Master of Science

20251 Fixed Income (Advanced Methods)

Lecture 3

Yield Curve Stripping

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Outline

Introduction

Introduction

- In the valuation of fixed income securities, it is not the Treasury yield curve that is used as the basis for determining the appropriate discount rate for computing the present value of cash flows but the Treasury spot rates.
- The Treasury spot rates are derived from the Treasury yield curve using the bootstrapping process.
- Similarly, it is not the swap curve that is used for discounting cash flows when the swap curve is the benchmark but the spot rates. The spot rates are derived from the swap curve in exactly the same way, using the bootstrapping methodology.
- The resulting spot rate curve is called the LIBOR spot rate curve.
- Moreover, a forward rate curve can be derived from the spot rate curve. The same thing is done in the swap market. The forward rate curve that is derived is called the LIBOR forward rate curve.
- In the United States it is common to use the Treasury spot rate curve for purposes of valuation. In other countries, either a government spot rate curve is used (if a liquid market for the securities exists) or the swap curve is used (or as explained shortly, the LIBOR curve).

Reasons for Increased Use of Swap Curve[1]

- Investors and issuers use the swap market for hedging and arbitrage purposes, and the swap curve as a benchmark for evaluating performance of fixed income securities and the pricing of fixed income securities.
- Since the swap curve is effectively the LIBOR curve and investors borrow based on LIBOR, the swap curve is more useful to funded investors than a government yield curve.
- The increased application of the swap curve for these activities is due to its advantages over using the government bond yield curve as a benchmark.
- The drawback of the swap curve relative to the government bond yield curve could be poorer liquidity. In such instances, the swap rates would reflect a liquidity premium.
- Fortunately, liquidity is not an issue in many countries as the swap market has become highly liquid, with narrow bid-ask spreads for a wide range of swap maturities. In some countries swaps may offer better liquidity than that country's government bond market.

[1] Source: Fabozzi, Fixed Income Analysis, 193-196.

Advantages of the swap curve over a government bond yield curve I

- There is almost no government regulation of the swap marke, that makes makes swap rates across different markets more comparable.
  - In some countries, there are some sovereign issues that offer various tax benefits to investors and, as a result, for global investors it makes comparative analysis of government rates across countries difficult because some market yields do not reflect their true yield.
- The supply of swaps depends only on the number of counterparties that are seeking or are willing to enter into a swap transaction at any given time. Since there is no underlying government bond, there can be no effect of market technical factors that may result in the yield for a government bond issue being less than its true yield.
- Comparisons across countries of government yield curves is difficult because of the differences in sovereign credit risk. Sovereign risk is not present in the swap curve because, as noted earlier, the swap curve is viewed as an inter-bank yield curve or AA yield curve.

Advantages of the swap curve over a government bond yield curve II

- There are more maturity points available to construct a swap curve than a government bond yield curve.
  - More specifically, what is quoted in the swap market are swap rates for 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, and 30 year maturities. Thus, in the swap market there are 10 market interest rates with a maturity of 2 years and greater.
  - In contrast, in the U.S. Treasury market, for example, there are only three market interest rates for on-the-run Treasuries with a maturity of 2 years or greater (2, 5, and 10 years) and one of the rates, the 10-year rate, may not be a good benchmark because it is often on special in the repo market. Moreover, because the U.S. Treasury has ceased the issuance of 30-year bonds, there is no 30-year yield available.

Term Structure Stripping

Term Structure stripping[2]

- The spot term structure is constructed from shorter maturities to longer maturities using market instruments like Libor, futures, FRAs and swaps.
- Then it can be used for pricing basic derivatives instruments, like coupon bonds.
- The procedure that allows the construction of the term structure is called bootstrapping.
- It calculates discount factors at certain "grid points" and at the intermediate dates linearly interpolates between values at the grid points.
- The necessary inputs for this process are:
  1. a set of money market rates,
  2. a set of futures contracts,
  3. a set of par swap rates.

[2] This part is based on James and Webber, pagg. 129-138.

Main Steps in bootstrapping

- The instruments in the procedure are:
  1. Money market instruments: the most commonly used maturities are O/N, 1W, 1M, 2M (and but not used: 3M and 6M).
  2. Futures that are quoted for cycles of 3 months.
  3. Swaps are quoted for 1Y, 2Y, 3Y, 4Y, 5Y, 7Y and 10Y (12Y, 15Y, 20Y, 25Y and 30Y) maturities.
- Libor rates: the discount factors are calculated directly.
- Futures are treated as if they were FRAs
- US and Japan: Swaps are semi-semi, i.e. fixed and floating cashflows are made semi-annually, so they require discount factors at six-months intervals. Day-count convention: ACT/365.
- Euro: Swaps are annual-semi, i.e. fixed (floating) cashflows are made annually (semi-annually). We need discount factors at yearly intervals. Day-count convention: 30/360.

Market Data (Euro)

Euro Data as from September 19, 2006					
LIBOR %		Futures		Swaps %	
o/n	3.04188	Dec-06	96.35	1	3.808
1w	3.06275	Mar-07	96.28	2	3.88
2w	3.06563	Jun-07	96.28	3	3.883
1m	3.19325	Sep-07	96.32	4	3.892
2m	3.2935	Dec-07	96.355	5	3.906
3m	3.37025	Mar-08	96.385	6	3.925
		Jun-08	96.395	7	3.947
		Sep-08	96.385	8	3.973
		Dec-08	96.365	9	4.001
				10	4.029
				12	3.999
				15	4.066
				20	4.202
				25	4.218
				30	4.214

Market Data (Yen)

Yen data as from January 9, 1996					
Libor %		Futures		Swaps %	
o/n	0.49	Mar 96	99.34	2y	1.14
1w	0.50	Jun 96	99.25	3y	1.60
1m	0.53	Sep 96	99.10	4y	2.04
2m	0.55	Dec 96	98.90	5y	2.43
3m	0.56			7y	3.01
				10y	3.36

Procedure

- Calculating Discount Factors from LIBOR
- Calculating Discount Factors from Futures
  - Interpolation for the stub date
  - Extract futures rates from futures price and treat them as forward rates (no convexity adjustment)
- Calculating Discount Factors from Swaps
  - Interpolating for 1yr and 2yrs discount factors using discount factors from futures
  - Computing swap rates at missing dates by linear interpolation of swap rates.

Market Data for Bootstrapping (Euro)

Market Data for yield curve construction (Euro)

Value date 21/09/2006

Days	Period	Rate %	Instrument	Day Count	Date	
	spot					
1	o/n	3.04188	Cash	ACT/360	*24-Sep-06*	
7	1w	3.06275	Cash	ACT/360	*30-Sep-06*	
15	2w	3.06563	Cash	ACT/360	*7-Oct-06*	
32	1m	3.19325	Cash	ACT/360	*21-Oct-06*	
61	2m	3.2935	Cash	ACT/360	*21-Nov-06*	
	3m	3.37025	Cash	ACT/360	*21-Dec-06*	
	stub		Futures	ACT/360	*20-Dec-06*	3rd Wedn. of each listed Month
91	Dec-06	96.35	Futures	ACT/360	*21-Mar-07*	3rd Wedn. of each listed Month
91	Mar-07	96.28	Futures	ACT/360	*20-Jun-07*	3rd Wedn. of each listed Month
91	Jun-07	96.28	Futures	ACT/360	*19-Sep-07*	3rd Wedn. of each listed Month
91	Sep-07	96.32	Futures	ACT/360	*19-Dec-07*	3rd Wedn. of each listed Month
91	Dec-07	96.355	Futures	ACT/360	*19-Mar-08*	3rd Wedn. of each listed Month
91	Mar-08	96.385	Futures	ACT/360	*18-Jun-08*	3rd Wedn. of each listed Month
91	Jun-08	96.395	Futures	ACT/360	*17-Sep-08*	3rd Wedn. of each listed Month
91	Sep-08	96.385	Futures	ACT/360	*17-Dec-08*	3rd Wedn. of each listed Month
91	Dec-08	96.365	Futures	ACT/360	*19-Mar-09*	3rd Wedn. of each listed Month
	1	3.808	Swap	ACT/365	*21-Sep-07*	
	2	3.88	Swap	ACT/365	*21-Sep-08*	
	3	3.883	Swap	ACT/365	*21-Sep-09*	
	4	3.892	Swap	ACT/365	*21-Sep-10*	
	5	3.906	Swap	ACT/365	*21-Sep-11*	
	6	3.925	Swap	ACT/365	*21-Sep-12*	
	7	3.947	Swap	ACT/365	*21-Sep-13*	
	8	3.973	Swap	ACT/365	*21-Sep-14*	
	9	4.001	Swap	ACT/365	*21-Sep-15*	
	10	4.029	Swap	ACT/365	*21-Sep-16*	
	12	3.999	Swap	ACT/365	*21-Sep-18*	
	15	4.066	Swap	ACT/365	*21-Sep-21*	
	20	4.202	Swap	ACT/365	*21-Sep-26*	
	25	4.218	Swap	ACT/365	*21-Sep-31*	
	30	4.214	Swap	ACT/365	*21-Sep-36*	

Date	Event	
19-Sep-06	Trade date	today
21-Sep-06	Settl. Date	2 bus. Days (spot date)
24-Sep-06	o/n	
30-Sep-06	1w	
7-Oct-06	2w	
21/10/2006	1m	
21-Nov-06	2m	
20/12/2006	stub	geom. interpolation on d.f.
21-Dec-06	3m	not used
20-Jun-07	Mar-07	
19-Sep-07	Jun-07	
21-Sep-07	Y1	not used
19-Dec-07	Sep-07	
19-Mar-08	Dec-07	
18-Jun-08	Mar-08	
17-Sep-08	Jun-08	
21-Sep-08	Y2	not used
17-Dec-08	Sep-08	
18-Mar-09	Dec-08	
21-Sep-09	Y3	
21-Sep-10	Y4	
21-Sep-11	Y5	
21-Sep-12	Y6	
21-Sep-13	Y7	
21-Sep-14	Y8	
21-Sep-15	Y9	
21-Sep-16	Y10	
21-Sep-17		linear interpolation of swap rates
21-Sep-18	Y12	
21-Sep-19		linear interpolation of swap rates
21-Sep-20		linear interpolation of swap rates
21-Sep-21	Y15	
21-Sep-22		linear interpolation of swap rates
21-Sep-23		linear interpolation of swap rates
21-Sep-24		linear interpolation of swap rates
21-Sep-25		linear interpolation of swap rates
21-Sep-26	Y20	
21-Sep-27		linear interpolation of swap rates
21-Sep-28		linear interpolation of swap rates
21-Sep-29		linear interpolation of swap rates
21-Sep-30		linear interpolation of swap rates
21-Sep-31	Y25	
21-Sep-32		linear interpolation of swap rates
21-Sep-33		linear interpolation of swap rates
21-Sep-34		linear interpolation of swap rates
21-Sep-35		linear interpolation of swap rates
21-Sep-36	Y30	

In gray the dates at which we do not observe a market rate and we need interpolation.

