}_2^{-\frac{1}{2}}$ are the diagonal matrices of out- and indegrees.

The normalized adjacency matrix $\mathbf{N}$ can be used in the singular value decomposition, too:

$$\mathbf{N} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}} \qquad \texttt{svd-n}$$

5.4.3 Eigenvectors and Eigenvalues

While the eigenvalue decomposition is not defined in the general case for an asymmetric matrix $\mathbf{A}$, its eigenvectors and eigenvalues are well-defined, if we distinguish between left and right eigenvectors. `diag`

Thus, we define the method `diag`, which is a decomposition in the KONECT sense, but not in the strict mathematical sense. $\mathbf{U}$ and $\mathbf{V}$ are then defined as the matrices containing the left and right eigenvectors of $\mathbf{A}$, and $\mathbf{\Lambda}$ is the diagonal matrix of corresponding eigenvalues.

Note that while left and right eigenvectors differ, their eigenvalues are identical.

The largest absolute eigenvalue $\pi = |\Lambda_{11}|$ is zero if and only if the directed graph is acyclic. We call it the cyclic eigenvalue in KONECT. In the general case, the eigenvalues of the asymmetric $\mathbf{A}$ may be complex; if they are, they come in complex conjugate pairs. (This is a consequence of the fact that the matrix $\mathbf{A}$ itself is real.)

5.4.4 Eigenvectors and Eigenvalues (Normalized)

This corresponds to finding the eigenvectors of the asymmetric matrix $\mathbf{N}$. The same comments as for the non-normalized case apply. `diag-n`

5.4.5 Stochastic Adjacency Matrix (P)

The matrix $\mathbf{P}$ can be defined in the directed case in the same way as for undirected graphs. Properties given in Section 5.1.3 remain valid.

5.4.6 Skew Adjacency Matrix (Y)

While an asymmetric matrix $\mathbf{A}$ may be transformed to $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$ to give a symmetric matrix, we can also use $\mathbf{Y} = \mathbf{A} - \mathbf{A}^{\mathrm{T}}$ to get a skew-symmetric matrix. A skew symmetric matrix $\mathbf{X}$ is a matrix such that $\mathbf{X} = -\mathbf{X}^{\mathrm{T}}$.

One motivation for considering the matrix $\mathbf{A} - \mathbf{A}^{\mathrm{T}}$ is simply that this matrix is skew-symmetric, which makes it a normal matrix, and thus the eigenvalue decomposition is well-defined. Beyond this, there is also an argument for why $\mathbf{A} - \mathbf{A}^{\mathrm{T}}$ in particular may be used – if it was only for deriving a normal matrix from $\mathbf{A}$, then after all $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$ would be perfectly adequate: When $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$ is used, the underlying assumption is that the direction of edges is unimportant and can be ignored. This is usually a good assumption in directed networks in which edges are very often reciprocated anyway, i.e., in highly symmetric graphs. Examples of such graphs are communication networks such as

email networks. In other directed networks however, the semantics of edge directions are different. For instance, a sports result network in which nodes are teams and a directed edge denotes that a team has won against another team: In such networks, we may interpret "A has won against B" as equivalent to "B has lost against A". Thus, we can consider such a network as a signed directed network, making $\mathbf{A} - \mathbf{A}^{\mathrm{T}}$ the natural way to *normalize* the network. This has the advantage of taking into account who has won each game, as opposed to just considering the "who has played whom" network that is implicitly given by the symmetrical $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$. This explains that the matrix $\mathbf{Y}$ is used in situations such as sports results which are inherently skew symmetric, i.e., where an edge in one direction is equivalent to a negated edge in the other direction. Another example of such networks are dominance networks between animals, for instance cattle (MA). Such networks are indicated by the tag `#skew`, as defined in Section 2.7.

Skew symmetric matrices have well-defined eigenvalue decompositions, which are however complex, as both the eigenvectors and eigenvalues will be complex numbers. The eigenvectors and eigenvalues however follow a specific pattern, that we can exploit to represent such a decomposition using only real numbers:

$$\mathbf{Y} = \mathbf{A} - \mathbf{A}^{\mathrm{T}} = \mathbf{Q}\mathbf{R}\mathbf{Q}^{\mathrm{T}} \tag{67}$$

such that

$$\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{U} + i\mathbf{V} \\ \mathbf{U} - i\mathbf{V} \end{bmatrix} \tag{68}$$

$$\mathbf{R} = \begin{bmatrix} i\mathbf{D} & \mathbf{0} \\ \mathbf{0} & -i\mathbf{D} \end{bmatrix} \tag{69}$$

where $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{D}$ are real matrices. In fact, this decomposition can be equivalently written as

$$\mathbf{Y} = \mathbf{A} - \mathbf{A}^{\mathrm{T}} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}} - \mathbf{V}\mathbf{D}\mathbf{U}^{\mathrm{T}}. \qquad \texttt{skew}$$

This decomposition is equivalent to that given by Constantine & Gower (1978). It shows that the skew-symmetric matrix $\mathbf{Y}$ has eigenvalues that are purely imaginary, come in pairs $\{\pm i\lambda\}$ that are the negative of each other (or equivalently, the complex conjugate), and their corresponding eigenvectors are also complex conjugates of each other.

Note also that the number of columns of both $\mathbf{U}$ and $\mathbf{V}$ is at most $\lfloor n \rfloor$, and thus, if n is odd, the skew-symmetric matrix $\mathbf{Y}$ has the eigenvalue zero, which is the complex conjugate of itself. Also, the expression $\mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}$ is *not* the singular value decomposition of $\mathbf{A}$, even if the form of the decomposition is the same. In particular, the matrix $\mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}$ has at most rank $\lfloor n/2 \rfloor$, while $\mathbf{A}$ itself may have rank up to n.

In some cases, we may be interested in the matrix

$$i\mathbf{Y} = i(\mathbf{A} - \mathbf{A}^{\mathrm{T}}), \qquad \texttt{skewi}$$

which has purely real eigenvalues distributed symmetrically around zero.

5.4.7 Normalized Skew Adjacency Matrix

This corresponds to the matrix

$$\mathbf{N} - \mathbf{N}^{\mathrm{T}} = \mathbf{D}_1^{-\frac{1}{2}}\mathbf{A}\mathbf{D}_2^{-\frac{1}{2}} - \mathbf{D}_2^{-\frac{1}{2}}\mathbf{A}^{\mathrm{T}}\mathbf{D}_1^{-\frac{1}{2}} \qquad \texttt{skewn}$$

5.4.8 Hermitian Adjacency Matrix (H)

In certain contexts, for instance when constructing the Hamiltonian of a system, it is necessary to specify Hermitian matrices, i.e., diagonalizable matrices that have only real eigenvalues. To take into account all connectivity information of a directed graph in a directed graph, the symmetric and skew-symmetric adjacency matrices can be combined in the following way:

$$\mathbf{H} = \frac{1}{\sqrt{2}}\left[\mathbf{A} + \mathbf{A}^{\mathrm{T}} + i(\mathbf{A} - \mathbf{A}^{\mathrm{T}})\right]. \qquad \texttt{herm}$$

The matrix $\mathbf{H}$ is Hermitian, i.e., it equals the complex conjugate of its transpose, $\mathbf{H} = \bar{\mathbf{H}}^{\mathrm{T}} = \mathbf{H}^{\dagger}$. Furthermore, the Hermitian adjacency matrix can be defined by

$$\mathbf{H}_{uv} = \exp\left\{i\frac{\pi}{4}\mathbf{Y}_{uv}\right\}, \tag{70}$$

which additionally justifies the factor $1/\sqrt{2}$ in the initial definition of $\mathbf{H}$, as $e^{i\frac{\pi}{4}} = 1/\sqrt{2} + i/\sqrt{2}$. Using this form, the Hermitian adjacency matrix can be generalized to the matrix $\mathbf{H}_\theta$ where $0 \leq \theta \leq \pi/2$ is a real parameter, giving

$$[\mathbf{H}_\theta]_{uv} = \exp\left\{i\theta\mathbf{Y}_{uv}\right\}, \tag{71}$$

from which the following special cases can be recovered:

$$\mathbf{H}_0 = \mathbf{A} + \mathbf{A}^{\mathrm{T}}, \tag{72}$$
$$\mathbf{H}_{\pi/4} = \mathbf{H}, \tag{73}$$
$$\mathbf{H}_{\pi/2} = i\mathbf{Y}. \tag{74}$$

These matrices appear in the modeling of quantum walks, as used for instance by Tödtli et al. (2016).

5.4.9 Normalized Hermitian Matrix

The normalized Hermitian matrix is

$$\frac{1}{\sqrt{2}}\mathbf{D}^{-\frac{1}{2}}\mathbf{H}\mathbf{D}^{-\frac{1}{2}}. \qquad \texttt{hermn}$$

(We currently do not have a symbol for this matrix.)

5.4.10 Directed Laplacian (Chung)

This variant of a Laplacian matrix for directed graphs is given by Chung (2005). It is itself symmetric, but takes into account edge directions. `lapd`

A normalized variant is also possible. `lapd-n`

5.4.11 Magnetic Laplacian (T)

The matrix

$$\mathbf{T}_\theta = \mathbf{D} - \mathbf{H}_\theta \tag{75}$$

is the magnetic Laplacian (Fanuel et al. 2016). It is Hermitian and positive-semidefinite.

