Computing Distributions of Economic Models via Simulation

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Structure of the Talk

- Review of LLN, CLT
- Review of Markov chains
- Outline of the problem
- Common solution techniques
- A better technique
- Our contribution

Quick Review: LLN and CLT

Let $(Y_i)_{i \geq 1}$ be an IID sequence of random variables in $\mathbb{R}$.

Let $\bar{Y}_n := \frac{1}{n} \sum_{i=1}^{n} Y_i$ and let $\mu := \mathbb{E}Y$.

Under suitable moment conditions:

$$(\bar{Y}_n - \mu) \to 0 \text{ with probability one}$$

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$$(\bar{Y}_n - \mu) = O_P(n^{-1/2})$$

Definition: if $X_n = O_P(n^{-1/2})$ then

$$\forall \varepsilon > 0,\ \exists M < \infty \text{ s.t. } \sup_{n\in\mathbb{N}} \mathbb{P}\{|X_n| > Mn^{-1/2}\} \leq \varepsilon$$

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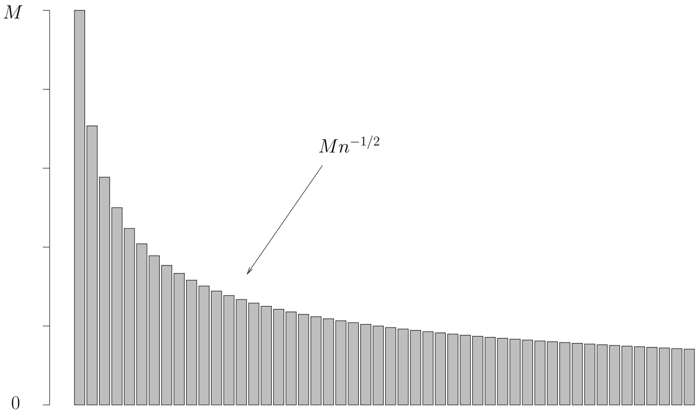
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Now consider the case of vectors in $\mathbb{R}^k$.

$$\langle x, y\rangle = \sum_{i=1}^{k} x_i y_i \quad \text{and} \quad \|x\| := \langle x, x\rangle^{1/2}$$

Let $Y = (Y_1, \ldots, Y_k) \in \mathbb{R}^k$ be a random vector, $\mathbb{E}\|Y\| < \infty$.

Expectation is $\mathcal{E}Y := (\mathbb{E}Y_1, \ldots, \mathbb{E}Y_k) \in \mathbb{R}^k$. Equivalently:

$$\langle \mathcal{E}Y, x\rangle = \mathbb{E}\langle Y, x\rangle \quad \forall x \in \mathbb{R}^k$$

Under suitable moment conditions LLN and CLT still hold:

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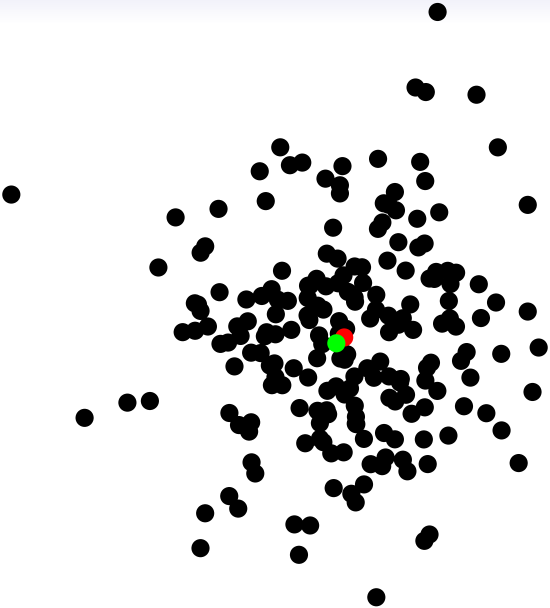

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$$\langle f, g\rangle = \int f(x)g(x)dx \quad \text{and} \quad \|f\| := \langle f, f\rangle^{1/2}$$

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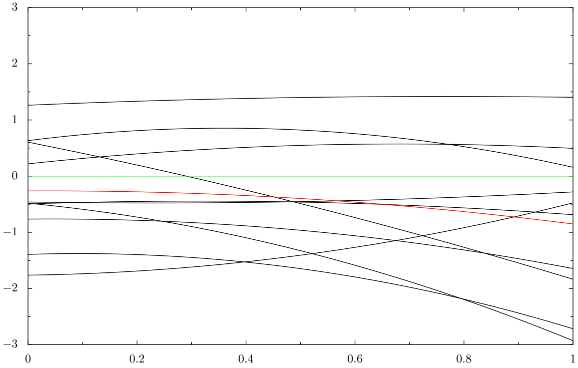
Y = the function $f(x) = W_0 + W_1 x + W_2 x^2$; $W_i \sim N(0,1)$.