Market Data for Bootstrapping (Yen)

Market Data for yield curve construction

Source: James and Webber, Interest Rate Modelling, pp 130-133

Days	Period	Rate %		Day Count	
	spot				
1	o/n	0,49	Cash	ACT/360	
7	1w	0,5	Cash	ACT/360	
33	1m	0,53	Cash	ACT/360	
60	2m	0,55	Cash	ACT/360	
	3m	0,56	Cash	ACT/360	
69	stub		Futures	ACT/360	3rd Wedn. Of each listed Month
91	mar-96	99,34	Futures	ACT/360	
91	giu-96	99,25	Futures	ACT/360	
91	set-96	99,1	Futures	ACT/360	
91	dic-96	98,9	Futures	ACT/360	
	18m		Swap	ACT/365	
	2	1,14	Swap	ACT/365	
	2,5		Swap	ACT/365	
	3	1,6	Swap	ACT/365	
	3,5		Swap	ACT/365	
	4	2,04	Swap	ACT/365	
	4,5		Swap	ACT/365	
	5	2,43	Swap	ACT/365	
	5,5		Swap	ACT/365	
	6		Swap	ACT/365	
	6,5		Swap	ACT/365	
	7	3,01	Swap	ACT/365	
	10	3,36			

Date	Event	
09/01/1996	Trade date:	today
11/01/1996	Settl. Date:	2 bus. Days (spot date)
12/01/1996	o/n	
18/01/1996	1w	
13/02/1996	1m	
11/03/1996	2m	
20/03/1996	*stub*	geom. Interpolation on d.f.
11/04/1996	3m	
11/07/1996	*6m*	geom. Interpolation on d.f.
19/06/1996	mar-96	
18/09/1996	giu-96	
18/12/1996	set-96	
13/01/1997	*1 yr*	geom. Interpolation on d.f.
19/03/1997	dic-96	
11/07/1997	*18m*	linear interpolation of swap rates
12/01/1998	2	
13/07/1998	*2,5*	linear interpolation of swap rates
11/01/1999	3	
12/07/1999	*3,5*	linear interpolation of swap rates
11/01/2000	4	
11/07/2000	*4,5*	linear interpolation of swap rates
11/01/2001	5	
11/07/2001	*5,5*	linear interpolation of swap rates
11/01/2002	6	
11/07/2002	*6,5*	linear interpolation of swap rates
13/01/2003	7	
11/07/2003	7,5	linear interpolation of swap rates
12/01/2004	8	linear interpolation of swap rates
12/07/2004	8,5	linear interpolation of swap rates
11/01/2005	9	linear interpolation of swap rates
11/07/2005	9,5	linear interpolation of swap rates
11/01/2006	10	

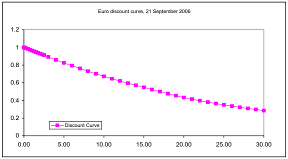
In Italics and gray the dates at which we do not observe a market rate and we need interpolation.

The Bootstrapping (Euro)

Computation of the Euro term structure, 21 September 2006

Trade date 19/09/2006
Value Date 21/09/2006 5

Days	Period	Rate %	FRA Rate	Stub date	Date	Day Week	Date	gg	Basis	yr	Discount Factor
Market Data								Time to maturity			
	spot				21/09/2006	5	21/09/2006			0.00	1
1	o/n	3.04188			22/09/2006	6	22/09/2006	1	2	0.00	0.999916
7	1w	3.06275			28/09/2006	5	28/09/2006	7	2	0.02	0.999405
	2w	3.06563			06/10/2006	6	06/10/2006	15	2	0.04	0.998724
33	1m	3.19325			21/10/2006	7	23/10/2006	32	2	0.09	0.997170
60	2m	3.2935			21/11/2006	3	21/11/2006	61	2	0.17	0.994450
69	Stub				20/12/2006	4	20/12/2006	90	2	0.25	0.991649
	3m	3.37025			21/03/2007	4	21/03/2007	181	2	0.50	0.982583
91.00	Dec-06	96.35	0.0365	0.990858	20/06/2007	4	20/06/2007	272	2	0.76	0.973430
91.00	Mar-07	96.28	0.0372	0.9906843	19/09/2007	4	19/09/2007	363	2	1.01	0.964362
91.00	Jun-07	96.28	0.0372	0.9906843	19/12/2007	4	19/12/2007	454	2	1.26	0.955474
91.00	Sep-07	96.32	0.0368	0.9907835	19/03/2008	4	19/03/2008	545	2	1.51	0.946751
91.00	Dec-07	96.355	0.03645	0.9908704	18/06/2008	4	18/06/2008	636	2	1.77	0.938178
91.00	Mar-08	96.385	0.03615	0.9909448	17/09/2008	4	17/09/2008	727	2	2.02	0.929706
91.00	Jun-08	96.395	0.03605	0.9909697	17/12/2008	4	17/12/2008	818	2	2.27	0.921287
91.00	Sep-08	96.385	0.03615	0.9909448	18/03/2009	4	18/03/2009	909	2	2.53	0.912899
91.00	Dec-08	96.365	0.03635	0.9908952	21/09/2009	2	21/09/2009	1096	2	3.04	0.891846
	1	3.808			21/09/2010	3	21/09/2010	1461	4	4.00	0.8581942
	2	3.88			21/09/2011	4	21/09/2011	1826	4	5.00	0.8254424
	3	3.883			21/09/2012	6	21/09/2012	2192	4	6.00	0.7934503
	4	3.892			21/09/2013	7	23/09/2013	2559	4	7.01	0.7620475
	5	3.906			21/09/2014	1	22/09/2014	2923	4	8.00	0.7314983
	6	3.925			21/09/2015	2	21/09/2015	3287	4	9.00	0.7016125
	7	3.947			21/09/2016	4	21/09/2016	3653	4	10.00	0.672432
	8	3.973			21/09/2017	5	21/09/2017	4018	4	11.00	0.6457647
	9	4.001			21/09/2018	6	21/09/2018	4383	4	12.00	0.6200421
	10	4.029			21/09/2019	7	23/09/2019	4750	4	13.01	0.595104
	11				21/09/2020	2	21/09/2020	5114	4	14.00	0.5713062
	12	3.999			21/09/2021	3	21/09/2021	5479	4	15.00	0.5482334
	13				21/09/2022	4	21/09/2022	5844	4	16.00	0.5237722
	14				21/09/2023	5	21/09/2023	6209	4	17.00	0.5000053
	15	4.066			21/09/2024	7	23/09/2024	6577	4	18.01	0.4768183
	16				21/09/2025	1	22/09/2025	6941	4	19.00	0.4544669
	17				21/09/2026	2	21/09/2026	7305	4	20.00	0.4327778
	18				21/09/2027	3	21/09/2027	7670	4	21.00	0.4148985
	19				21/09/2028	5	21/09/2028	8036	4	22.00	0.3977158
	20	4.202			21/09/2029	6	21/09/2029	8401	4	23.00	0.3812031
	21				21/09/2030	7	23/09/2030	8768	4	24.01	0.3652527
	22				21/09/2031	1	22/09/2031	9132	4	25.00	0.3500469
	23				21/09/2032	3	21/09/2032	9497	4	26.00	0.3360381
	24				21/09/2033	4	21/09/2033	9862	4	27.00	0.3225635
	25	4.218			21/09/2034	5	21/09/2034	10227	4	28.00	0.3096389
	26				21/09/2035	6	21/09/2035	10592	4	29.00	0.2972417
	27				21/09/2036	1	22/09/2036	10959	4	30.00	0.2853184
	28										
	29										
	30	4.214									



Calculation of discount factors from swap rates

	Period	Date	Basis (30/360)	Time to mat.	Interpolation Dates prior	Interpolation Dates next	Disc. Factor	Accrual Factor	Annuity	Swap Rate
	spot	21/09/2006	4	0						
6	1	21/09/2007	4	1	19/09/2007	19/12/2007	0.964161	1.0000	0.964161	3.7171%
2	2	22/09/2008	4	2.0027778	17/09/2008	17/12/2008	0.929214	1.0028	1.895956	3.7335%
2	3	21/09/2009	4	3			0.891846	0.9972	2.785325	3.8830%
3	4	21/09/2010	4	4			0.858194	1.0000	3.643519	3.8920%
4	5	21/09/2011	4	5			0.825442	1.0000	4.468961	3.9060%
6	6	21/09/2012	4	6			0.793450	1.0000	5.262412	3.9250%
2	7	23/09/2013	4	7.0055556			0.762047	1.0056	6.028693	3.9470%
2	8	22/09/2014	4	8.0027778			0.731498	0.9972	6.758159	3.9730%
2	9	21/09/2015	4	9			0.701613	0.9972	7.457823	4.0010%
4	10	21/09/2016	4	10			0.672432	1.0000	8.130255	4.0290%
5	11	21/09/2017	4	11			0.645765	1.0000	8.77602	4.0364%
6	12	21/09/2018	4	12			0.620042	1.0000	9.396062	4.0438%
2	13	23/09/2019	4	13.005556			0.595104	1.0056	9.994472	4.0512%
2	14	21/09/2020	4	14			0.571306	0.9944	10.5626	4.0586%
3	15	21/09/2021	4	15			0.548233	1.0000	11.11084	4.0660%
4	16	21/09/2022	4	16			0.523772	1.0000	11.63461	4.0932%
5	17	21/09/2023	4	17			0.500005	1.0000	12.13461	4.1204%
2	18	23/09/2024	4	18.005556			0.476818	1.0056	12.61408	4.1476%
2	19	22/09/2025	4	19.002778			0.454467	0.9972	13.06729	4.1748%
2	20	21/09/2026	4	20			0.432778	0.9972	13.49886	4.2020%
3	21	21/09/2027	4	21			0.414899	1.0000	13.91376	4.2052%
5	22	21/09/2028	4	22			0.397716	1.0000	14.31148	4.2084%
6	23	21/09/2029	4	23			0.381203	1.0000	14.69268	4.2116%
2	24	23/09/2030	4	24.005556			0.365253	1.0056	15.05996	4.2148%
2	25	22/09/2031	4	25.002778			0.350047	0.9972	15.40904	4.2180%
3	26	21/09/2032	4	26			0.336038	0.9972	15.74414	4.2172%
4	27	21/09/2033	4	27			0.322563	1.0000	16.0667	4.2164%
5	28	21/09/2034	4	28			0.309639	1.0000	16.37634	4.2156%
6	29	21/09/2035	4	29			0.297242	1.0000	16.67358	4.2148%
2	30	22/09/2036	4	30.002778			0.285318	1.0028	16.9597	4.2140%