For the case $\theta = 0$, this reduces to the ordinary undirected Laplacian, effectively ignoring the edge directions.

$$\mathbf{T}_0 = \mathbf{D} - \mathbf{H}_0 = \mathbf{D} - (\mathbf{A} + \mathbf{A}^{\mathrm{T}}) = \mathbf{L} \tag{76}$$

In the case $\theta \neq 0$, the magnetic Laplacian T_θ is distinct from the ordinary Laplacian, even if the directed graph is symmetric, i.e., when all edges are reciprocated.

For the case $\theta = \pi/4$, this can be called the Hermitian Laplacian, and is simply denoted $\mathbf{T}$, in the same way that $\mathbf{H}_{\pi/4}$ is denoted $\mathbf{H}$:

$$\mathbf{T} = \mathbf{T}_{\pi/4} = \mathbf{D} - \mathbf{H} \qquad \texttt{lapherm}$$

Note that we define a bare $\mathbf{T}$ to be $\mathbf{T}_{\pi/4}$ rather than $\mathbf{T}_0$ – this is the reason why use the distinct symbol $\mathbf{T}$, as otherwise we could just use $\mathbf{L}_\theta$ and define $\mathbf{L}_0 = \mathbf{L}$.

For $\theta = \pi/2$ this is called the skew Laplacian:

$$\mathbf{T}_{\pi/2} = \mathbf{D} - i(\mathbf{A} - \mathbf{A}^{\mathrm{T}}) = \mathbf{D} - i\mathbf{Y} \qquad \texttt{lapskew}$$

For any value of θ, the matrix $\mathbf{T}_\theta$ is Hermitian, and therefore has real spectrum. As is the ordinary Laplacian matrix, the magnetic Laplacian matrix is positive-semidefinite, i.e., all eigenvalues are nonnegative, for all values of θ.

5.4.12 Unidirectional Laplacian

This is the generally non-normal matrix

$$\vec{\mathbf{L}} = \mathbf{D}_1 - \vec{\mathbf{A}}, \qquad \texttt{lapdiag2}$$

where $\mathbf{D}_1$ is the diagonal outdegree matrix, and $\vec{\mathbf{A}}$ is the generally asymmetric adjacency matrix of the given directed graph.

This matrix is in general non-normal and as such its eigenvalue decomposition cannot be considered. However, the eigenvalues and eigenvectors are well-defined. The eigenvalues have nonnegative real part, and are either real, or come in complex conjugate pairs.

5.4.13 Decomposition into Directed Components

DEDICOM (decomposition into directed components) refers to a class of matrix decompositions for directed networks of the form

$$\mathbf{A} = \mathbf{U}\mathbf{X}\mathbf{U}^{\mathrm{T}}, \tag{77}$$

in which $\mathbf{U}$ is orthogonal and $\mathbf{X}$ is asymmetric, and thus non-diagonal. These represent a family of decompositions for which there is no single best one, and in particular the problem of finding the best rank-k such decompositions is not given by the truncation to rank k of the solution to the full-rank problem. An overview of algorithms is given in (Kunegis & Fliege 2012).

5.4.14 Impractical Decompositions

The decompositions presented in this section apply to the (generally) asymmetric adjacency matrix $\mathbf{A}$ of directed graphs. They all have practical problems that forbids their application to almost all networks in KONECT: Some are fully dense in the sense that they need $O(n^2)$ memory to be represented. Others are numerically unstable. Thus, these decomposition can only be used with the smallest of networks. As a result, they are not of practical interest in KONECT, but rather of theoretical importance.

As a general rule, implementations of these decompositions only exist with full rank, and no low-rank approximation is possible, as is the case with the eigenvalue and singular value decompositions.

A useful overview of rare and esoteric matrix decompositions and decomposition-like methods for generally asymmetric matrices is given by Moler & Van Loan (2003) in the context of computing the matrix exponential.

The **Schur decomposition** reduces a matrix to cycle-free form `schur`

$$\mathbf{A} = \mathbf{U}\mathbf{R}\mathbf{U}^{\mathrm{T}}, \tag{78}$$

with $\mathbf{U}$ orthogonal and $\mathbf{R}$ triangular. This is the preferred method used in (Moler & Van Loan 2003). The Octave/Matlab function `schur()` only takes dense a matrix and computes a full decomposition.

The **orthogonal Hessenberg decomposition** (Blum et al. 2004) is defined as

$$\mathbf{A} = \mathbf{U}\mathbf{H}\mathbf{U}^{\mathrm{T}}, \tag{79}$$

where $\mathbf{U}$ is orthogonal and $\mathbf{H}$ is an upper Hessenberg matrix, i.e., an upper triangular matrix in which additionally the subdiagonal contains non-zeroes. This corresponds to the function `OTHES` in EISPACK, and to `hess()` in Octave/Matlab. It is only implemented for dense matrices.

The **Jordan decomposition** is defined as

$$\mathbf{U}\mathbf{F}\mathbf{U}^{-1}. \tag{80}$$

This decomposition is not stable, i.e., it is not robust against multiple eigenvalues. An implementation exists in Matlab only in the symbolic toolbox; due to its instability, an implementation for floating point values would not work.

The **companion decomposition** or Frobenius normal form (Moler & Van Loan 2003, p. 19) is given by

$$\mathbf{A} = \mathbf{Y}\mathbf{C}\mathbf{Y}^{\mathrm{T}}, \tag{81}$$

in which $\mathbf{C}$ is a companion matrix.

6 Plots

Plots are drawn to visualize a certain aspect of a dataset. These plots can be used to compare several network visually, or to illustrate the definition of a certain numerical statistic.

As a running example, we show the plots for the Wikipedia elections network (EL), or for larger networks in some cases. Plots for all networks (in which computation was feasible) are shown on the KONECT website[17]. The KONECT Toolbox contains Matlab code for generating these plot types.

[17] konect.cc/plots

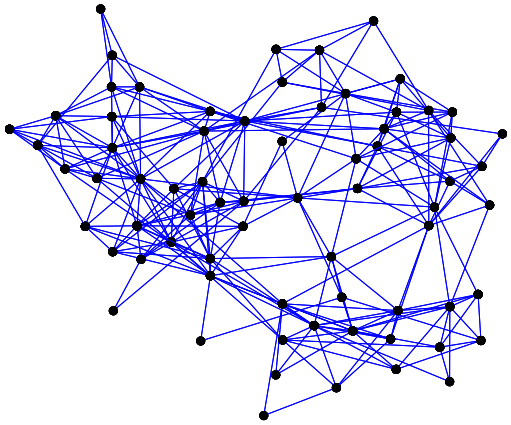


Figure 4: The layout of the highschool social network given by Coleman (1964) (MH), generated using the algorithm of Fruchterman & Reingold (1991). This network has 70 vertices, and for such small networks, drawing a graph leads to sensible plot. For large graphs, graph drawing usually lead to "hairball"-like plots and are thus less useful.

6.1 Layout

Layout plots show the nodes and edges of a graph in a way that makes features if the graph visible. Usually, this only makes sense for small graphs.[18] In KONECT, we use the algorithm of Fruchterman & Reingold (1991). An example is shown in Figure 4.

6.2 Temporal Distribution

The temporal distributions shows the distribution of edge creation times. It is only defined for networks with known edge creation times. The X axis is the time, and the Y axis is the number of edges added during each time interval. We compute two plots: the distribution, and the cumulative distribution. Examples are shown for the BItcoin OTC network (BO) in Figure 5.

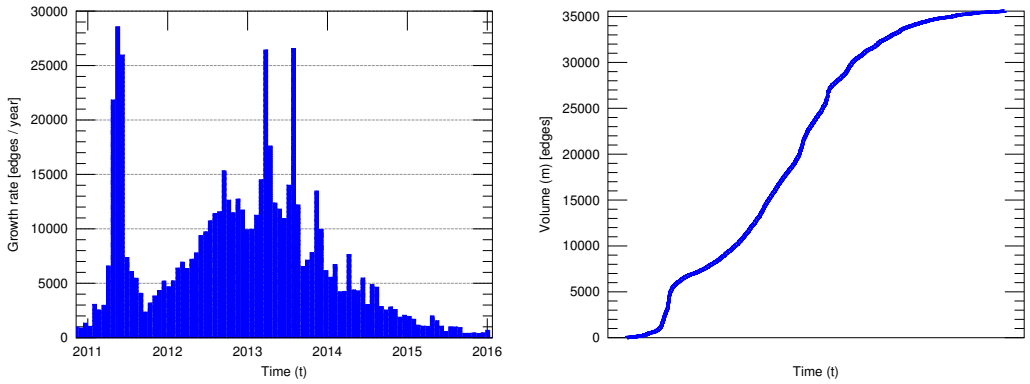
6.3 Edge Weight and Multiplicity Distribution

The edge weight and multiplicity distribution plots show the distribution of edge weights and of edge multiplicities, respectively. They are not generated for unweighted networks. The X axis shows values of the edge weights or multiplicities, and the Y axis shows frequencies. Edge multiplicity distributions are plotted on doubly logarithmic scales.