Quick Review: Markov Chains

Consider an $\mathbb{R}$-valued process $(X_t)_{t \geq 0}$ defined by

$$X_t = h(X_{t-1}) + W_t, \quad (W_t)_{t \geq 1} \overset{\text{IID}}{\sim} N(0, \sigma^2), \quad X_0 = x_0 \qquad (1)$$

This is an example of a discrete time Markov chain.

Note $X_1 = h(x_0) + W_1$, $X_2 = h(h(x_0) + W_1) + W_2$, etc.

Thus, for some F we have $X_t = F(x_0, W_1, \ldots, W_t)$.

Notation: $\psi_t \overset{\mathscr{D}}{=} X_t$ for all $t \geq 1$.

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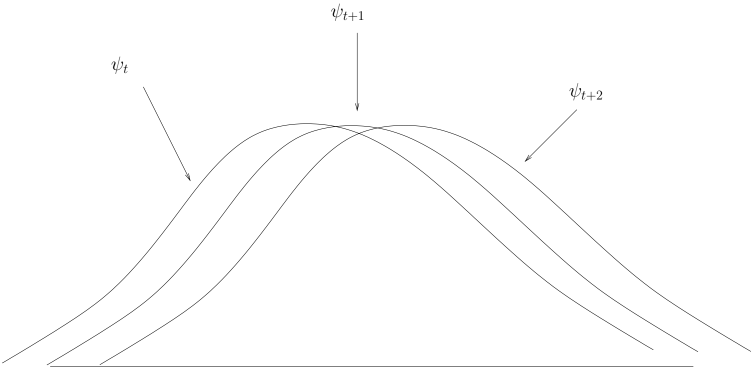
ψ_{t+1}
ψ_t
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$$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{[y-h(X_{t-1})]^2}{2\sigma^2}\right\}$$

$$\because \quad X_t \,|\, X_{t-1} = h(X_{t-1}) + N(0,\sigma^2) = N(h(X_{t-1}),\sigma^2)$$

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Proof.

For any two random variables X, Y we have

$$\frac{p_{X,Y}(x,y)}{p_X(x)} = p_{Y|X}(y|x)$$

$$\therefore \quad \int p_{X,Y}(x,y)dx = \int p_{Y|X}(y|x)p_X(x)dx$$

Recall that: $\int p_{X,Y}(x,y)dx = p_Y(y)$

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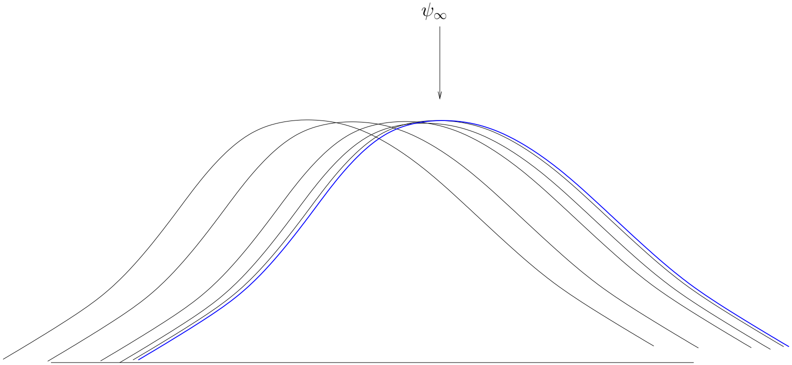
Sometimes the sequence $(\psi_t)_{t\geq 1}$ converges: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Stationary Distribution

The limit ψ_∞ is called stationary, satisfies

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Example: For $X_t = h(X_{t-1}) + N(0, \sigma^2)$, if

$$\exists \lambda < 1,\ L < \infty \ \text{ s.t. } \ |h(x)| \le \lambda |x| + L,\ \forall x \in \mathbb{R}$$

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Computing Distributions

Motivating Example:

- Suppose we are studying the price of a commodity.
- Price at time t denoted by X_t.
- Fix model : $X_t = h(X_{t-1}) + W_t$ where $(W_t)_{t \geq 1} \overset{\text{IID}}{\sim} N(0, \sigma^2)$.
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- How to compute ψ_∞ numerically if it exists?

Computing Distributions

Motivating Example:

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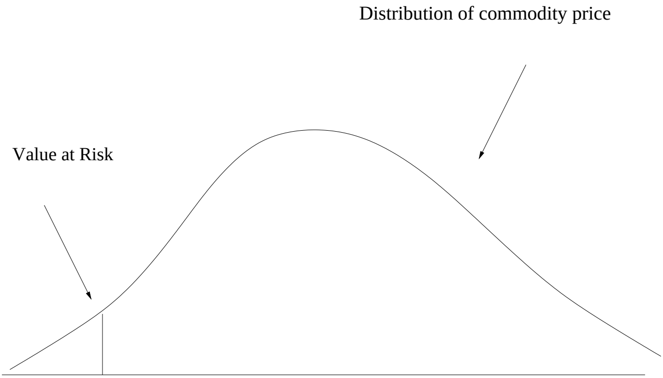
Distribution of commodity price
Value at Risk

Computation of ψ_T

Common technique 1: Discretization.