Calculation of stub discount factor

Period	Rate	Date	N. Days	Weight	Discount factor
2m	3.2935	21/11/2006	61	0.03333	0.994450318
3m	3.37025	21/12/2006	91	0.96667	0.991552721
Stub date		20/12/2006	90		0.991649172

=(D63-D64)/(D63-D62)

=(F62^E62)*(F63^(1-E62))

The Bootstrapping (Yen)

Computation of the JPY yield curve, 9 January 1996

Source: James and Webber, Interest Rate Modelling, pp 130-133

Trade date 09/01/1996
Value Date 11/01/1996 5

Market Data					Date	Day Week	Date	Time to maturity		Discount Factor		
Days	Period	Rate %						gg	yr			
	spot				11/01/1996	5	11/01/1996			1		
1	o/n	0,49			12/01/1996	6	12/01/1996	1	0,00	0,999986		
7	1w	0,5			18/01/1996	5	18/01/1996	7	0,02	0,999903		
33	1m	0,53		????	13/02/1996	3	13/02/1996	33	0,09	0,999514		
60	2m	0,55			11/03/1996	2	11/03/1996	60	0,17	0,999084		
69	stub		FRA Rate	***Stub date***	19/03/1996	3	20/03/1996	69	0,19	0,998940		3rd wed of the month
91	mar-96	99,34	0,0066	0,9983344	19/06/1996	4	19/06/1996	160	0,44	0,997276		3rd wed of the month
91	giu-96	99,25	0,0075	0,9981078	19/09/1996	5	18/09/1996	251	0,70	0,995389		3rd wed of the month
91	set-96	99,1	0,009	0,9977302	19/12/1996	5	18/12/1996	342	0,95	0,993129		3rd wed of the month
91	dic-96	98,9	0,011	0,9972272	19/03/1997	4	19/03/1997	433	1,20	0,990376		3rd wed of the month
	18m				11/07/1997	6	11/07/1997	547	1,50	0,9858533		
	2	1,14			11/01/1998	1	12/01/1998	732	2,01	0,9774097		
	2,5				11/07/1998	7	13/07/1998	914	2,50	0,9662513		
	3	1,6			11/01/1999	2	11/01/1999	1096	3,00	0,9529825		
	3,5				11/07/1999	1	12/07/1999	1278	3,50	0,9380052		
	4	2,04			11/01/2000	3	11/01/2000	1461	4,00	0,9210904	5	
	4,5				11/07/2000	3	11/07/2000	1643	4,50	0,9034789	2	
	5	2,43			11/01/2001	5	11/01/2001	1827	5,01	0,8842259		
	5,5				11/07/2001	4	11/07/2001	2008	5,50	0,8662562		
	6				11/01/2002	6	11/01/2002	2192	6,01	0,8471097		
	6,5				11/07/2002	5	11/07/2002	2373	6,50	0,8272069		
	7	3,01			11/01/2003	7	13/01/2003	2559	7,01	0,8060973		
	7,5				11/07/2003	6	11/07/2003	2738	7,50	0,7904453		
	8				11/01/2004	1	12/01/2004	2923	8,01	0,7741924		
	8,5				11/07/2004	1	12/07/2004	3105	8,51	0,7579424		
	9				11/01/2005	3	11/01/2005	3288	9,01	0,7414522		
	9,5				11/07/2005	2	11/07/2005	3469	9,50	0,724933		
	10	3,36			11/01/2006	4	11/01/2006	3653	10,01	0,7080797		

Calculation of 6 monthly spot rates

Period	Date	Time to mat.	Interpolation Dates prior	Interpolation Dates next	Disc. Factor	Annuity	Swap Rate
spot	11/01/1996	0					
6m	11/07/1996	0,50	19/06/1996	18/09/1996	0,996819	0,497044	0,006399
1yr	13/01/1997	1,01	18/12/1996	19/03/1997	0,992342	1,002731	0,007637
18m	11/07/1997	1,50			0,985853	1,486204	0,009519
2	12/01/1998	2,01			0,977410	1,981603	0,0114
2,5	13/07/1998	2,50			0,966251	2,463405	0,0137
3	11/01/1999	3,00			0,952983	2,938591	0,016
3,5	12/07/1999	3,50			0,938005	3,406309	0,0182
4	11/01/2000	4,00			0,921090	3,868116	0,0204
4,5	11/07/2000	4,50			0,903479	4,318618	0,02235
5	11/01/2001	5,01			0,884226	4,764364	0,0243
5,5	11/07/2001	5,50			0,866256	5,193933	0,02575
6	11/01/2002	6,01			0,847110	5,620969	0,0272
6,5	11/07/2002	6,50			0,827207	6,031173	0,02865
7	13/01/2003	7,01			0,806097	6,441951	0,0301
7,5	11/07/2003	7,50			0,790445	6,829594	0,030683
8	12/01/2004	8,01			0,774192	7,221993	0,031267
8,5	12/07/2004	8,51			0,757942	7,599926	0,03185
9	11/01/2005	9,01			0,741452	7,971668	0,032433
9,5	11/07/2005	9,50			0,724933	8,331155	0,033017
10	11/01/2006	10,01			0,708080	8,688105	0,0336

Calculation of stub discount factor

Period	Rate	Date	N. Days	Weight	Discount factor
2m	0,55	11/03/1996	60	0,70968	0,999084173
3m	0,56	11/04/1996	91	0,29032	0,998586445
Stub date		20/03/1996	69		0,998939646

=(D41-D42)/(D41-D40)

=(F40^E40)*(F41^(1-E40))

Discount Factors from LIBOR rates

Calculating Discount Factors from LIBOR

- $L(t, t+\tau)$ is the Libor rate at time t with tenor τ.
- Libor is bought for 1 at time t and at time $t+\tau$ pays

$$1 + L(t, t+\tau)\,\alpha_{t,t+\tau}$$

  where $\alpha_{t,t+\tau}$ is the day count function in the LIBOR market.
- The day count convention is actual/360, i.e. $\alpha_{t,t+\tau} = \frac{\tau}{360}$

At time t, the discount factor for time τ is:

$$P(t, t+\tau) = \frac{1}{1 + L(t, t+\tau)\,\alpha_{t,t+\tau}}.$$

Table 2: Discount factors from Euro LIBOR

Maturity	τ	$L(t, t+\tau)$	$\alpha_{t,t+\tau}$	$P(t, t+\tau)$
o/n	1	0.030419	$\frac{1}{360}$	$\frac{1}{1+0.030419\,\frac{1}{360}} = 0.999916$
1w	7	0.030628	$\frac{7}{360}$	$\frac{1}{10.030628\,\frac{7}{360}} = 0.999405$
2w	15	0.030656	$\frac{15}{360}$	$\frac{1}{1+0.030656\,\frac{15}{360}} = 0.998724$
1m	32	0.031933	$\frac{32}{360}$	$\frac{1}{1+0.031933\,\frac{32}{360}} = 0.99717$
2m	61	0.032935	$\frac{61}{360}$	$\frac{1}{1+0.032935\,\frac{61}{360}} = 0.99445$
3m	91	0.033703	$\frac{91}{360}$	$\frac{1}{1+0.033703\,\frac{91}{360}} = 0.991553$

Discount Factors from Futures

Calculating Discount Factors from Futures I

- Futures are treated as if they are FRA's: margin payments are ignored.
- On the delivery date they deliver LIBOR at the rate implied by the futures price.

Table 3: Implied Euro FRA rates

Maturity	Futures	FRA rate
20-Dec-06	96.35	$\frac{100-96.35}{100} = 0.0365$
21-Mar-07	96.28	$\frac{100-96.28}{100} = 0.0372$
20-Jun-07	96.28	$\frac{100-96.28}{100} = 0.0372$
19-Sep-07	96.32	$\frac{100-96.32}{100} = 0.0368$
19-Dec-07	96.355	$\frac{100-96.355}{100} = 0.03645$
19-Mar-08	96.385	$\frac{100-96.385}{100} = 0.03615$
18-Jun-08	96.395	$\frac{100-96.395}{100} = 0.03605$
17-Sep-08	96.385	$\frac{100-96.385}{100} = 0.03615$
17-Dec-08	96.365	$\frac{100-96.365}{100} = 0.03635$

Stub date

- To calculate discount factors, we need a discount factor for the first expiry date, i.e. December 20, 2006.
- This is called the **stub date**.
- We geometrically interpolate between discount factors obtained from 2m and 3m LIBOR.
- This is equivalent to a linear interpolation on (annually compounded) spot rates.

Interpolation for the stub date

Maturity	τ	$P(t, t+\tau)$
2m	61	0.994450318
20-Dec-06	90	$(0.994450318)^{w}(0.991552721)^{1-w}$
3m	91	0.991552721

where w solves:

$$61w + 91(1-w) = 90,$$

i.e.:

$$w = \frac{91-90}{91-61} = 0.03333.$$

- The discount factor for the stub date is found by geometric interpolation:

$$(0.994450318)^{0.03333}(0.991552721)^{1-0.03333} = 0.991649172.$$

Obtaining Discount Factors from futures

- The discount factors for the futures contract maturity are then obtained using the forward discount factor $P_{t,t+\tau_1,t+\tau_1+\tau_2}$ by the relationship:

$$P_{t,t+\tau_1+\tau_2} = P_{t,t+\tau_1} P_{t,t+\tau_1,t+\tau_1+\tau_2}.$$

- $P_{t,t+\tau_1,t+\tau_1+\tau_2}$ can be expressed in terms of the FRA rate, so we have:

$$P_{t,t+\tau_1+\tau_2} = P_{t,t+\tau_1} \frac{1}{1 + F_{t,t+\tau_1,t+\tau_1+\tau_2} \alpha_{t+\tau_1,t+\tau_1+\tau_2}}.$$

  where $F\left(t, t+\tau_1, t+\tau_1+\tau_2\right)$ is the FRA rate and $\alpha_{t+\tau_1,t+\tau_1+\tau_2}$ is the day count convention.