6.4 Degree Distribution

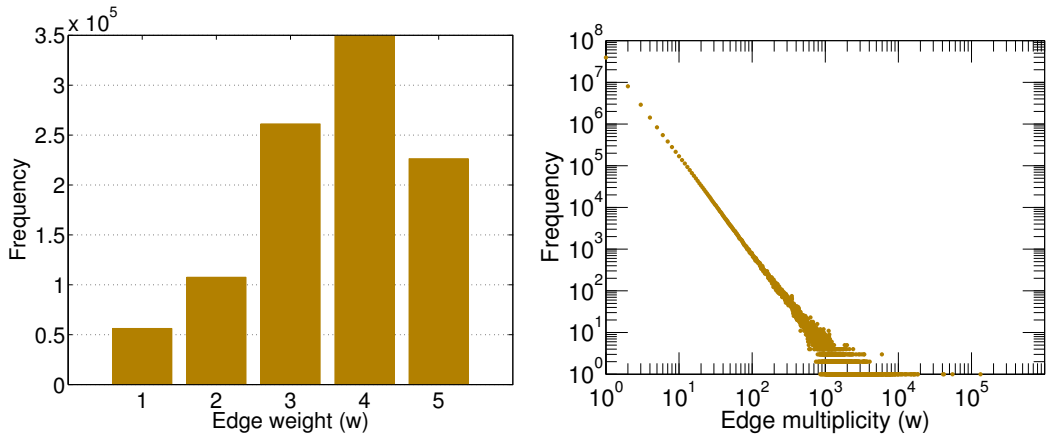
The distribution of degree values $d(u)$ over all vertices u characterizes the network as a whole, and is often used to visualize a network. In particular, a power law is often assumed, stating that the number of nodes with n neighbors is proportional to $n^{-\gamma}$, for a constant γ (Barabási & Albert

[18] See networkscience.wordpress.com/2016/06/22/no-hairball-the-graph-drawing-experiment for an explanation.



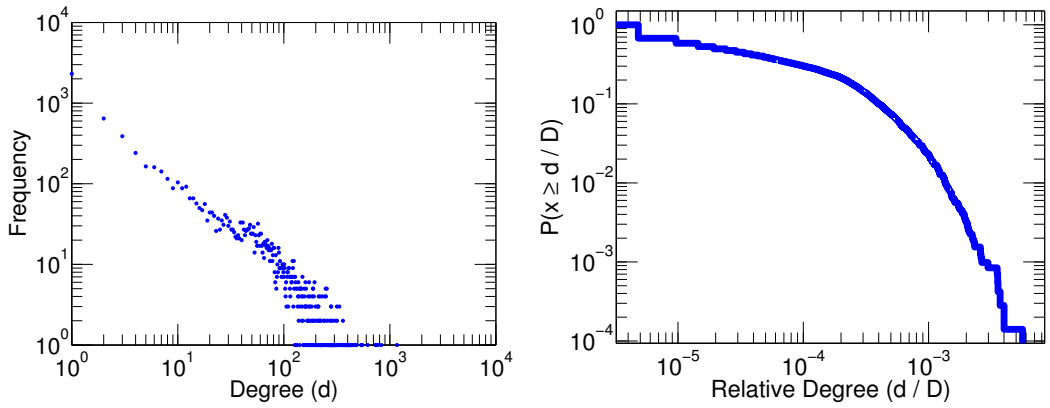
(a) Temporal distribution (b) Cumulative temporal distribution

Figure 5: The temporal distribution and cumulative temporal distribution of the Bitcoin OTC network (BO).



(a) Edge weight distribution (b) Edge multiplicity distribution

Figure 6: The distribution of (a) edge weights for the MovieLens rating network (M2) and (b) edge multiplicities for the German Wikipedia edit network (de).



(a) Degree distribution

(b) Cumulative degree distribution

Figure 7: The degree distribution and cumulative degree distribution for the Wikipedia election network (EL).

1999). This assumption can be inspected visually by plotting the degree distribution on a doubly logarithmic scale, on which a power law renders as a straight line. KONECT supports two different plots: The degree distribution, and the cumulative degree distribution. The degree distribution shows the number of nodes with degree n, in function of n. The cumulative degree distribution shows the probability that the degree of a node picked at random is larger than n, in function of n. Both plots use a doubly logarithmic scale.

Another visualization of the degree distribution supported by KONECT is in the form of the Lorenz curve, a type of plot to measure inequality originally used in economics (not shown).

The Lorenz curve is a tool originally from economics that visualizes statements of the form "X% of nodes with smallest degree account for Y% of edges". The set of values (X, Y) thus defined is the Lorenz curve. In a network the Lorenz curve is a straight diagonal line when all nodes have the same degree, and curved otherwise (Kunegis & Preusse 2012). The area between the Lorenz curve and the diagonal is half the Gini coefficient (see above).

6.5 Out/indegree Comparison

The out/indegree comparison plots show the joint distribution of outdegrees and indegrees of all nodes of directed graphs. The plot shows, for one directed network, each node as a point, which the outdegree on the X axis and the indegree on the Y axis.

An example is shown in Figure 9 for the Wikipedia elections network.

6.6 Assortativity Plot

In some networks, nodes with high degree are more often connected with other nodes of high degree, while nodes of low degree are more often connected with other nodes of low degree. This property is called assortativity, i.e., such networks are said to be assortativity. On the other hand, some networks, are dissortative, i.e., in them nodes of high degree are more often connected to nodes

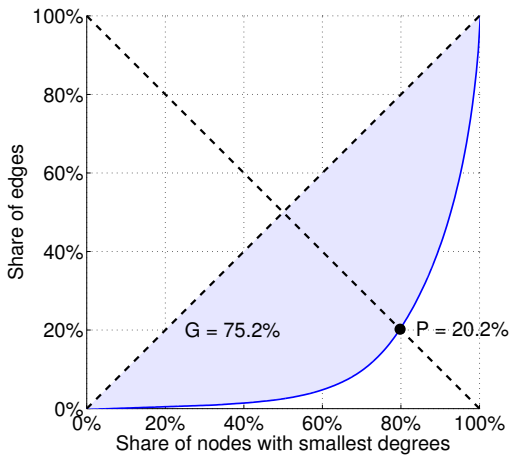


Figure 8: The Lorenz curve for the Wikipedia election network (EL).

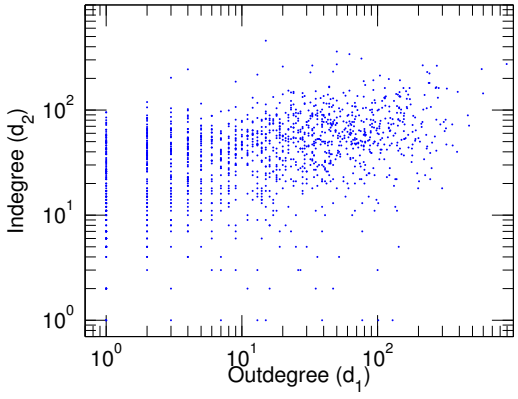


Figure 9: The out/indegree comparison plot of the Wikipedia election network (EL).

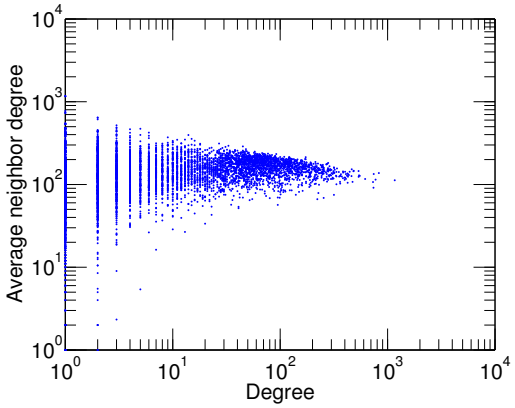


Figure 10: The assortativity plot of the Wikipedia election network (EL).

of low degree and vice versa. In addition to the assortativity ρ defined as the Pearson correlation coefficient between the degrees of connected nodes, the assortativity or dissortativity of networks may be analyse by plotting all nodes of a network by their degree and the average degree of their neighbors. Thus, the assortativity plot of a network shows all nodes of a network with the degree on the X axis, and the average degree of their neighbors on the Y axis.

An example of the assortativity plot is shown for the Wikipedia elections network in Figure 10.

6.7 Clustering Coefficient Distribution

In Section 4.7, we defined the clustering coefficient of a node in a graph as the propotion of that node's neighbors that are connected, and proceeded to define the clustering coefficient as the corresponding measure applied to the whole network. In some case however, we may be interested in the distribution of the clustering coefficient over the nodes in the network. For instance, a network could have some very clustered parts, and some less clustered parts, while another network could have many nodes with a similar, average clustering coefficient. Thus, we may want to consider the distribution of clustering coefficient. This distribution can be plotted as a cumulated plot.

6.8 Spectral Plot

The eigenvalues of a network's characteristic matrices $\mathbf{A}$, $\mathbf{N}$ and $\mathbf{L}$ are often used to characterize the network as a whole. KONECT supports computing and visualizing the spectrum (i.e., the set of eigenvalues) of a network in multiple ways. Two types of plots are supported: Those showing the top-k eigenvalues computed exactly, and those showing the overall distribution of eigenvalues, computed approximately. The eigenvalues of $\mathbf{A}$ are positive and negative reals, the eigenvalues of $\mathbf{N}$ are in the range $[-1, +1]$, and the eigenvalues of $\mathbf{L}$ are all nonnegative. For $\mathbf{A}$ and $\mathbf{N}$, the largest absolute eigenvalues are used, while for $\mathbf{L}$ the smallest eigenvalues are used. The number of eigenvalue shown k depends on the network, and is chosen by KONECT such as to result in reasonable runtimes for the decomposition algorithms.