- Discretize state space onto grid of size n.
- Replace $X_t = h(X_{t-1}) + N(0, \sigma^2)$ with “similar” model taking values on the grid.
- Solve for time T distribution ψ_T^n.

- But how far is ψ_T^n from target ψ_T for given n?

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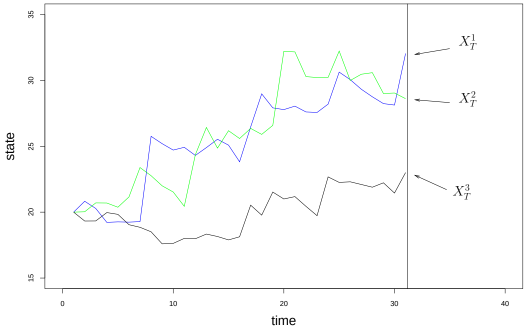
Generate n draws of X_T using $X_t = h(X_{t-1}) + W_t$, $X_0 = x_0$:



Now use $(X_T^1, \ldots, X_T^n)$ to generate some approximation ψ_T^n.

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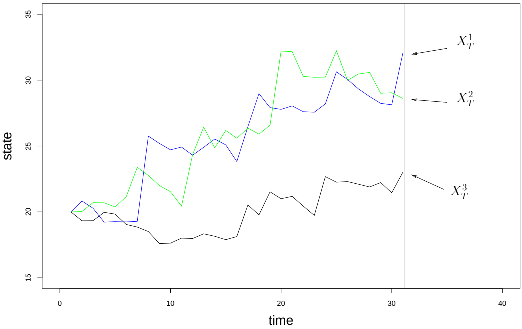

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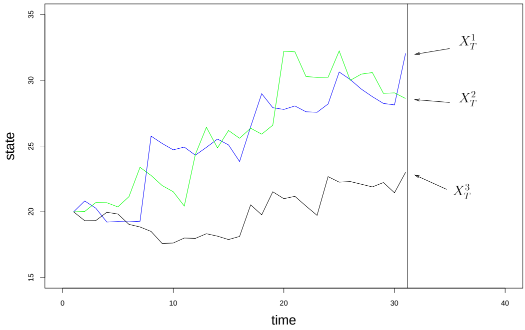

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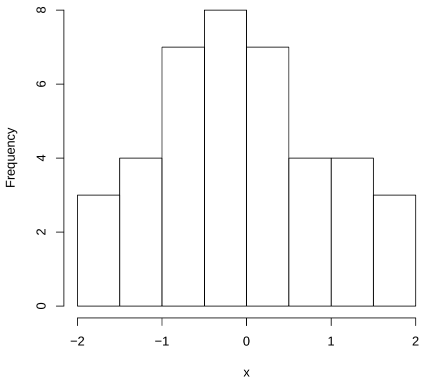
Example 1: Histogram



Example 2: Kernel Density Estimator

Has the expression

$$f_T^n(y) = \frac{1}{n\delta_n} \sum_{i=1}^{n} K\left(\frac{y - X_T^i}{\delta_n}\right). \tag{4}$$

Here

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- $\delta_n > 0$ the bandwidth, chosen s.t. $\delta_n \to 0$.

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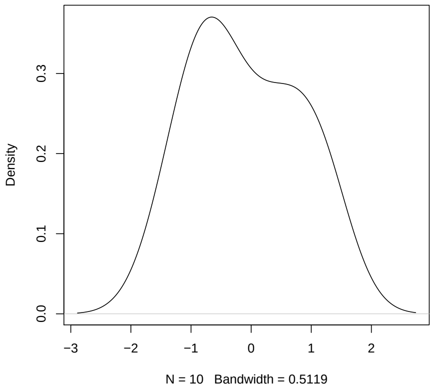
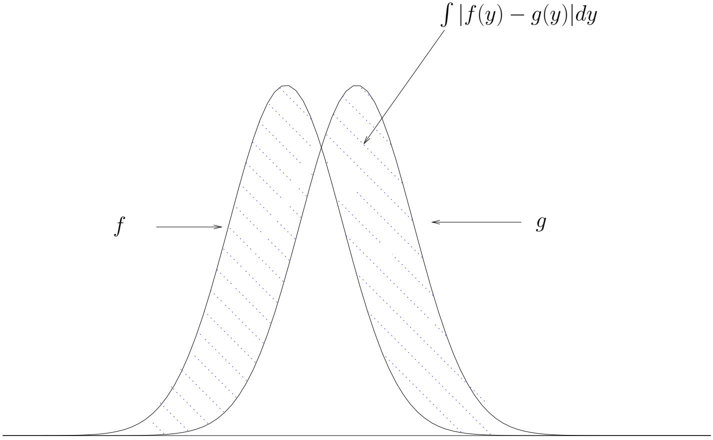