- For FRA's:

$$\alpha_{t+\tau_1,t+\tau_1+\tau_2} = \frac{91}{360}.$$

Discount Factors from Futures

- Now we can recursively compute the discount factors for date $t + \tau_1 + \tau_2$:

Starting Date $t+\tau_1$	Ending Date $t+\tau_2$	$P_{t,t+\tau_1}$	$\times$	$P_{t,t+\tau_1,t+\tau_1+\tau_2}$	$=$	$P_{t,t+\tau_1+\tau_2}$
20-Dec-06	*21-Mar-07*	0.991649	$\times$	0.990858	$=$	0.982583
21-Mar-07	*20-Jun-07*	0.982583	$\times$	0.990684	$=$	0.97343
20-Jun-07	*19-Sep-07*	0.97343	$\times$	0.990684	$=$	0.964362
19-Sep-07	*19-Dec-07*	0.964362	$\times$	0.990784	$=$	0.955474
19-Dec-07	*19-Mar-08*	0.955474	$\times$	0.99087	$=$	0.946751
19-Mar-08	*18-Jun-08*	0.946751	$\times$	0.990945	$=$	0.938178
18-Jun-08	*17-Sep-08*	0.938178	$\times$	0.99097	$=$	0.929706
17-Sep-08	*17-Dec-08*	0.929706	$\times$	0.990945	$=$	0.921287
17-Dec-08	*19-Mar-09*	0.921287	$\times$	0.990895	$=$	0.912899

Discount Factors from Swaps

Calculating Discount Factors from Swaps

- Some care is due to the cashflows timing.
- Euro swaps have (fixed) cashflows at yearly intervals so we need discount factors at twelve-month intervals.
- We need three stages:
  1. Interpolate for the 1y discount factor using futures quotations, and then compute the 1y swap rates.
  2. Interpolate for the 2y discount factor using futures quotations, and then compute the 2y swap rates.
  3. Interpolate swap rates at missing maturities (e.g. 11y, 13y,...) and compute the corresponding discount factors.

Interpolating the 1y discount factors

- For computing the 1y d.f., occurring on Friday the 21-Sep-2007, we use the d.f. for date 19-Sep-2007 (2 days) and 19-Dec-2007 (89 days).
- The weight factor is then:

$$w = \frac{89}{89+2} = 0.978021978.$$

- The interpolated 1y discount factor is then:

$$0.964362^{0.978021978} 0.955474^{0.021978022} = 0.964166.$$

Interpolating for 2y discount factors

- For interpolating the 2y df, occurring on Monday the 22-Sep-2008, we use the d.f.s for date 17-Sep-2006 (5 days) and 17-Dec-08 (86 days).
- The weight factor is

$$w = \frac{86}{91} = 0.945054945.$$

- The interpolated 2y discount factor is then:

$$0.929706^{0.945054945} 0.921287^{0.054945055} = 0.929241.$$

Computing the 1y and the 2y swap rates

Par swap rate is calculated from discount factor using the formula:

$$S(t, T_n) = \frac{1 - P(t, T_n)}{\sum_{i=1}^{N} \alpha_{T_{i-1}, T_i} P(t, T_i)},$$

and the convention is $\alpha_{T_{i-1}, T_i} = \mathrm{ACT}/365$.

- For the 1y swap rate, we have:

$$S\left(t, t + \frac{365}{365}\right) = \frac{1 - P\left(t, \frac{365}{365}\right)}{\frac{365}{365} P\left(t, \frac{365}{365}\right)} = \frac{1 - 0.964166}{\frac{365}{365} 0.964161} = 0.037166.$$

- For the 2y swap rate, we have:

$$S\left(t, t + \frac{732}{365}\right) = \frac{1 - P\left(t, \frac{732}{365}\right)}{\frac{365}{365} P\left(t, \frac{365}{365}\right) + \frac{367}{365} P\left(t, \frac{732}{365}\right)} = \frac{1 - 0.929241}{\frac{365}{365} 0.037166 + \frac{367}{365} 0.929241}$$
$$= 0.037320.$$

Calculating Discount Factors from Swap

The remaining discount factors are obtained by recursion:

$$P(t, T_i) = \frac{1 - S(t, T_i) \sum_{j=1}^{i-1} P(t, T_j) \alpha_{T_{j-1}, Tj}}{1 + S(t, T_i) \alpha_{T_{i-1}, T_i}}$$

- Whenever the swap rate is missing we compute it using linear interpolation.

The procedure

	N	O	P	Q	R	S	T	U	V	W
26	**Calculation of discount factors from swap rates**									
27	Period	Date	Basis	Time to mat.	Interpolation Dates		Disc. Factor	Accrual Factor	Annuity	Swap Rate
28			(30/360)		prior	next				
29	spot	21/09/2006	4	0						
30	1	21/09/2007	4	1	19/09/2007	19/12/2007	0.9641655	1.0000	0.964166	3.7166%
31	2	22/09/2008	4	2.0027778	17/09/2008	17/12/2008	0.9292410	1.0028	1.895988	3.7320%
32	3	21/09/2009	4	3			0.8918447	0.9972	2.785355	3.8830%
33	4	21/09/2010	4	4			0.8581931	1.0000	3.643548	3.8920%
34	5	21/09/2011	4	5			0.8254413	1.0000	4.468989	3.9060%
35	6	21/09/2012	4	6			0.7934493	1.0000	5.262439	3.9250%
36	7	23/09/2013	4	7.0055556			0.7620465	1.0056	6.028719	3.9470%
37	8	22/09/2014	4	8.0027778			0.7314973	0.9972	6.758184	3.9730%
38	9	21/09/2015	4	9			0.7016115	0.9972	7.457847	4.0010%
39	10	21/09/2016	4	10			0.6724311	1.0000	8.130278	4.0290%
40	11	21/09/2017	4	11			0.6476538	1.0000	8.777932	4.0140%
41	12	21/09/2018	4	12			0.6240161	1.0000	9.401948	3.9990%
42	13	23/09/2019	4	13.005556			0.5977455	1.0056	10.00301	4.0213%
43	14	21/09/2020	4	14			0.5724905	0.9944	10.57232	4.0437%
44	15	21/09/2021	4	15			0.5478536	1.0000	11.12018	4.0660%
45	16	21/09/2022	4	16			0.5234049	1.0000	11.64358	4.0932%
46	17	21/09/2023	4	17			0.4996502	1.0000	12.14323	4.1204%
47	18	23/09/2024	4	18.005556			0.4764752	1.0056	12.62236	4.1476%
48	19	22/09/2025	4	19.002778			0.4541353	0.9972	13.07523	4.1748%
49	20	21/09/2026	4	20			0.4324575	0.9972	13.50649	4.2020%
50	21	21/09/2027	4	21			0.4145909	1.0000	13.92108	4.2052%
51	22	21/09/2028	4	22			0.3974204	1.0000	14.3185	4.2084%
52	23	21/09/2029	4	23			0.3809194	1.0000	14.69942	4.2116%
53	24	23/09/2030	4	24.005556			0.3649804	1.0056	15.06642	4.2148%
54	25	22/09/2031	4	25.002778			0.3497853	0.9972	15.41524	4.2180%
55	26	21/09/2032	4	26			0.3357871	0.9972	15.75009	4.2172%
56	27	21/09/2033	4	27			0.3223227	1.0000	16.07241	4.2164%
57	28	21/09/2034	4	28			0.3094079	1.0000	16.38182	4.2156%
58	29	21/09/2035	4	29			0.2970201	1.0000	16.67884	4.2148%
59	30	22/09/2036	4	30.002778			0.2851058	1.0028	16.96474	4.2140%

A note on geometric (or exponential) and linear interpolation

- The simplest approach is to consider linear interpolation on the discount factors.
- This choice forgets that the discount function looks like an exponential function $e^{-r\tau}$: it is more convenient to have exponential interpolation of the discount factors.

Exponential interpolation of discount factors

$$P(t, t+\tau) = P(t, t+\tau_1)^{\frac{(\tau_2-\tau)}{(\tau_2-\tau_1)}} P(t, t+\tau_2)^{\frac{(\tau-\tau_1)}{(\tau_2-\tau_1)}}.$$

- Another possibility is linearly interpolating spot rates (LIBOR or continuously compounded ones).

Linear interpolation of spot rates

$$L(t, t+\tau) = L(t, t+\tau_1) \times \frac{(\tau_2-\tau)}{(\tau_2-\tau_1)} + L(t, t+\tau_2) \times \frac{(\tau-\tau_1)}{(\tau_2-\tau_1)}.$$

Remarks

Problems with the method

- The bootstrapping procedure is based on rates from three different sources that do not have a common underlying curve.
- This results in steps in the yield curve when rates derived from LIBOR give over to rates derived from futures, and when rates from futures give over to those from swaps.
- There are also credit considerations. Swaps have less default than LIBOR, since the cashflows are less than those for a LIBOR transaction: e.g. the principal is not exchanged.
- The method uses linear interpolation which produces characteristics problems with forward rates calculated from the curve that is very ragged and sensitive to slight variations in prices
- Futures are treated as forward contracts.

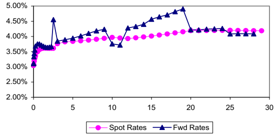
The Euro discount curve

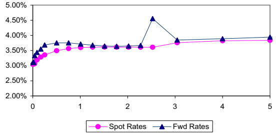
	F	G	H	I	J	K	L
6	Date	Day Week	Date	Time to maturity			Discount Factor
7				gg	Basis	yr	
8	21/09/2006	5	21/09/2006			0.00	1
9	22/09/2006	6	22/09/2006	1	2	0.00	0.999916
10	28/09/2006	5	28/09/2006	7	2	0.02	0.999405
11	05/10/2006	5	05/10/2006	14	2	0.04	0.998809
12	21/10/2006	7	23/10/2006	32	2	0.09	0.997170
13	21/11/2006	3	21/11/2006	61	2	0.17	0.994450
14	20/12/2006	4	20/12/2006	90	2	0.25	0.991649
15	21/03/2007	4	21/03/2007	181	2	0.50	0.982583
16	20/06/2007	4	20/06/2007	272	2	0.76	0.973430
17	19/09/2007	4	19/09/2007	363	2	1.01	0.964362
18	19/12/2007	4	19/12/2007	454	2	1.26	0.955474
19	19/03/2008	4	19/03/2008	545	2	1.51	0.946751
20	18/06/2008	4	18/06/2008	636	2	1.77	0.938178
21	17/09/2008	4	17/09/2008	727	2	2.02	0.929706
22	17/12/2008	4	17/12/2008	818	2	2.27	0.921287
23	18/03/2009	4	18/03/2009	909	2	2.53	0.912899
24	21/09/2009	2	21/09/2009	1096	2	3.04	0.891845
25	21/09/2010	3	21/09/2010	1461	4	4.00	0.8581931
26	21/09/2011	4	21/09/2011	1826	4	5.00	0.8254413
27	21/09/2012	6	21/09/2012	2192	4	6.00	0.7934493
28	21/09/2013	7	23/09/2013	2559	4	7.01	0.7620465
29	21/09/2014	1	22/09/2014	2923	4	8.00	0.7314973
30	21/09/2015	2	21/09/2015	3287	4	9.00	0.7016115
31	21/09/2016	4	21/09/2016	3653	4	10.00	0.6724311
32	21/09/2017	5	21/09/2017	4018	4	11.00	0.6476538
33	21/09/2018	6	21/09/2018	4383	4	12.00	0.6240161
34	21/09/2019	7	23/09/2019	4750	4	13.01	0.5977455
35	21/09/2020	2	21/09/2020	5114	4	14.00	0.5724905
36	21/09/2021	3	21/09/2021	5479	4	15.00	0.5478536
37	21/09/2022	4	21/09/2022	5844	4	16.00	0.5234049
38	21/09/2023	5	21/09/2023	6209	4	17.00	0.4996502
39	21/09/2024	7	23/09/2024	6577	4	18.01	0.4764752
40	21/09/2025	1	22/09/2025	6941	4	19.00	0.4541353
41	21/09/2026	2	21/09/2026	7305	4	20.00	0.4324575
42	21/09/2027	3	21/09/2027	7670	4	21.00	0.4145909
43	21/09/2028	5	21/09/2028	8036	4	22.00	0.3974204
44	21/09/2029	6	21/09/2029	8401	4	23.00	0.3809194
45	21/09/2030	7	23/09/2030	8768	4	24.01	0.3649804
46	21/09/2031	1	22/09/2031	9132	4	25.00	0.3497853
47	21/09/2032	3	21/09/2032	9497	4	26.00	0.3357871
48	21/09/2033	4	21/09/2033	9862	4	27.00	0.3223227
49	21/09/2034	5	21/09/2034	10227	4	28.00	0.3094079
50	21/09/2035	6	21/09/2035	10592	4	29.00	0.2970201
51	21/09/2036	1	22/09/2036	10959	4	30.00	0.2851058