Two plots are generated: the non-cumulative eigenvalue distribution, and the cumulative eigenvalue distribution. For the non-cumulative distribution, the absolute λ_i are shown in function of i

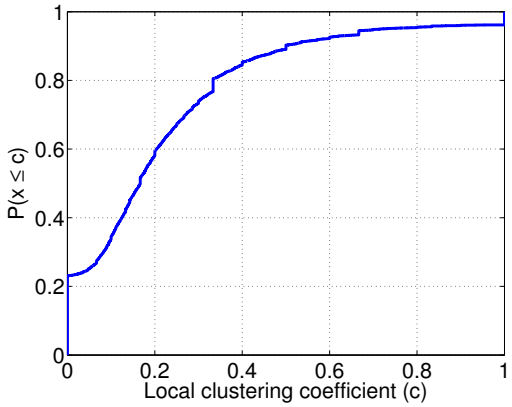
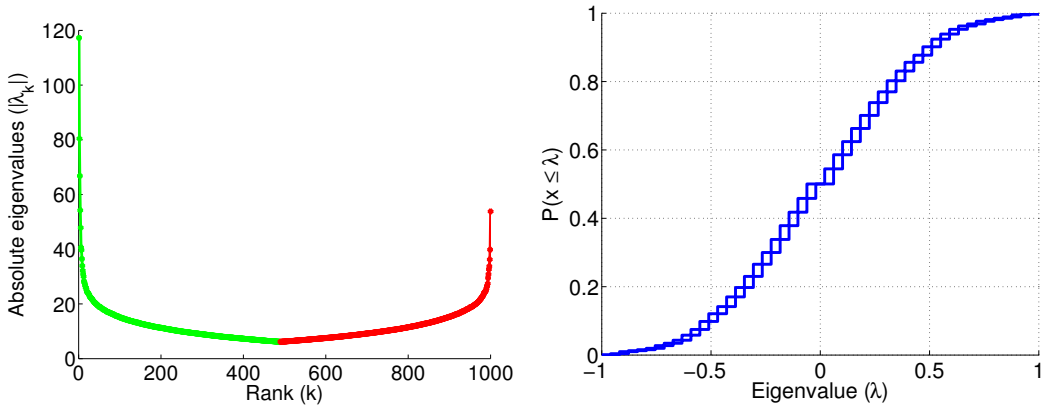


Figure 11: The clustering coefficient distribution for Facebook link network (OI).



(a) Top-k eigenvalues of $\mathbf{A}$ (b) Cumulative eigenvalue distribution of $\mathbf{N}$

Figure 12: The top-k eigenvalues of $\mathbf{A}$ and the cumulative spectral distribution of $\mathbf{N}$ for the Wikipedia election network (EL). In the first plot (a), positive eigenvalues are shown in green and negative ones in red.

for $1 \leq i \leq k$. The sign of eigenvalues (positive and negative) is shown by the color of the points (green and red). For the cumulated eigenvalue plots, the range of all eigenvalues is computed, divided into 49 bins (an odd number to avoid a bin limit at zero for the matrix $\mathbf{N}$), and then the number of eigenvalues in each bin is computed. The result is plotted as a cumulated distribution plot, with boxes indicating the uncertainty of the computation, due to the fact that eigenvalues are not computed exactly, but only in bins.

6.9 Complex Eigenvalues Plot

The adjacency matrix of an undirected graph is symmetric and therefore its eigenvalues are real. For directed graphs however, the adjacency matrix $\mathbf{A}$ is asymmetric, and in the general case its eigenvalues are complex. We thus plot, for directed graphs, the top-k complex eigenvalues by absolute value of the adjacency matrix $\mathbf{A}$.

Three properties can be read off the complex eigenvalues: whether a graph is nearly acyclic, whether a graph is nearly symmetric, and whether a graph is nearly bipartite. If a directed graph is acyclic, its adjacency matrix is nilpotent and therefore all its eigenvalues are zero. The complex eigenvalue plot can therefore serve as a test for networks that are nearly acyclic: the smaller the absolute value of the complex eigenvalues of a directed graph, the nearer it is to being acyclic. When a directed network is symmetric, i.e., all directed edges come in pairs connecting two nodes in opposite direction, then the adjacency matrix $\mathbf{A}$ is symmetric and therefore all its eigenvalues are complex. Thus, a nearly symmetric directed network has complex eigenvalues that are near the real line. Finally, the eigenvalues of a bipartite graph are symmetric around the imaginary axis. In other words, if $a + bi$ is an eigenvalue, then so is $-a + bi$ when the graph is bipartite. Thus, the amount of symmetric along the imaginary axis is an indicator for bipartivity. Note that bipartivity here takes into account edge directions: There must be two groups such that all (or most) directed edges go from the first group to second. Figure 13 shows two examples of such plots.

6.10 Distance Distribution Plot

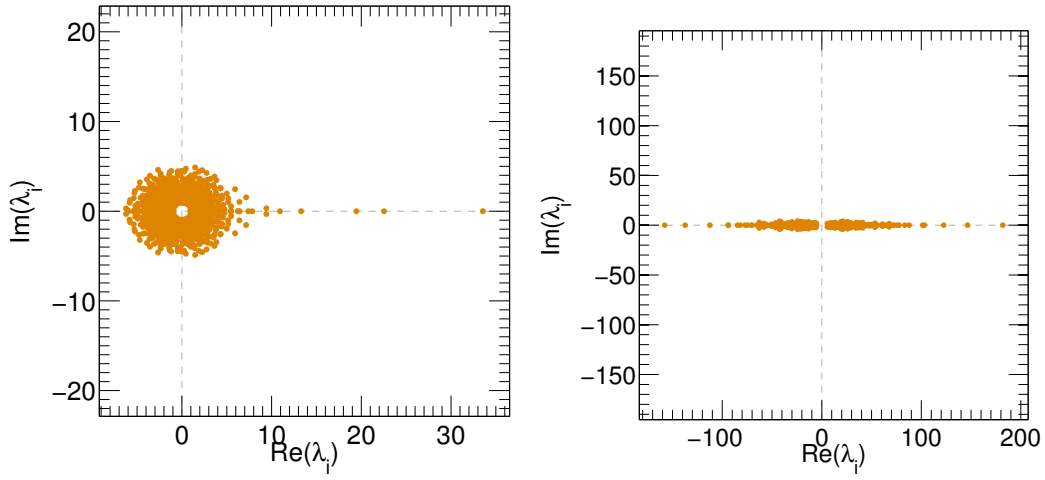
Distance statistics can be visualized in the distance distribution plot. The distance distribution plot shows, for each integer k, the number of node pairs at distance k from each other, divided by the total number of node pairs. The distance distribution plot is also called the *hop plot*. The distance distribution plot can be used to read off the diameter, the median path length, and the 90-percentile effective diameter (see Section 4.4). For temporal networks, the distance distribution plot can be shown over time.

The non-temporal distance distribution plot shows the cumulated distance distribution function between all node pairs (u, v) in the network, including pairs of the form (u, u), whose distance is zero.

The temporal distance distribution plot shows the same data in function of time, with time on the X axis, and each colored curve representing one distance value.

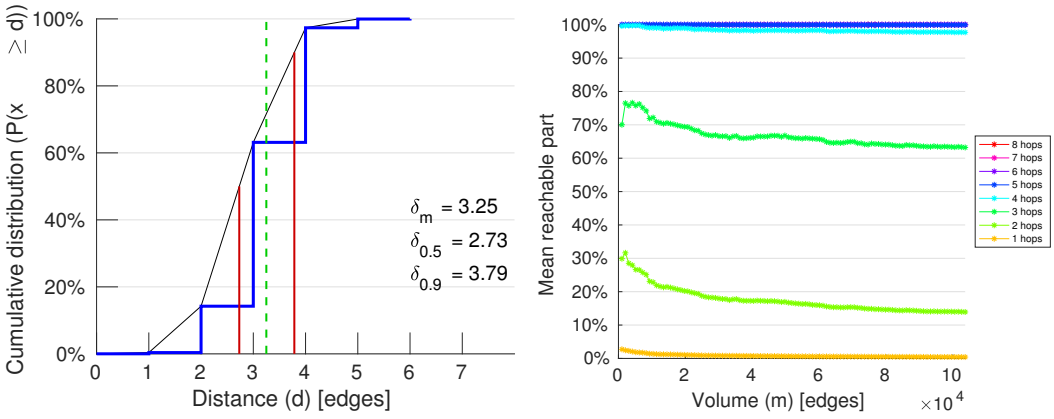
6.11 Graph Drawings

A graph drawing is a representation of a graph, showing its vertices and egdes laid out in two (or three) dimensions in order for the graph structure to become visible. Graph drawings are easy to produce when a graph is small, and become harder to generate and less useful when a graph is larger.