Density
0.0
0.1
0.2
0.3
−3
−2
−1
0
1
2
N = 10 Bandwidth = 0.5119

Properties of Kernel Density Estimator

Recall the notion of L_1 distance:



Properties of Kernel Density Estimator

Asymptotic properties excellent:

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The Look-Ahead Estimator

Due to Glynn and Henderson (2001), MOR.

Recall:

$$p(\cdot|X_{t-1}) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{[y - h(X_{t-1})]^2}{2\sigma^2}\right\}$$

Notice that p encodes model $X_t = h(X_{t-1}) + N(0, \sigma^2)$.

Definition

The look ahead estimator of ψ_T is

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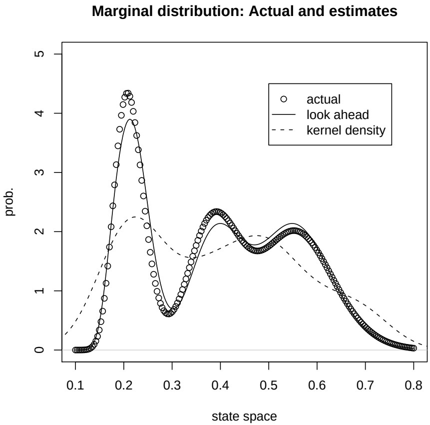
Recall that the CLT gives us the $O_P(n^{-1/2})$ rate of convergence.

In particular,

$$(\psi_T^n(y) - \psi_T(y)) = O_P(n^{-1/2})$$


Marginal distribution: Actual and estimates
actual
look ahead
kernel density
prob.
state space
0
1
2
3
4
5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

Stationary Densities

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- convergence of $\psi_T^n(y) \to \psi_T(y)$, and
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Can prove that the L_2 expectation is $\mathcal{E}Y = \psi_T$

The L_2 LLN and CLT now give

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A Markov Chain Hilbert space CLT (Stachurski, 2006) gives

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A Hypothesis Test

Take a (suitably stable) Markov model

$$X_t = h(X_{t-1}) + W_t, \quad (W_t)_{t\geq 1} \overset{\text{IID}}{\sim} N(0,\sigma^2), \quad X_1 = x_1 \tag{6}$$

Let $(\hat{X}_t)_{t=1}^n$ be observed data.

Null hypothesis: data is generated by (6).

Let $\psi_\infty^n(y) := \frac{1}{n}\sum_{t=1}^n p(y \,|\, \hat{X}_t)$.

Under the null, $n\|\psi_\infty^n - \psi_\infty\|^2 \xrightarrow{\mathscr{D}} T := \sum_{\ell\geq 1}^\infty \lambda_\ell Z_\ell^2$.

Reject null if $n\|\psi_\infty^n - \psi_\infty\|^2 \geq$ the critical value of T.

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An Experiment

Applied the test to Vasicek model of interest rates:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dB_t$$

To implement test, computed critical value of $\sum_{\ell \geq 1}^{\infty} \lambda_\ell Z_\ell^2$.

Requires the eigenvalues $(\lambda_\ell)_{\ell \geq 1}$ of C, the asymptotic covariance function of $\sqrt{n}(\psi_\infty^n - \psi_\infty)$.

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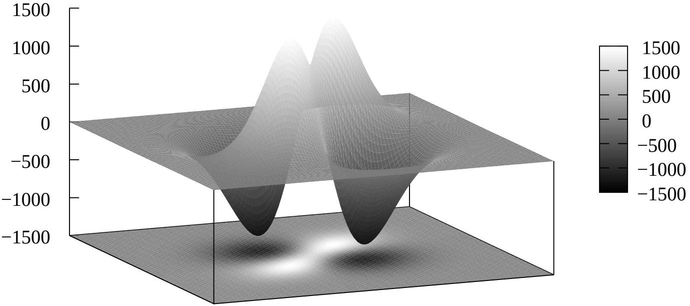

Figure: Covariance Function, Vasicek Model

Computed the eigenvalues $(\lambda_\ell)_{\ell\geq 1}$ of C by Galerkin projection.

Computed the critical value of $T := \sum_{\ell\geq 1}^{\infty} \lambda_\ell Z_\ell^2$ by simulation.

Results: statistic attains the asymptotic distribution much faster than test proposed by Aït-Sahalia (1996).

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