Spot and Forward Curve: Euro market

The term structure of spot and forward interest rates

	Dates	Time To Mat.	Disc. Factor	Spot Rates	Fwd Rates
73	21/09/2006	0	1		
74	22/09/2006	0.003	0.99992	3.0418%	3.1078%
75	28/09/2006	0.019	0.99940	3.0618%	3.1083%
76	05/10/2006	0.039	0.99881	3.0638%	3.3315%
77	23/10/2006	0.089	0.99717	3.1887%	3.4369%
78	21/11/2006	0.169	0.99445	3.2843%	3.5503%
79	20/12/2006	0.250	0.99165	3.3544%	3.6837%
80	21/03/2007	0.503	0.98258	3.4946%	3.7540%
81	20/06/2007	0.756	0.97343	3.5642%	3.7540%
82	19/09/2007	1.008	0.96436	3.5989%	3.7139%
83	19/12/2007	1.261	0.95547	3.6117%	3.6787%
84	19/03/2008	1.514	0.94675	3.6145%	3.6486%
85	18/06/2008	1.767	0.93818	3.6122%	3.6385%
86	17/09/2008	2.019	0.92971	3.6093%	3.6486%
87	17/12/2008	2.272	0.92129	3.6081%	3.6687%
88	18/03/2009	2.525	0.91290	3.6091%	4.5543%
89	21/09/2009	3.044	0.89184	3.7597%	3.8463%
90	21/09/2010	4.000	0.85819	3.8232%	3.8911%
91	21/09/2011	5.000	0.82544	3.8367%	3.9420%
92	21/09/2012	6.000	0.79345	3.8561%	4.0162%
93	23/09/2013	7.006	0.76205	3.8790%	4.1026%
94	22/09/2014	8.003	0.73150	3.9069%	4.1828%
95	21/09/2015	9.000	0.70161	3.9375%	4.2364%
96	21/09/2016	10.000	0.67243	3.9686%	3.7543%
97	21/09/2017	11.000	0.64765	3.9491%	3.7180%
98	21/09/2018	12.000	0.62402	3.9298%	4.2777%
99	23/09/2019	13.006	0.59775	3.9567%	4.3288%
100	21/09/2020	14.000	0.57249	3.9840%	4.3988%
101	21/09/2021	15.000	0.54785	4.0116%	4.5653%
102	21/09/2022	16.000	0.52340	4.0462%	4.6447%
103	21/09/2023	17.000	0.49965	4.0815%	4.7105%
104	23/09/2024	18.006	0.47648	4.1173%	4.8152%
105	22/09/2025	19.003	0.45414	4.1539%	4.9046%
106	21/09/2026	20.000	0.43246	4.1914%	4.2192%
107	21/09/2027	21.000	0.41459	4.1927%	4.2182%
108	21/09/2028	22.000	0.39742	4.1944%	4.2407%
109	21/09/2029	23.000	0.38092	4.1964%	4.2511%
110	23/09/2030	24.006	0.36498	4.1987%	4.2641%
111	22/09/2031	25.003	0.34979	4.2013%	4.0842%
112	21/09/2032	26.000	0.33579	4.1972%	4.0924%
113	21/09/2033	27.000	0.32232	4.1933%	4.0893%
114	21/09/2034	28.000	0.30941	4.1896%	4.0860%
115	21/09/2035	29.000	0.29702	4.1861%	4.0716%
116	22/09/2036	30.003	0.28511	4.1826%	





Bootstrapping the Yen curve

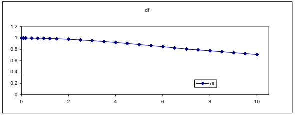
Computation of the JPY term structure, 9 January 1996

Source: James and Webber, Interest Rate Modelling, pp 130-133

Trade date 09/01/1996
Value Date 11/01/1996 5

Row	Days	Period	Rate %	FRA Rate	*Stub date*	Date	Day Week	Date	gg	Basis	yr	df	
8		spot				11/01/1996	5	11/01/1996			0	1	
9	1	o/n	0.49			12/01/1996	6	12/01/1996	1	2	0.00	0.999986	
10	7	1w	0.5			18/01/1996	5	18/01/1996	7	2	0.02	0.999903	
11	33	1m	0.53		????	13/02/1996	3	13/02/1996	33	2	0.09	0.999514	
12	60	2m	0.55			11/03/1996	2	11/03/1996	60	2	0.17	0.999084	
13	69	stub		FRA Rate	*Stub date*	20/03/1996	4	20/03/1996	69	2	0.19	0.998940	3rd wed of the month
14	91	mar-96	99.34	0.0066	0.9983344	19/06/1996	4	19/06/1996	160	2	0.44	0.997276	3rd wed of the month
15	91	giu-96	99.25	0.0075	0.9981078	18/09/1996	4	18/09/1996	251	2	0.70	0.995389	3rd wed of the month
16	91	set-96	99.1	0.009	0.9977302	18/12/1996	4	18/12/1996	342	2	0.95	0.993129	3rd wed of the month
17	91	dic-96	98.9	0.011	0.9972272	19/03/1997	4	19/03/1997	433	2	1.20	0.990376	3rd wed of the month
18		18m				11/07/1997	6	11/07/1997	547	3	1.49863	0.9858533	
19		2	1.14			11/01/1998	1	12/01/1998	732	3	2.01	0.9774097	
20		2.5				11/07/1998	7	13/07/1998	914	3	2.50	0.9662513	
21		3	1.6			11/01/1999	2	11/01/1999	1096	3	3.00	0.9529825	
22		3.5				11/07/1999	1	12/07/1999	1278	3	3.50	0.9380052	
23		4	2.04			11/01/2000	3	11/01/2000	1461	3	4.00	0.9210904	
24		4.5				11/07/2000	3	11/07/2000	1643	3	4.50	0.9034789	
25		5	2.43			11/01/2001	5	11/01/2001	1827	3	5.01	0.8842259	
26		5.5				11/07/2001	4	11/07/2001	2008	3	5.50	0.8662562	
27		6				11/01/2002	6	11/01/2002	2192	3	6.01	0.8471097	
28		6.5				11/07/2002	5	11/07/2002	2373	3	6.50	0.8272069	
29		7	3.01			11/01/2003	7	13/01/2003	2559	3	7.01	0.8060973	
30		7.5				11/07/2003	6	11/07/2003	2738	3	7.50	0.7904453	
31		8				11/01/2004	1	12/01/2004	2923	3	8.01	0.7741924	
32		8.5				11/07/2004	1	12/07/2004	3105	3	8.51	0.7579424	
33		9				11/01/2005	3	11/01/2005	3288	3	9.01	0.7414522	
34		9.5				11/07/2005	2	11/07/2005	3469	3	9.50	0.724933	
35		10	3.36			11/01/2006	4	11/01/2006	3653	3	10.0082	0.7080797	

(Market Data: Days, Period, Rate %; Date, Day Week, Date, Time to maturity: gg, Basis, yr; Discount Factor: df)



Calculation of stub discount factor

	Period	Rate	Date	N. Days	Weight	Discount factor
40	2m	0.55	11/03/1996	60	0.70968	0.999084173
41	3m	0.56	11/04/1996	91	0.29032	0.998586445
42	Stub date		20/03/1996	69		0.998939646

=(D41-D42)/(D41-D40)

=(F40^E40)*(F41^(1-E40))

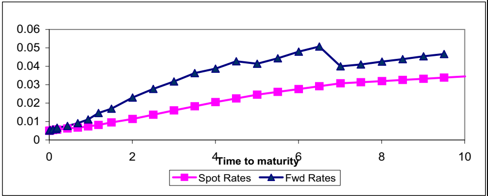
Calculation of 6 monthly spot rates

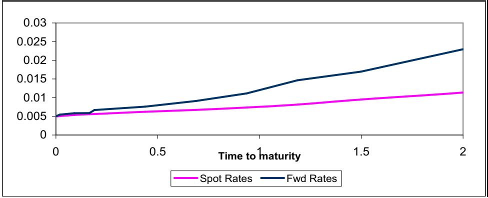
	Period	Date	Basis	Time to mat.	Interpolation Dates prior	Interpolation Dates next	Disc. Factor	Accrual Factor	Annuity	Swap Rate
	spot	11/01/1996	3	0						
5	6m	11/07/1996	3	0.50	19/06/1996	18/09/1996	0.996819	0.4986	0.497044	0.006399
2	1yr	13/01/1997	3	1.01	18/12/1996	19/03/1997	0.992342	0.5096	1.002731	0.007637
6	18m	11/07/1997	3	1.50			0.985853	0.4904	1.486204	0.009519
2	2	12/01/1998	3	2.01			0.977410	0.5068	1.981603	0.0114
2	2.5	13/07/1998	3	2.50			0.966251	0.4986	2.463405	0.0137
2	3	11/01/1999	3	3.00			0.952983	0.4986	2.938591	0.016
2	3.5	12/07/1999	3	3.50			0.938005	0.4986	3.406309	0.0182
3	4	11/01/2000	3	4.00			0.921090	0.5014	3.868116	0.0204
3	4.5	11/07/2000	3	4.50			0.903479	0.4986	4.318618	0.02235
5	5	11/01/2001	3	5.01			0.884226	0.5041	4.764364	0.0243
4	5.5	11/07/2001	3	5.50			0.866256	0.4959	5.193933	0.02575
6	6	11/01/2002	3	6.01			0.847110	0.5041	5.620969	0.0272
5	6.5	11/07/2002	3	6.50			0.827207	0.4959	6.031173	0.02865
2	7	13/01/2003	3	7.01			0.806097	0.5096	6.441951	0.0301
6	7.5	11/07/2003	3	7.50			0.790445	0.4904	6.829594	0.030683
2	8	12/01/2004	3	8.01			0.774192	0.5068	7.221993	0.031267
2	8.5	12/07/2004	3	8.51			0.757942	0.4986	7.599926	0.03185
3	9	11/01/2005	3	9.01			0.741452	0.5014	7.971668	0.032433
2	9.5	11/07/2005	3	9.50			0.724933	0.4959	8.331155	0.033017
4	10	11/01/2006	3	10.0			0.708080	0.5041	8.688105	0.0336