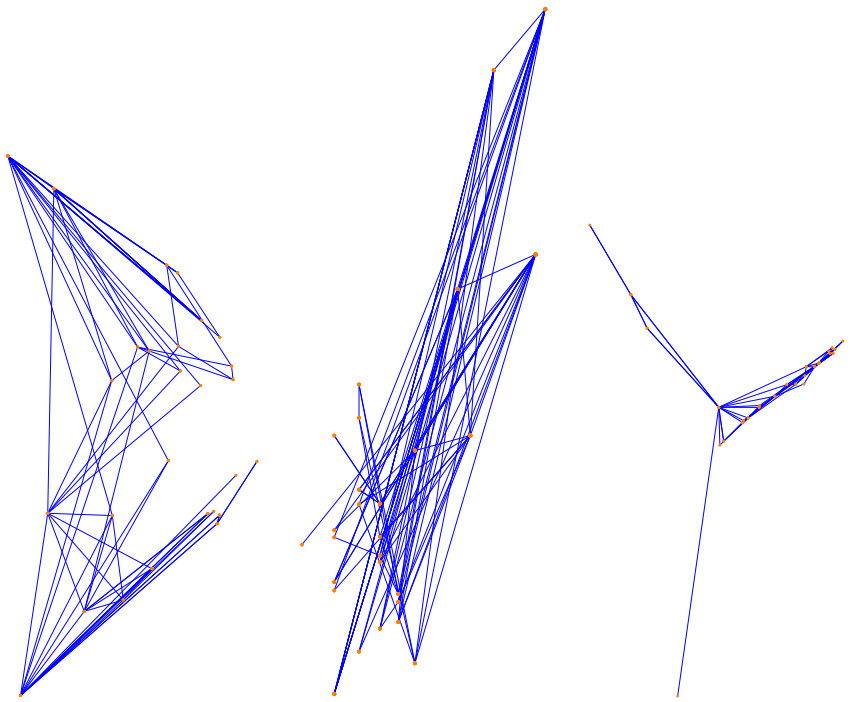
(a) Wikipedia elections (b) UC Irvine messages

Figure 13: The top-k complex eigenvalues λ_i of the asymmetric adjacency matrix $\mathbf{A}$ of the directed Wikipedia election (EL) and UC Irvine messages (UC) networks.



(a) Distance distribution plot (b) Temporal distance distribution plot

Figure 14: The distance distribution plot and temporal distance distribution plot of the Wikipedia election network (EL).



(a) Adjacency matrix $\mathbf{A}$ (b) Normalized adjacency matrix $\mathbf{N}$ (c) Laplacian $\mathbf{L}$

Figure 15: Drawings of the Zachary karate club social network (ZA) using (a) the adjacency matrix $\mathbf{A}$, (b) the normalized adjacency matrix $\mathbf{N}$, (c) the Laplacian matrix $\mathbf{L}$.

Given a graph, a graph drawing can be specified by the placement of its vertices in the plane. To determine such a placement is a non-trivial problem, for which many algortihms exist, depending on the required properties of the drawing. For instance, each vertex should be placed near to its neighbors, vertices should not be drawn to near to each other, and edges should, if possible, not cross each other. It is clear that it is impossible to fulfill all these requirements at once, and thus no best graph drawing exists.

In KONECT, we show drawings of small graphs only, such that vertices and edges remain visible. The graph drawings in KONECT are spectral graph drawings, i.e., they are based on the eigenvectors of characteric graph matrices. In particular, KONECT included graph drawings based on the adjacency matrix $\mathbf{A}$, the normalized adjacency matrix $\mathbf{N}$ and the Laplacian matrix $\mathbf{L}$ (Koren 2003). Let $\mathbf{x}$ and $\mathbf{y}$ be the two chosen eigenvector of each matrix, then the coordinate of the node $u \in V$ is given by $\mathbf{x}_u$ and $\mathbf{y}_u$.

For the adjacency matrix $\mathbf{A}$ and the normalized adjacency matrix $\mathbf{N}$, we use the two eigenvector with largest absolute eigevalue. For the Laplacian matrix $\mathbf{L}$, we use the two eigenvectors with smallest nonzero eigenvalue. Examples for the Zachary karate club social network (ZA) are shown in Figure 15.

7 Other Definitions

This section contains other definitions used in KONECT.

7.1 Node Features

A feature is a numerical characteristic of a node, such as the degree and the eccentricity. Features have multiple uses, such as to measure the centrality or the influence of a node in a network.

The degree is defined as the number of neighbors of a node. In directed networks, we can distinguish the indegree, the outdegree and the degree difference (indegree minus outdegree, notes `degreediff`). `degree`

Certain features are spectral, i.e., they are defined as the eigenvectors of certain matrices. For instance, the PageRank vector is defined as the dominant eigenvector of the matrix $\mathbf{G} = (1-\alpha)\mathbf{P} + \alpha\mathbf{1}$, and the eigenvector centrality as the component in the dominant eigenvector of the adjacency matrix $\mathbf{A}$ (**?**). `pagerank`

The local clustering coefficients give the clustering coefficient distribution (Seshadhri et al. 2012). `cluscod`

Other node features include:

- Average degree of neighbors
- Betweenness centrality
- k-Core number: largest k such that a node is still part of the graph's k-core. The distribution of k-core values over all nodes is then the k-core distribution, which can also be seen as the size of the k-core for all $k \leq 0$.

7.2 Measures

What is called a *measure* in KONECT is a measure of link prediction accuracy.

- The **average precision**. Measured over the whole test set. The best value is 1; the worst value depends on the ratio of true and false elements in the test set (for equally many, it is $1 - \ln 2$. When a proportion p of pairs in the test set is true, then the average precision of a random guess is p. `ap`
- The **mean average precision**. This is the average precision measured for each node separately, and then averaged over all nodes. The value is between 0 and 1. This is very slow to compute. `map`
- The **area under the curve** (Bradley 1997). This is the area under the ROC curve. The value is between 0 and 1. The area under the curve of a random is always $\frac{1}{2}$, regardless of the test set true/false ratio. This is very similar to the average precision, but can be used when the relative size of the true/false test sets is different from run to run. `auc`
- The **mean area under the curve**. This is the area under the curve computed for each node, and then averaged over all nodes. This is very slow to compute. `mauc`
- The **Pearson correlation coefficient**. This is probably the fastest to compute, but is not justified by any model. We used it successfully in an early paper (Kunegis & Lommatzsch 2009). `corr`

- The **Kendall rank correlation coefficient**. Very slow to compute and not justified by any model. `kendall`
- **Spearman's rank correlation coefficient**. Very slow to compute and not justified by any model. `spear`

Recommendations: We performing machine learning, use `auc`. When evaluating a recommender system, use `mauc`. If `mauc` is too slow, use `auc` instead.

8 The KONECT Toolbox

The KONECT Toolbox[19] for Matlab is a set of functions for the Matlab programming language[20] containing implementations of statistics, plots and other network analysis methods. The KONECT Toolbox is used to generate the numerical statistics and plots in this handbook as well as on the KONECT website.

Installation The KONECT Toolbox is provided as a directory containing *.m files. The directory can be added to the Matlab path using addpath() to be used.

Usage All functions have names beginning with `konect_`.

8.1 Examples

This section gives short example for using the toolbox. The examples can be executed in Matlab.

Load a unipartite dataset This example loads the Slashdot signed social network.

```
T = load('out.slashdot-zoo');
n = max(max(T(:,1:2)));
A = sparse(T(:,1), T(:,2), T(:,3), n, n);
```

This loads the weighted adjacency matrix of the Slashdot Zoo into the matrix `A`.

8.2 Variables

Naming variables can be quite complicated and hard to read in Octave or MAtlab. Therefore KONECT code follows specific rules as laid out in this section. As a general rule, variables with the same name always denote the same thing. Thus a variable `network` will always be a string denoting the internal name of a network (such as `ucidata-zachary` for the famous Karate Club), and a variable `statistic` will always be a string representing the internal name of a statistic (such as `clusco` for the clustering coefficient). Likewise, `value` will always be a numerical variable denoting the value of a statistic (such as 0.234 for a clustering coefficient).

Long variable names (containing full words) are in all-lowercase. Words are separated by underscore. When refering to a variable in comments, the variable is written in all-uppercase. Short variable names (letters) are lowercase for numbers and vectors, and uppercase for matrices.

[19] https://github.com/kunegis/konect-toolbox
[20] www.mathworks.com/products/matlab

Table 9: Long variable names of string type used in KONECT.

`network`	The internal network name, e.g., "advogato". The internal network name is used in the names of files related to the network. These names do not contain periods, slashes, spaces, or colons, but may otherwise contain dashes, spaces, etc.
`class`	The internal name for a set of networks, e.g., "test", "1", "2", "3". The class "N" includes the $10 \times N$ smallest networks.
`code`	The 1/2/3-character code for a network, e.g., "EN" for Enron.
`curve`	The internal name of a curve fitting method.
`decomposition`	The internal of a matrix decomposition, as passed to the function `konect_decomposition()`, e.g., "sym", "asym" and "lap".
`feature`	The internal name of a feature, e.g., "degree" and "decomp.sym".
`filename`	A filename.
`format`	The network format in lower case as defined in the function `konect_consts()`, e.g., "sym" and "bip".
`label`	The readable name of things used in plots, tables, etc.
`measure`	The internal name of a measure of link prediction accuracy, e.g., "map" and "auc".
`method`	The internal name of a link prediction method.
`statistic`	The internal name of a network statistic, e.g., "power" and "alcon".
`transform`	The name of a transform, e.g. "simple" and "lcc".
`type`	The internal name of the computation type. This can be "split" or "full". This decides which version of a network gets used, in particular for time-dependent analyses.
`weights`	The edge weight type as defined in the function `konect_consts()`, e.g., "unweighted" and "signed".