Spot and Forward Curve: Yen market

The term structure of spot and forward interest rates

	Dates	Time To Mat.	Disc. Factor	Spot Rates	Fwd Rates
51	11/01/1996	0	1		
52	12/01/1996	0.00	0.99999	0.00497	0.00509
53	18/01/1996	0.02	0.99990	0.00507	0.00545
54	13/02/1996	0.09	0.99951	0.00537	0.00582
55	11/03/1996	0.16	0.99908	0.00557	0.00587
56	20/03/1996	0.19	0.99894	0.00561	0.00669
57	19/06/1996	0.44	0.99728	0.00622	0.00760
58	18/09/1996	0.69	0.99539	0.00672	0.00911
59	18/12/1996	0.94	0.99313	0.00736	0.01114
60	19/03/1997	1.19	0.99038	0.00815	0.01465
61	11/07/1997	1.50	0.98585	0.00951	0.01697
62	12/01/1998	2.01	0.97741	0.01139	0.02303
63	13/07/1998	2.50	0.96625	0.01371	0.02773
64	11/01/1999	3.00	0.95298	0.01604	0.03177
65	12/07/1999	3.50	0.93801	0.01828	0.03630
66	11/01/2000	4.00	0.92109	0.02054	0.03872
67	11/07/2000	4.50	0.90348	0.02255	0.04273
68	11/01/2001	5.01	0.88423	0.02458	0.04140
69	11/07/2001	5.50	0.86626	0.02610	0.04434
70	11/01/2002	6.01	0.84711	0.02763	0.04794
71	11/07/2002	6.50	0.82721	0.02918	0.05073
72	13/01/2003	7.01	0.80610	0.03074	0.03998
73	11/07/2003	7.50	0.79045	0.03135	0.04099
74	12/01/2004	8.01	0.77419	0.03196	0.04254
75	12/07/2004	8.51	0.75794	0.03258	0.04387
76	11/01/2005	9.01	0.74145	0.03321	0.04544
77	11/07/2005	9.50	0.72493	0.03385	0.04666
78	11/01/2006	10.01	0.70808	0.03449	





Constraining Forward Rates

- We have assumed that the par swap rate at each intermediate coupon date lies on a straight line between the 5 and 10Y rates. This method is named, Linear Swap Rates (LSR).
- A second method, Constant Forward Rates (CFR), constraints the problem by enforcing that all one year forward rates, effective at 11, 12, 13, 14 years, be equal. This method is nowadays fairly standard and it is the simplest market standard methodology. It requires the solution of a simple linear equation.
- Other possibilities consist in constraining Forward Rates to lie on a line or on a parabola.
- An useful discussion can be found at http://www.fincad.com/news/assets/pdfs/dec05/curve_building.pdf

The convexity adjustment

- This adjustment helps in transforming forward rates in futures rates.
- Futures rates are greater than forward rates: if rates rise, then margin payments on the futures contract are received immediately whereas the loss on the FRA will only occur at the maturity date.
- The converse happens if rates fall.
- The convexity adjustment is the amount by which the futures rate need to be decreased and is determined by the market's expectations of future changes in rates.
- The computation of the convexity adjustment requires a term structure model, i.e. some assumption about the dynamics of interest rates.

Application to trades and positions

- The discount rates provide the tools by which we can now analyse deals and dealing positions.
- The value of any deal in a portfolio will be the sum of the present values of each individual cash flow of which that deal comprises,
- However, the discount curve has been computed only at the grid points. In many cases, cash flows will not fall on the node points.
- Some methodology for connecting the nodes to calculate discount factors for all time period is required.
- Linear interpolation is reputed too inaccurate and significant resources need to be allocated to provide an interpolation methodology.

Interpolating the discount curve

Interpolating the discount curve

- Once we have constructed the discount curve at grid points we would like to extend it to some other points. For example the dates where a bond pays its coupons.
- Two schools have been proposed: the parametric and the spline methods.
- The first method models the forward curve by a parametric function. The parameters are fitted using a minimization routine.
- With the spline method, we use a piecewise polynomial function (e.g. a piecewise cubic polynomial), and we join the so-called knot points, where the function and its first derivative are continuous.
- Ioannides (2003) provides a detailed comparison among the different estimation techniques.

Interpolating the discount curve: the parametric approach

A parsimonious representation of the term structure

- Nelson and Siegel (1987) model the instantaneous forward curve $f(t, t+\tau)$ at any maturity τ as follows:

$$f(t, t+\tau) = \beta_0 + (\beta_1 + \beta_2\tau)\exp(-\tau k),$$

where β_0, β_1, β_2 and τ_1 are the parameters to be estimated.

  - β_0 specifies the long rate to which the fwd rate horizontally asymptotically,
  - β_1 is the weight attached to the short term component (spread spread short/long-term),
  - β_2 is the weight attached to the medium term component,
  - k measures the point of the beginning of decay:

- In particular, the short and long end of the curve are related to the NS parameters:

$$\begin{aligned}
\lim_{\tau\to 0} f(t, t+\tau) &= \beta_0 + \beta_1, \\
\lim_{\tau\to \infty} f(t, t+\tau) &= \beta_0.
\end{aligned}$$

The Nelson-Siegel Discount function

- The spot rate is obtained by integrating the eq. above over the time to maturity τ :

$$R\left(t,t+\tau\right)=\frac{\int_0^{\tau} f\left(t,u\right)u}{\tau}=\beta_0+\left(\beta_1+\frac{\beta_2}{k}\right)\frac{1-\exp\left(-\tau k\right)}{\tau k}-\frac{\beta_2}{k}\exp\left(-\tau k\right).$$

- The discount function becomes:

$$P\left(t,t+\tau\right)=\exp\left(-\tau R\left(t,t+\tau\right)\right).$$

Calibrating the Nelson-Siegel curve

- Given the constructed discount curve at the grid points, compute $R_{mkt}(t, t+\tau)$ and choose the parameters that minimizes some distance function between $R_{mkt}(t, t+\tau)$ and $R(t, t+\tau)$:

$$\min_{\beta_0,\beta_1,\beta_2,k} \sum_{i=1}^{n} \left(R(t, t+\tau_i) - R_{mkt}(t, t+\tau_i)\right)^2 .$$

- The fitting could be defined in terms of price rather than yield errors.
- This is the preffered approach when we deal with bond markets data.
- This is the approach followed by the BCE.[3]

[3] BCE, Technical Notes, http://www.ecb.europa.eu/stats/money/yc/html/technical_notes.pdf.

Nelson Siegel vs. Market Curve: Euro Market

Nelson-Siegel Interpolation of the term structure of spot rates

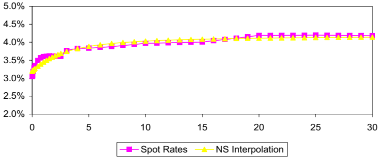
Dates	Time To Mat.	Disc. Factor	Spot Rates	NS Interpolation
21/09/2006	0.000	1.000000		
22/09/2006	0.003	0.999916	3.0418%	3.1968%
28/09/2006	0.019	0.999405	3.0618%	3.2019%
06/10/2006	0.042	0.998724	3.0637%	3.2086%
23/10/2006	0.089	0.997170	3.1887%	3.2226%
21/11/2006	0.169	0.994450	3.2843%	3.2460%
20/12/2006	0.250	0.991649	3.3544%	3.2686%
21/03/2007	0.503	0.982583	3.4946%	3.3350%
20/06/2007	0.756	0.973430	3.5642%	3.3950%
19/09/2007	1.008	0.964362	3.5989%	3.4494%
19/12/2007	1.261	0.955474	3.6117%	3.4987%
19/03/2008	1.514	0.946751	3.6145%	3.5435%
18/06/2008	1.767	0.938178	3.6122%	3.5842%
17/09/2008	2.019	0.929706	3.6093%	3.6213%
17/12/2008	2.272	0.921287	3.6081%	3.6552%
18/03/2009	2.525	0.912899	3.6091%	3.6861%
21/09/2009	3.044	0.891846	3.7597%	3.7417%
21/09/2010	4.000	0.858194	3.8231%	3.8218%
21/09/2011	5.000	0.825442	3.8367%	3.8833%
21/09/2012	6.000	0.793450	3.8561%	3.9293%
23/09/2013	7.006	0.762047	3.8790%	3.9646%
22/09/2014	8.003	0.731498	3.9069%	3.9919%
21/09/2015	9.000	0.701613	3.9375%	4.0136%
21/09/2016	10.000	0.672432	3.9685%	4.0314%
21/09/2017	11.000	0.645765	3.9756%	4.0460%
21/09/2018	12.000	0.620042	3.9831%	4.0583%
23/09/2019	13.006	0.595104	3.9907%	4.0687%
21/09/2020	14.000	0.571306	3.9988%	4.0776%
21/09/2021	15.000	0.548233	4.0070%	4.0853%
21/09/2022	16.000	0.523772	4.0419%	4.0921%
21/09/2023	17.000	0.500005	4.0773%	4.0981%
23/09/2024	18.006	0.476818	4.1133%	4.1035%
22/09/2025	19.003	0.454467	4.1501%	4.1082%
21/09/2026	20.000	0.432778	4.1877%	4.1125%
21/09/2027	21.000	0.414899	4.1891%	4.1164%
21/09/2028	22.000	0.397716	4.1910%	4.1199%
21/09/2029	23.000	0.381203	4.1931%	4.1231%
23/09/2030	24.006	0.365253	4.1956%	4.1261%
22/09/2031	25.003	0.350047	4.1983%	4.1288%
21/09/2032	26.000	0.336038	4.1943%	4.1313%
21/09/2033	27.000	0.322563	4.1906%	4.1336%
21/09/2034	28.000	0.309639	4.1870%	4.1358%
21/09/2035	29.000	0.297242	4.1835%	4.1378%
22/09/2036	30.003	0.285318	4.1801%	4.1396%

=beta0+(beta1+beta2/kappa)*(1-EXP(-kappa*B20))/(kappa*B20)-beta2*EXP(-kappa*B20)/kappa

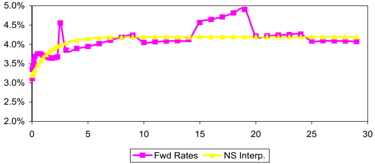
Parameters			
beta0	0.041939	beta2	-3.34E-07
beta1	-0.00998	kappa	0.612566
	f. obj.	2.881E-05	

=SOMMA.Q.DIFF(D5:D47,E5:E47)

$$R(t,t+\tau)=\beta_0+\left(\beta_1+\frac{\beta_2}{k}\right)\frac{1-\exp(-\tau k)}{\tau k}-\frac{\beta_2}{k}\exp(-\tau k).$$



$$f(t,t+\tau)=\beta_0+(\beta_1+\beta_2\tau)\exp(-\tau k)$$



Nelson Siegel in Matlab

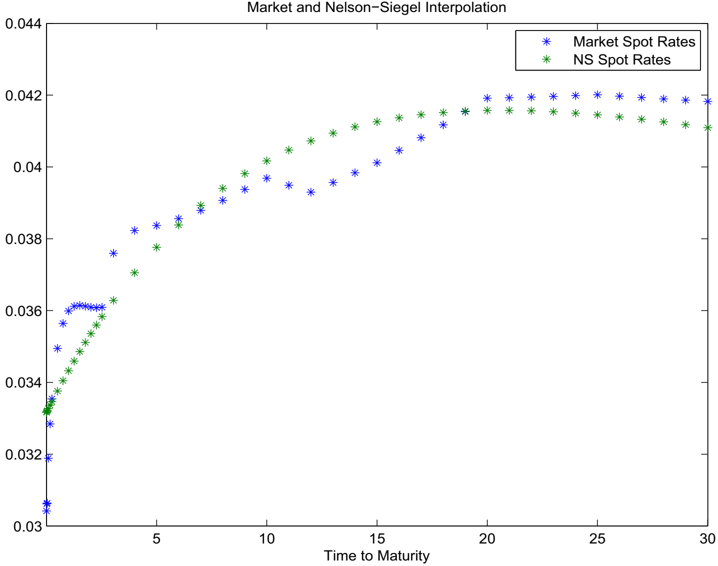
```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%DEFINING THE NS SPOT AND DISCOUNT CURVE%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [ns_spot,ns_df]=ns_curve(parameters,ttm)
beta0=parameters(1);
beta1=parameters(2);
beta2=parameters(3);
kappa=parameters(4);
ns_spot=beta0+(beta1+beta2/kappa)*(1-exp(-kappa.*ttm))./(kappa.*ttm)-...
beta2*exp(-kappa.*ttm)/kappa;
ns_df=exp(-ns_spot.*ttm); %end function

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%DEFINING THE FITTING ERROR %%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [ns_error]=ns_fit(parameters, mktdata)
%mkdata array containing market time to maturity and spot rats
ttm=mktdata(:,1);
mkt_spot=mktdata(:,2);
ns_spot=ns_curve(parameters,ttm);
ns_error=ns_spot-mkt_spot; %end function
```

Nelson Siegel in Matlab

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%FITTING THE NS CURVE%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%READ MARKET DATA FROM EXCEL
filename='StrippingYield2010.xls';
sheetname='NSInterpolation (Euro)'; datarange='B5:D47';
mktcurve=xlsread(filename,sheetname,datarange);
%extract time to maturities and discount factors
mkt_date=mktcurve(:,1); mkt_df=mktcurve(:,2); mkt_spot=mktcurve(:,3);
mktdata=[mktcurve(:,1) mktcurve(:,3)];
%PERFORM NON-LINEAR LEAST SQUARES
[ns_parameter_fit, lsfit] = ...
lsqnonlin(@(x)ns_fit(x,mktdata),[0.04 0.02 0.01 0.001],[0 0 0 0])
%GENERATE A PLOT
h=figure(1);
plot(mktcurve(:,1), [mktcurve(:,3)  ns_curve(ns_parameter_fit ,mktcurve(:,1))],'*')
title('Market and Nelson-Siegel Interpolation')
legend('Market Spot Rates','NS Spot Rates')
xlabel('Time to Maturity')
xlim([mktcurve(1,1) mktcurve(end,1)])
set(h,'Color',[1 1 1])
print(h,'-dpdf','MktvsNS.pdf')
```

Nelson Siegel vs. Market Curve: Euro Market



Parsimonious Models – Pros and Cons

- We can capture as much as possible of the term structure with a few parameters as possible.
- There is a balance between goodness of fit on the one hand and parsimony on the other: they lack of flexibility, i.e. cannot account for all possible shapes of the TS we see in practice.
- Alternative approach like spline models are more flexible, better for pricing but much less parsimonious.

Interpolating the discount curve: the spline method

Cubic Spline

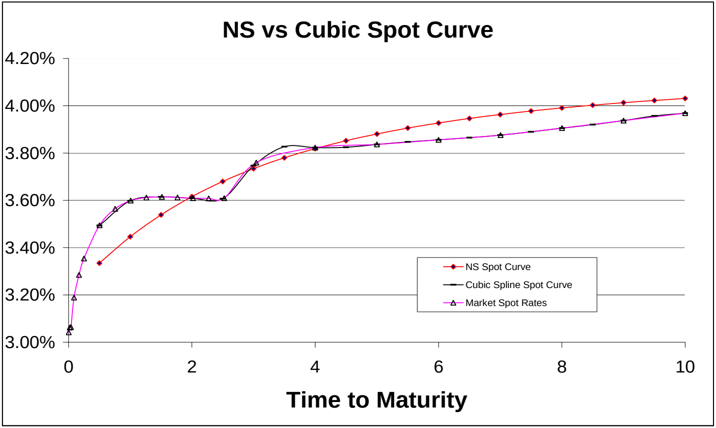
- The cubic spline parameterization was first used by McCulloch (1971) to estimate the nominal term structure. He later added taxes in McCulloch (1975).
- The methodology is described for instance in Fabozzi, Interest rate, term structure, and valuation modeling, from page 158 to page 183.
- Here we just illustrate how to do it in Matlab.
- The following example is available in the *.m files maincubic.m and mycurve.m
- Let us suppose that you have a Matlab function mycurve(), that provide you with data referring to the time to maturity and to the correspond discount factors.

Cubic Spline in Matlab

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%CUBIC SPLINE INTERPOLATION AND BOND PRICING%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Read From Excel bootstrapped data
filename='StrippingYield2010.xls';sheetname='NSInterpolation (Euro)';
datarange='B5:D47';
mktcurve=xlsread(filename,sheetname,datarange);
%extract time to maturities and discount factors
mkt_date=mktcurve(:,1); mkt_df=mktcurve(:,2); mkt_spot=mktcurve(:,3);
%assign dates where to interpolate spot and df
tenor1=0.25; maturity1=5; coupon1=0.05; notional1=1;
cpndates1=[tenor1:tenor1:maturity1];
%a coupon bond paying a 5% coupon semiannually for 10 years
tenor2=0.5; maturity2=10; coupon2=0.05;notional2=1;
cpndates2=[tenor2:tenor2:maturity2];

interpchoice='spline'
%example 1
ci_spot1=interp1(mkt_date, mkt_spot,cpndates1,interpchoice);
ci_df1=exp(-ci_spot1.*cpndates1);
ci_price_1=sum(coupon1*ci_df1)*tenor1*notional1+notional1*ci_df1(end)
%example 2
cpndates=[tenor2:tenor2:maturity2];
ci_spot2=interp1(mkt_date, mkt_spot,cpndates2,interpchoice);
```

Spot Curve: NS vs Cubic



Pricing a Coupon Bond: NS vs Cubic

Nelson_Siegel Interpolation

Pricing a Coupon Bond paying quarterly a 5% coupon. Maturity 5 years.

Cpn dates	Interpolated Spot	DF	Tenor	Coupon	Notional	PV()
0						
0.25	0.032698	0.991859	0.25	0.05		0.012398
0.5	0.033344	0.983466	0.25	0.05		0.012293
0.75	0.03393	0.974873	0.25	0.05		0.012186
1	0.034462	0.966125	0.25	0.05		0.012077
1.25	0.034946	0.957258	0.25	0.05		0.011966
1.5	0.035387	0.948304	0.25	0.05		0.011854
1.75	0.035789	0.93929	0.25	0.05		0.011741
2	0.036156	0.93024	0.25	0.05		0.011628
2.25	0.036492	0.921173	0.25	0.05		0.011515
2.5	0.0368	0.912106	0.25	0.05		0.011401
2.75	0.037082	0.903052	0.25	0.05		0.011288
3	0.037341	0.894024	0.25	0.05		0.011175
3.25	0.037579	0.885032	0.25	0.05		0.011063
3.5	0.037799	0.876083	0.25	0.05		0.010951
3.75	0.038001	0.867184	0.25	0.05		0.01084
4	0.038188	0.858342	0.25	0.05		0.010729
4.25	0.038361	0.849561	0.25	0.05		0.01062
4.5	0.038522	0.840846	0.25	0.05		0.010511
4.75	0.03867	0.832199	0.25	0.05		0.010402
5	0.038809	0.823623	0.25	0.05	1	0.833918
				PV(Cpn Bond)		**1.050556**

Cubic Spline Interpolation

Pricing a Coupon Bond paying quarterly a 5% coupon. Maturity 5 years.

Cpn dates	Interpolated Spot	DF	Tenor	Coupon	Notional	PV()
0						
0.25	0.033544	0.991649	0.25	0.05		0.012396
0.5	0.034936	0.982684	0.25	0.05		0.012284
0.75	0.035631	0.973631	0.25	0.05		0.01217
1	0.035982	0.964658	0.25	0.05		0.012058
1.25	0.036114	0.955861	0.25	0.05		0.011948
1.5	0.036145	0.947226	0.25	0.05		0.01184
1.75	0.036125	0.938739	0.25	0.05		0.011734
2	0.036093	0.930357	0.25	0.05		0.011629
2.25	0.036086	0.922015	0.25	0.05		0.011525
2.5	0.036068	0.913775	0.25	0.05		0.011422
2.75	0.036616	0.90421	0.25	0.05		0.011303
3	0.037464	0.893693	0.25	0.05		0.011171
3.25	0.03804	0.883708	0.25	0.05		0.011046
3.5	0.038268	0.874645	0.25	0.05		0.010933
3.75	0.038286	0.866258	0.25	0.05		0.010828
4	0.038232	0.858193	0.25	0.05		0.010727
4.25	0.038216	0.850085	0.25	0.05		0.010626
4.5	0.038247	0.841887	0.25	0.05		0.010524
4.75	0.038303	0.83365	0.25	0.05		0.010421
5	0.038367	0.825441	0.25	0.05	1	0.835759
				PV(Cpn Bond)		**1.052346**