8.2.1 Strings

Table 9 shows common variable names used for string variables. As a general rules, strings as used for what we call *internal names*. These are canonical names given to individual networks, matrix decompositions, statistics, error measures, etc. These internal names are stable: We never change them, because that will break things. As a result, many internal names may appear inaccurate, inconsistent, or may simply look like having bad typography. For instance, some variants of the diameter statistic include the name `diam`, while others include the name `diameter`. This is unfortunate, but backward compatibility is tantamount here. In "user-facing" material such as the website and papers, we use pretty names for all of these, which we can change as we want.

8.2.2 Scalars

Table 10 shows variable names used for scalar values. As a general rule, we try to use individual lowercase latin letters for scalar variables, in keeping with Octave/Matlab usage. This aligns nicely with the mathematical aspects of KONECT, but can lead to ambiguities, and there we also use longer variable names in certain cases.

Table 10: Variable names used for scalars in KONECT.

`n, n1, n2`	Row/column count in matrices, left/right vertex count
`r`	Rank of a decomposition
`m`	Edge count
`i, j`	Vertices as integer, i.e., indexes in rows and columns.
`prediction`	A link prediction score, i.e., a value returned by a link prediction algorithm for a given node pair.
`precision`	The prediction accuracy value, typically between 0 and 1.
`means`	Values used for additive (de)normalization, as a structure.

8.2.3 Matrices

Table 11 shows variable names used for matrix-valued variables. As a general rules, matrices have uppercase names consisting of individual letters, as is the usage in Octave/Matlab, and in mathematics.

Note that when the adjacency matrix of an undirected graph is stored in a variable, each edge is usually stored just once, instead of twice. In other words, the variable `A` for undirected networks does not equal the matrix $\mathbf{A}$, instead the expression `A + A'` does.

8.2.4 Compound Types

A struct containing elements whose names are of a specific type are named `[VALUETYPE]s_[KEYTYPE]`. For instance, a struct with labels used for methods is named as follows:

```
labels_method.('auc') = 'Area under the curve';
```

Note:

- The first element is the name of the content type.
- The plural is used only for the content type.

8.2.5 IDs

Variables named `method`, `decomposition`, etc. are always strings. If a method, decomposition or any other type is represented as an integer (e.g., as an index into an array), then `_id` is appended to the variable name. For instance:

```
decomposition = 'sym'; decomposition_id = 2;
```

This means that an array of values by ID of keys is called for instance:

```
labels_decomposition_id{1} = 'Eigenvalue decomposition';
labels_decomposition_id{2} = 'Singular value decomposition';
```

Table 11: Variable names used for matrices and vectors in KONECT. As a general rule, matrices have upper-case names and vectors have lower-case names.

`A`	$(n \times n)$ Adjacency matrix (in code where the adjacency and biadjacency matrix are distinguished)
`A`	$(n \times n$ or $n_1 \times n_2)$ Adjacency or biadjacency matrix (in code where the two are not distinguished)
`B`	$(n_1 \times n_2)$ Biadjacency matrix (in code where the adjacency and biadjacency matrix are distinguished)
`D`	$(r \times r)$ Central matrix; e.g., eigenvalues; as matrix
`dd`	$(r \times 1)$ Diagonal of the central matrix
`E`	$(e \times 2)$ Test set for link prediction, stored in the same way as `T`
`L`	$(n \times n)$ Laplacian matrix
`M, N`	Normalized (bi)adjacency matrix
`T`	$(m \times 2$ or $m \times 3$ or $m \times 4)$ Compact adjacency matrix, as stored in `out.*` files, and such that it can be converted to a sparse matrix using `konect_spconvert()`. First column: row IDs Second column: column IDs Third column (optional): edge weights (1 if not present) Fourth column (optional): timestamps in Unix time
`U`	$(n \times r$ or $n_1 \times r)$ Left part of decomposition; e.g., left eigenvectors
`V`	$(n \times r$ or $n_2 \times r)$ Right part of decomposition; e.g., right eigenvectors
`X`	$(r \times r)$ Central matrix, when explicitly nondiagonal
`Z`	$(n \times n)$ Normalized Laplacian matrix

9 File Formats

Due to the ubiquity of networks in many areas, there are a large number of file formats for storing graphs and graph-like structures. Some of these are well-suited for accessibility from many different programming languages (mostly line-oriented text formats), some are well-suited for integration with other formats (semantic formats such as RDF and XML-based ones), while other formats are optimized for efficient access (binary formats). In KONECT, we thus use three file formats covering the three cases:

- TSV format: This format is text-based and uses tab- (or space-) separated values. This is the main KONECT data format from which the two others are derived. The format has the advantage that it can be read easily from many different programming languages and environments.
- RDF format: Datasets are also available as RDF, as text-based format used in the *Semantic Web* community. This format was deprecated as it did not have any users, and the files were generally an order of magnitude larger than other formats.
- Matlab format: To compute statistics and plots and perform experiments, we use Matlab's own binary format (version 7.3), which can be accessed efficiently from within Matlab.

9.1 TSV Format

In the following, we describe KONECT's TSV format. Each network `$network` is represented by the file `out.$network`.

The edges are stored as tab-separated values (TSV). The file is a text file, and each line contains information about one edge. Each line contains two, three or four numbers represented textually, and separated by any sequence of whitespace. The preferred separator is a single tab (`\t`). The first two columns are mandatory and contain the source and destination node ID of the edge. The third column is optional and contains the edge weight. When the network is dynamic, the third column contains $+1$ for added edges and -1 for removed edges. For unweighted, non-temporal networks, multiple edges may be aggregated into a single line containing, in the third column, the number of aggregated edges. The fourth column is optional and contains the edge creation time, and is stored as UNIX time, i.e., the number of seconds since 1 January 1970. The fourth column is usually an integer, but may contain floating point numbers. If the fourth column is present, the third column must also be given, and may be `1` if it is otherwise not needed. The beginning of the file contains additional comment lines with the following information:

```
% FORMAT WEIGHTS
% RELATIONSHIP-COUNT SUBJECT-COUNT OBJECT-COUNT
```

where `FORMAT` is the internal name for the format as given in Table 1, `WEIGHTS` is the internal name for the weight types as given in Table 2, `RELATIONSHIP-COUNT` is the number of data lines in the file, and `SUBJECT-COUNT` and `OBJECT-COUNT` both equal the number of nodes n in unipartite networks, and the number of left and right nodes n_1 and n_2 in bipartite networks. The first line is mandatory; the second line is optional.

9.2 Meta File

In addition to the data files (TSV, etc.), each network in KONECT has a file `meta.$network` associated with it. This is a short text file that contains all information about the dataset that are not structural. In other words, all structural information is stored in the TSV and other data formats, while meta information is stored in this file. The file contains metadata about the network that is independent of the mathematical structure of the network. The file is a text file coded in UTF-8. Each line contains one key/value pair, written as the key, a colon and the value. Whitespace is allowed everywhere, and is ignored. The following metadata fields are used:

- `name`: (obligatory) The name of the dataset. This contains only the name of the source, without description the type or category, e.g., "YouTube", "Wikipedia elections"). The name uses sentence case. When the name refers to the name of entities represented by the dataset, we use the plural. For instance, a dataset of relationships between monkeys should be named "Monkeys" rather than "Monkey", but note that many animal names are invariable in English. Note also that words such as "trust" do no refer to individual edges, but to the *type* of edge, and thus the plural is not used for them. Thus, a dataset may be named "MovieLens trust" rather than "MovieLens trusts". For networks with the same name, the source (e.g., the conference or author name) is added in parentheses. Within each category, all names must be distinct, but multiple networks can have the same `name` if they have different categories. We use the conference name for disambiguation if the dataset is recent, and the author name if it is older; this is however just a rule of thumb. The year number is used to disambiguate multiple versions of a dataset.

- `code`: (obligatory) The short code used in plots and narrow tables. The code consists of two or three alphanumeric characters. The first two characters are usually uppercase letters and denote the data source. The last character, if present, usually distinguishes the different networks from one source. The code is unique across all of KONECT, although it is not an error if they are not: It will only lead to certain plots being confusing. As KONECT grows, codes have become less and less unique; we may thus allow identical codes in the future.

- `category`: (obligatory) The internal name of the category, as given in the column "Internal name" in Table 3. It is important that this field is present, and corresponds to a valid category.

- `long-description`: (optional, recommended) A long descriptive text consisting of full sentences, and describing the dataset in a verbose way. HTML markup may be used sparingly (tags: `<I>`, `<CODE>`, etc.), usually only for absolutely necessary typography, such as setting species names in italics. Hyperlinks are not used, but URLs may be shown insode a `<CODE>` tags.

- `entity-names`: (optional, recommended) A comma-seperated list of entity names (e.g., "user, movie" or "protein"). Unipartite networks give a single name; bipartite networks give two. The names are in the singular and in lowercase. We usually follow the nomenclature given by the data sources in naming these.

- `relationship-names`: (optional, recommended) The name of the relationship represented by edges, as a lowercase substantive in the singular (e.g., "friendship", "road"). Again, we usually follow the nomenclature given by the data sources in naming these. The substantive may be a phrase, i.e., contain spaces (e.g., "tag assignment", "metabolic reaction"). If there are multiple relationship types not otherwise distinguished by the dataset, they are separated by a slash (e.g., "answer/comment").

- `extr`: (optional, recommended) The name of the subdirectory under the directory `konect-extr/extr/` that contains the extraction code for this dataset in the KONECT Extraction package.[21] If not given, it usually means that the dataset does not have extraction code, which is only the case with very old datasets. These were added to KONECT by hand without making the extraction reproducible; Stu was a long way off then.

- `url`: (optional, recommended for almost all networks) The URL(s) of the data sources, as a comma separated list. Most datasets have a single URL. The field is not present when datasets where given to us privately.

- `description`: (deprecated) A short description of the form "User–movie ratings". Note that the file should contain an actual en dash, coded in UTF-8. The field is now deprecated as this information is now covered by the `entity-names` and `relationship-names` fields.

- `cite`: (optional, recommended for almost all networks) The bibtex key of the citation(s) for this dataset, as a comma separated list. The given bibtex keys must correspond to the publications listed in the file `konect.bib` in the KONECT-Extr package. Most datasets have a single bibtex entry. This is usually the actual paper about the dataset. Many researchers have a preference about which of their papers should be cited for a dataset they have released, and we follow these.

[21] See https://github.com/kunegis/konect-extr/tree/master/extr

- `timeiso`: (optional) A single ISO timestamp denoting the date of the dataset or two timestamps separated by a slash (/) for a time range. The format is: `YYYY[-MM[-DD]][/YYYY[-MM[-DD]]]`, e.g., `2005-10-08/2006-11-03` or `2007`. In general, this should denote when the actual edges where created, but it often denotes only the date of data aggregation by the source from which KONECT gets the data. It does not denote the date at which the data is added to KONECT.

- `tags`: (optional) A space-separated list of hashtags describing the network. The list of tags is given in Section 2.7. Properly chosen tags are important to define the unit tests that KONECT will perform, in order to detect erroneously detected datasets.

- `fullname`: (optional, semi-deprecated) A longer name to disambiguate different datasets from the same source, e.g., "Youtube ratings" and "Youtube friendships". Uses sentence case. These names are unique across KONECT, falling back to the `name` when the fullname is not given. This field is only recommended when the `name` field is not unique.

- `n3-*`: (optional, deprecated) Metadata which is used for the generation of RDF files. The symbol {n} in the name of the meta key represents an order by unique, sequential numbers starting at one. These are deprecated.

  - `n3-add-prefix{n}` (optional): Used to define additional N3 prefixes. The default prefixes are specified in this way.
  - `n3-comment-{n}` (optional): Add commentary lines which are placed at the beginning of the N3 file.
  - `n3-edgedata-{n}` (optional): Additional N3-data, to be displayed with each edge.
  - `n3-nodedata-m-{n}` (optional): Additional N3-data, to be displayed with the first occurence of the source ID.
  - `n3-nodedata-n-{n}` (optional): Additional N3-data, to be displayed with the first occurence of the target ID.
  - `n3-prefix-m`: N3-prefix for the source IDs.
  - `n3-prefix-n` (optional): N3-prefix for the target IDs. If this field is left out, the value of {n3-prefix-m} is used.
  - `n3-prefix-j` (optional): Additional prefix which can be used with the source id, if there is an entity to be represented with the same id.
  - `n3-prefix-k` (optional): Additional prefix which can be used with the target id, if there is an entity to be represented with the same id. This is used for example in meta.facebook-wosn-wall for the representation of users walls.
  - `n3-prefix-l` (optional): N3-prefix for the edges, if they are to be represented by some N3-entity.
  - `n3-type-l` (optional): RDF-type for the edges.
  - `n3-type-m`: RDF-type for source IDs.
  - `n3-type-n` (optional): RDF-type for target IDs.

  The following fields are used in the n3-expressions for edgedata and nodedata:

`$m` : n3-prefix-m + source ID

`$n` : n3-prefix-n (or n3-prefix-m if the other is undefined) + target ID

`$j` : source ID

`$k` : target ID

`$l` : edge ID

`$timestamp` : edge timestamp

9.3 Semantic Data Format

For datasets that contain multiple relationship types, we use a more elaborative scheme. This scheme is not used systematically in KONECT, because the scope of KONECT is to handle each network separately. However, this scheme is used by certain families of datasets, and in particular by the extraction code.

This data format is semantic in the sense that it is able to represent almost all types of data in a structured fashion. It should not be confused with *semantic web* formats. This format is close in spirit to relational databases, only that it systematically distinguishes between entity types and relationship types.

As all KONECT data formats, the goal is to enable batch processing, and to allow reading and writing from many different programming languages, and to allow multiple programming languages and libraries to be composed easily. This goes at the price of compactness (no binary representation) and updatability (all IDs are continuous).

In this format, a dataset consists of an arbitrary number of entities types and relationship types. For each entity types, all entities have an integer ID ranging from one to the number of entities of that type. All information about entities (i.e., what would be contained in the corresponding entity table in a relational database) is contained in individual text files, separated by entity type and attribute. All information about relationships is stored in text files with one line per relationship, with one file per relationship type.

In the following, `$network` is the internal name of the semantic network. For one semantic network, there are multiple `ent.*` files, and multiple `rel.*` files, as described in the following.

Entities For each attribute `$att` of the entity type `$ent` related to the relationship type `$rel`, there is one file named `ent.$network_$rel[_$rel]_$ent_$att`. This is not strictly semantic, as the relationship type name should not appear. However, this is done in order to facilitate conversion to individual datasets.

The `$rel` string is present twice for files in `extr/` and only once for files in {`uni/`,`dat/`}.

The entity attributes file contain one line per entity, and do *not* contain the entity IDs – these are implicit in the line numbers. In addition, there is a header:

```
% <empty>
[% COUNT]
```

`COUNT` represents the biggest ID, and may be omitted.

An empty line indicates that there is no data available for this specific entity. There is no need to quote or escape characters as there is always only one attribute per line.

Relationships For each relationship type `$rel`, there is a file named `rel.$network_$rel`. The first line of the file is a comment line that defines the columns:
`% ent.NAME_A ent.NAME_B [WEIGHT_TYPE.NAME] dat.NAME*`
`WEIGHT_TYPE` can be:

- `weight`: an unspecified type
- `double`, `float`, `int`, `short`, `byte`: the corresponding type as in C, except that `char` is written as `byte`. (This last convention is influenced by the Java programming language.)

Data can be integers, floating point numbers or strings. Only numbers with well-defined numerical semantics are stored as numbers (e.g., zip codes are stored as strings).

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The picture of a pair of cherries in Figure 3 was created by the authors of Wikimedia Commons and is released under the Creative Commons CC-SA 3.0 license.[23]

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[22] http://konect.cc/networks/

[23] https://commons.wikimedia.org/wiki/File:Cherry_Stella444.jpg

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A Glossary of Terms

Some terms related to graph theory are well established in mathematics, network theory and computer science, while other terms do not have a widely-used definition. The choices made in this work are those of the authors, and were chosen to reflect best practices and to avoid confusion.

Adjacency matrix The matrix describing a network, usually denoted $\mathbf{A}$. To be contrasted with the half-adjacency matrix (for undirected unipartite networks, also denoted $\mathbf{A}$) and the biadjacency matrix (for bipartite networks, denoted $\mathbf{B}$). The adjacency matrix is always square, and for undirected networks it is symmetric.

All The term *all* in KONECT is the name of a group used when a certain method is applied to all KONECT networks individually. When a method is applied to all networks at once, we use *everything*, which is a class rather than a group.

Arc A directed edge. In general, we consider arcs to be a special cases of edges, and thus we rarely use the term *arc* in favor of *directed edge*. (In other texts, an edge is taken to be undirected by definition, and the term *directed edge* is then a contradiction.)

Asymmetric A directed graph is called asymmetric if it contains at least one unreciprocated edge.

Biadjacency matrix The characteristic matrix of a bipartite network, usually denoted $\mathbf{B}$. The corresponding adjacency matrix is then $[\mathbf{0}, \mathbf{B}; \mathbf{B}^{\mathrm{T}}, \mathbf{0}]$.

Category Networks have a category, which describes the domain they apply to: social networks, transport networks, citation networks, etc.

Central matrix The matrix $\mathbf{X}$ in any decomposition of the form $\mathbf{UXV}^{\mathrm{T}}$, not necessarily diagonal or symmetric; a generalization of the diagonal eigenvalue matrix.

Class A class is a set of networks determined by the number of edges they contain. The networks of KONECT are divided into classes by their volume: Class 1 contains the ten smallest networks, Class 2 contains the next ten smallest networks, etc. The class named 'everything' contains all networks, and the class named 'test' contains a set of small representative networks that cover most important network types.

Claw Three edges sharing a single vertex. A claw can be understood as a 3-star.

Code The two- or three-character code representation of a network. These are used in scatter plots that show many networks.

Cross A pattern of four edges sharing a single endpoint. Also called a 4-star.

Curve A curve fitting method used for link prediction, when using the link prediction method described in (Kunegis & Lommatzsch 2009).

Cycle A cyclic sequence of connected edges, not containing any edge twice. A cycle contrasts with a tour, in which a single vertex can appear multiple times.

Decomposition In KONECT the word *decomposition* is used to denote the combination of a characteristic graph matrix (e.g. the adjacency matrix or Laplacian) with a matrix decomposition. As an extension, some other constructions are also called *decomposition*, such as LDA.

Density This word is avoided in KONECT. In the literature, it may refer to either the fill (probability that an edge exists), or to the average degree. The former definition is typically used in mathematical contexts, while the latter is used in computer science contexts.

Directed A directed graph contains directed edges. A directed graph can be symmetric (if all edges are reciprocated), or asymmetric (otherwise).

Edge A connection between two nodes. In mathematics, an edge is undirected and constrasts with an arc which is directed. In the context of KONECT, all types of connections between nodes are called *edges* and an arc is a special case of an edge.

Everything In KONECT, *everything* is the class of all networks. It is used when a method is applied to the set of all networks as a whole. When a method is applied to all networks individually, we use *all*, which is a group.

Family A family is a set of networks that were generated by the same extraction code. Each family corresponds to a directory under `konect-extr/extr/`.

Feature A node feature. I.e., a number assigned to each node. Examples are the degree, PageRank and the eccentricity. Equivalently, a node vector.

Fill The probability that two randomly chosen nodes are connected. Also called the *density* or *edge density*, in particular in a mathematical context. The fill is the sole parameter of the Erdős–Rényi random graph model. The word *fill* is specific to KONECT.

Format The format of a network determines its general structure, and whether edges are directed. There are three possible formats: unipartite and undirected; unipartite and directed; and bipartite. Directed bipartite networks are not possible in KONECT.

Group A group of networks is a set of networks that have similar attributes, as defined by their metadata such as format, weights, timestamps, etc. Examples are unipartite networks, network with multiple edges, and signed temporal networks. Each group has an internal name in KONECT, which is in all-uppercase, with words separated by underscore. See Section 2.6. The purpose of groups is to be able to specify the set of networks to which a statistic, plot or other type of analysis applies to.

Half-adjacency matrix The adjacency matrix $\mathbf{A}$ of an undirected graph contains two nonzero entries for each edge $\{i, j\}$: $\mathbf{A}_{ij}$ and $\mathbf{A}_{ji}$. To avoid this, KONECT code uses the half-adjacency matrix, which contains only one of the two nonzero entries. The half-adjacency matrix is therefore not unique, i.e., it is unspecified whether $\mathbf{A}_{ij}$ or $\mathbf{A}_{ji}$ is nonzero. In code, the half-adjacency matrix is denoted `A`. The term *half-adjacency matrix* is specific to KONECT, but the use of such a representation is widespread.

Loop A loop is an edge that connects a node with itself. A loop may be directed or undirected. They are often called *self-loops* in the literature, but that expression is quite redundant.

Measure A measure of the accuracy of link prediction methods, for instance the area under the curve or the mean average precision.

Method A link prediction method.

Normalization In KONECT, *normalization* is defined in its broad sense, in which some quantity is, at some level, divided by another to result in quantities that can be compared. This may be performed on the level of individual numbers, in which case we call it *relativization*, but also at higher levels such as on entire matrices. For instance, the matrix $\mathbf{N} = \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}$ is called the normalized adjacency matrix. Normalization and relativization can often be contrasted. As an example, the largest absolute eigenvalue of the normalized matrix $\mathbf{N}$ is always one while the relative largest eigenvalue of the non-normalized adjacency matrix would be α/n, a value in general not equal to one.

PageRank A node-based feature of a directed network, defined as the dominant eigenvector of the matrix $\mathbf{G} = (1 - \alpha)\mathbf{P} + \alpha\mathbf{1}$, with eigenvalue one.

Path A sequence of connected nodes, in which each node can appear only once. The extension that allows multiple nodes is called a walk. A path with identical start and end nodes is called a *cycle*.

Relative The adjective *relative* is applied to statistics when they are derived by dividing a given statistic by a reference value, often the maximally possible attainable value. For instance, the relative size of the largest connected component equals the size of the largest connected component divided by the number of nodes. Such statistics have values in the range $[0, 1]$ and are useful for comparing networks of different sizes. Relative statistics are a form of normalization.

Score A numerical value given to a node pair. Usually used for link prediction, but can also measure distance or similary between nodes.

Size The number of nodes in a network.

Statistic A statistic is a numerical measure of a network, i.e., a number that describes a network, such as the clustering coefficient, the diameter or the algebraic connectivity. All statistics are real numbers.

Symmetric A directed graph is called symmetric if all its edges are reciprocated. Symmetric directed graphs are equivalent to undirected graphs, but have double the node degrees. Thus, equations between the two are in general *not* exchangeable. For graphs, *symmetric* only applies to directed graphs.

Tour A cyclic sequence of connected nodes which may contain a single vertex multiple times. It can be considered a walk that returns to it starting point, or a generalization of a cycle that allows to visit nodes multiple times.

Trail A trail is a walk that does not visit any edge twice. Compare to a path, which must not visit any node twice. Every path is a trail and a walk, and every trail is a walk, but not the other way around.

Transform A transform is an operation that applies to a graph and that gives another graph. Examples are taking the largest connected component, removing multiple edges, and making a bipartite graph unipartite. Certain graph properties can be expressed as other graph properties applied to graph transforms. For instance, the size of the largest connected component is the size of the transform which keeps only the largest connected component.

Triangle Three nodes all connected with each other. The number of triangles in a network is a commonly used statistic, used for instance as the basis to compute the clustering coefficient. Counting the triangles in a network is a very common computational problem.

Volume The number of edges in a network.

Walk A sequence of connected nodes, which may contain a single node multiple times. The restriction to include a single node only once is called a path. If the endpoints of a walk are identical, then the walk is also a tour.

Wedge Two edges sharing a common node, i.e., two incident edges. The number of wedges in a network is an important network statistic, which characterizes that skewness of the degree distribution, and which can be easily calculated. A wedge can be seen as a 2-star or a 2-path.

Weights (always in the plural) The weights of a network describe the range of edge weights it allows. The list of possible edge weights is given in Table 2.

B Glossary of Mathematical Symbols

The following symbols are used in mathematical expessions throughout KONECT. Due to the large number of different measures used in graph theory and network analysis, many common symbols for measures overlap. For many measures, there is more than one commonly-used notation; the notation chosen in KONECT represents a reasonable balance between using established notation when it exists, and having distinct symbols for different measures.

In KONECT and literature written by its authors, matrices are written as bold uppercase letters, and vectors as bold lowercase letters. Individual numbers and sets are written in non-bold symbols.

Symbol	**Meaning**
a	algebraic connectivity
b	non-bipartivity
c	global clustering coefficient
$c(u)$	local clustering coefficient
d	average degree
$d(u)$	degree of a vertex
$d(u, v)$	shortest-path distance
e	edge
g	line count, data volume
l	loop count
m	volume, edge count
$\tilde{m}$	average multiplicity
$\bar{\bar{m}}$	number of unique edges
n	size, node count
p	fill
q	square count
r	rank of a decomposition
r	rating value
r	radius of a graph
s	wedge count
t	triangle count
u, v, w	vertices
w	edge weight
w	network weight
$w(\ldots)$	weight function

Symbol	Meaning
x	cross count
y	reciprocity
z	claw count
α	spectral norm
β	preferential attachment exponent
γ	power law exponent
δ	diameter
ϵ	eccentricity
ζ	negativity
η	dyadic conflict
λ	eigenvalue
μ	average edge weight
ν	operator 2-norm
ξ	algebraic conflict
π	cyclic eigenvalue
ρ	assortativity
σ	singular value
τ	triadic conflict
ϕ	spectral signed frustration
χ	algebraic non-bipartivity
C	controllability
C_k	k-cycle count
E	edge set
F	frustration
G	graph
G	Gini coefficient
H	entropy
K_k	k-clique count
N	size of largest connected component
P_k	k-path count
S_k	k-star count
T_k	k-tour count
V	vertex set
W_k	k-walk count
$\mathbf{0}$	zeroes matrix
$\mathbf{1}$	ones matrix
$\mathbf{A}$	adjacency matrix
$\mathbf{B}$	biadjacency matrix
$\mathbf{D}$	degree matrix
$\mathbf{E}$	incidence matrix
$\mathbf{F}$	line matrix
$\mathbf{G}$	PageRank matrix ("Google matrix")
$\mathbf{H}$	Hermitian adjacency matrix

Symbol	Meaning
$\mathbf{I}$	identity matrix
$\mathbf{K}$	signless Laplacian matrix
$\mathbf{L}$	Laplacian matrix
$\mathbf{M}$	normalized biadjacency matrix
$\mathbf{N}$	normalized adjacency matrix
$\mathbf{P}$	stochastic adjacency matrix
$\mathbf{S}$	stochastic Laplacian matrix
$\mathbf{T}$	magnetic Laplacian matrix
$\mathbf{U}, \mathbf{V}$	eigenvector matrices
$\mathbf{W}$	Seidel adjacency matrix
$\mathbf{X}$	central matrix
$\mathbf{Y}$	skew-symmetric adjacency matrix
$\mathbf{Z}$	normalized Laplacian matrix
$\mathbf{\Gamma}$	Moore–Penrose pseudoinverse of $\mathbf{L}$
$\mathbf{\Lambda}$	eigenvalue matrix
$\mathbf{\Sigma}$	singular value matrix
$\bar{G}$	unweighted graph
$\bar{\bar{G}}$	graph with unique edges
$\lvert G \rvert$	unsigned graph
$-G$	negated graph
$\bar{\mathbf{X}}$	explicitly undirected or symmetrized version of a characteristic matrix
$\hat{\mathbf{X}}$	repulsive version of a characteristic matrix
$\breve{\mathbf{X}}$	Hermitian version of a characteristic matrix
$\acute{\mathbf{X}}$	skew-Hermitian version of a characteristic matrix
$\vec{\mathbf{X}}$	unidirectional version of a characteristic matrix
$\grave{\mathbf{X}}$	bipartite version of a characteristic matrix