Pricing a Coupon Bond: NS vs Cubic

Nelson_Siegel Interpolation						
Pricing a Coupon Bond paying semiann a 5% coupon. Maturity 10 years.						
	Interpolated					
Cpn dates	Spot	DF	Tenor	Coupon	Notional	PV()
0		1				
0.5	0.033344	0.983466	0.5	0.05		0.024587
1	0.034462	0.966125	0.5	0.05		0.024153
1.5	0.035387	0.948304	0.5	0.05		0.023708
2	0.036156	0.93024	0.5	0.05		0.023256
2.5	0.0368	0.912106	0.5	0.05		0.022803
3	0.037341	0.894024	0.5	0.05		0.022351
3.5	0.037799	0.876083	0.5	0.05		0.021902
4	0.038188	0.858342	0.5	0.05		0.021459
4.5	0.038522	0.840846	0.5	0.05		0.021021
5	0.038809	0.823623	0.5	0.05		0.020591
5.5	0.039057	0.806692	0.5	0.05		0.020167
6	0.039273	0.790065	0.5	0.05		0.019752
6.5	0.039463	0.773749	0.5	0.05		0.019344
7	0.03963	0.757746	0.5	0.05		0.018944
7.5	0.039777	0.742058	0.5	0.05		0.018551
8	0.039908	0.726682	0.5	0.05		0.018167
8.5	0.040025	0.711617	0.5	0.05		0.01779
9	0.040131	0.696857	0.5	0.05		0.017421
9.5	0.040225	0.682398	0.5	0.05		0.01706
10	0.040311	0.668237	0.5	0.05	1	0.684942
				PV(Cpn Bond)		**1.077968**

Cubic Spline Interpolation						
Pricing a Coupon Bond paying semiann a 5% coupon. Maturity 10 years.						
	Interpolated					
Cpn dates	Spot	DF	Tenor	Coupon	Notional	PV()
0		1				
0.5	0.034936	0.982684	0.5	0.05		0.024567
1	0.035982	0.964658	0.5	0.05		0.024116
1.5	0.036145	0.947226	0.5	0.05		0.023681
2	0.036093	0.930357	0.5	0.05		0.023259
2.5	0.036068	0.913775	0.5	0.05		0.022844
3	0.037464	0.893693	0.5	0.05		0.022342
3.5	0.038268	0.874645	0.5	0.05		0.021866
4	0.038232	0.858193	0.5	0.05		0.021455
4.5	0.038247	0.841887	0.5	0.05		0.021047
5	0.038367	0.825441	0.5	0.05		0.020636
5.5	0.038471	0.809296	0.5	0.05		0.020232
6	0.038561	0.793449	0.5	0.05		0.019836
6.5	0.038666	0.777768	0.5	0.05		0.019444
7	0.038789	0.762219	0.5	0.05		0.019055
7.5	0.038926	0.746807	0.5	0.05		0.01867
8	0.039068	0.731581	0.5	0.05		0.01829
8.5	0.03921	0.716566	0.5	0.05		0.017914
9	0.039375	0.701612	0.5	0.05		0.01754
9.5	0.039568	0.686675	0.5	0.05		0.017167
10	0.039686	0.672431	0.5	0.05	1	0.689242
				PV(Cpn Bond)		**1.083205**

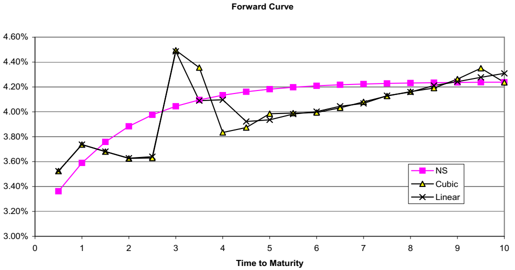
Cubic Spline: limits

- Unfortunately the term structure of forwad rates can have a very irregular shape, as shown in the following example.
- In the book by Fabozzi, possible solutions are discussed.
- See the Kamakuraco.com web site and look for forward curve and smoothness. There is interesting material by Van Deventer.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%MATLAB CODE TO EXTRACT FORWARD RATES%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%COMPARE FWD CURVES
ci_fwdcurve=(ci_df2(1:end-1)./ci_df2(2:end)-1)/tenor2
li_fwdcurve=(li_df2(1:end-1)./li_df2(2:end)-1)/tenor2
ns_fwdcurve=(ns_df2(1:end-1)./ns_df2(2:end)-1)/tenor2
h=figure(2);
plot(cpndates2(2:end), [ci_fwdcurve'  li_fwdcurve' ns_fwdcurve'])
title('Market and Nelson-Siegel Interpolation')
legend('Cubic','Linear','Nelson-Siegel')
xlabel('Time to Maturity')
xlim([cpndates2(2) cpndates2(end)])
set(h,'Color',[1 1 1])
print(h,'-dpdf','FwdCurveCubvsNS.pdf')
```

Forward Rate Curve: NS vs Cubic vs Linear

Time	Tenor	NS	Cubic	Linear
0				
0.5	0.5	3.36%	3.52%	3.52%
1	0.5	3.59%	3.74%	3.74%
1.5	0.5	3.76%	3.68%	3.68%
2	0.5	3.88%	3.63%	3.63%
2.5	0.5	3.98%	3.63%	3.64%
3	0.5	4.05%	4.49%	4.49%
3.5	0.5	4.10%	4.36%	4.09%
4	0.5	4.13%	3.83%	4.10%
4.5	0.5	4.16%	3.87%	3.92%
5	0.5	4.18%	3.98%	3.94%
5.5	0.5	4.20%	3.99%	3.98%
6	0.5	4.21%	3.99%	4.00%
6.5	0.5	4.22%	4.03%	4.04%
7	0.5	4.22%	4.08%	4.07%
7.5	0.5	4.23%	4.13%	4.13%
8	0.5	4.23%	4.16%	4.16%
8.5	0.5	4.23%	4.19%	4.21%
9	0.5	4.24%	4.26%	4.24%
9.5	0.5	4.24%	4.35%	4.28%
10	0.5	4.24%	4.24%	4.31%



Main References

1. Fabozzi F. (ed.), Interest Rate, Term Structure, and Valuation Modeling (Hardcover), Wiley; 1st edition (July 15, 2002).
2. Martellini Lionel, Priaulet Philippe and Priaulet Stéphane, Fixed Income Securities, Wiley Finance, 2003. `Chapter 4`.
3. Adams K. J. and Donald R. van Deventer, 1994, Fitting Yield Curves and Forward Rate Curves with Maximum Smoothness, The Journal of Fixed Income, June 1994, 52-62.
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5. Ioannides M., A comparison of yield curve estimation techniques using UK data, Journal of Banking and Finance, 27, 2003, 1-26.
6. James J. and N. Webber, Interest Rate Modelling, Wiley series in Financial Engineering, 2000.
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Additional useful references

8. Britten-Jones Mark, Fixed Income and Interest Rate Derivative Analysis, Butterworth-Heinemann Finance, 1998.
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Appendix: Computation of the yield curve in the Japanese market

Calculating Discount Factors from LIBOR

- $L(t, t+\tau)$ is the Libor rate at time t with tenor τ.
- Libor is bought for 1 at time t and at time $t+\tau$ pays

$$1 + L(t, t+\tau)\,\alpha_{t,t+\tau}$$

  where $\alpha_{t,t+\tau}$ is the day count function in the LIBOR market.
- The day count convention is actual/360, i.e. $\alpha_{t,t+\tau} = \frac{\tau}{360}$
- At time t,the discount factor for time τ is:

$$P(t, t+\tau) = \frac{1}{1 + L(t, t+\tau)\,\alpha_{t,t+\tau}}$$

Table 2: Discount factors from yen LIBOR

Maturity	τ	$L(t, t+\tau)$	$\alpha_{t,t+\tau}$	$P(t, t+\tau)$
o/n	1	0.0049	$\frac{1}{360}$	$\frac{1}{1+0.0049\frac{1}{360}} = 0.99999$
1w	7	0.0050	$\frac{7}{360}$	$\frac{1}{1+0.0050\frac{71}{360}} = 0.99901$
1m	33	0.0053	$\frac{33}{360}$	$\frac{1}{1+0.0053\frac{33}{360}} = 0.99951$
2m	60	0.0055	$\frac{60}{360}$	$\frac{1}{1+0.0055\frac{60}{360}} = 0.99908$
3m	91	0.0056	$\frac{91}{360}$	$\frac{1}{1+0.0056\frac{91}{360}} = 0.99859$

Calculating Discount Factors from Futures

- Futures are treated as if they are FRA's: margin payments are ignored.
- On the delivery date they deliver LIBOR at the rate implied by the futures price.

Table 3: Implied yen FRA rates

Maturity	Futures	FRA rate
20 Mar 96	99.34	$\frac{100-99.34}{100} = 0.0066$
19 Jun 96	99.25	$\frac{100-99.25}{100} = 0.0075$
18 Sep 96	99.10	$\frac{100-99.10}{100} = 0.009$
18 Dec 96	98.90	$\frac{100-98.90}{100} = 0.011$

- To calculate discount factors, we need a discount factor for the first expiry date, i.e. the 20 March 1996. This is called the **stub date**.
- We geometrically interpolate between discount factors obtained from 2m and 3m LIBOR. This is equivalent to a linear interpolation on (annually compounded) spot rates.

Interpolation for the stub date

Maturity	τ	$P(t, t+\tau)$
2m	60	0.99908
20-Mar-06	69	$(0.99908)^{w}(0.99859)^{1-w}$
3m	91	0.99859

where w solves:

$$60w + 91(1-w) = 69,$$

i.e.:

$$w = \frac{91-69}{91-60} = 0.70968$$

- The discount factor for the stub date is found by geometric interpolation:

$$(0.99908)^{.70968}(0.99859)^{1-.70968} = 0.99894$$

Discount Factors from Futures

- Now we can recursively compute the discount factors for date $t + \tau_1 + \tau_2$:

Start Date $t+\tau_1$	End Date $t+\tau_1+\tau_2$	$P_{t,t+\tau_1}$	Forward Factor	$P_{t,t+\tau_1+\tau_2}$
20 Mar	19/6	0.99894	$\frac{1}{1+0.0066\,\frac{91}{360}}$ $=0.99833$	0.99894*0.99833 $=0.997276$
19 Jun	18/9	0.997276	$\frac{1}{1+0.0075\,\frac{91}{360}}$ $=0.99811$	0.997276*0.99811 $=0.995389$
18 Sep.	18/12	0.995389	$\frac{1}{1+0.009\,\frac{91}{360}}$ $=0.99773$	0.995389*0.99773 $=0.993129$
18 Dec	19/3/97	0.993129	$\frac{1}{1+0.011\,\frac{91}{360}}$ $=0.99723$	0.993129*0.99723 $=0.990376$

Calculating Discount Factors from Swaps

- Some care is due to the cashflows timing.
- Yen swaps have cashflows at six-monthly intervals so we need discount factors at six-monthly intervals.
- We need three stages:
  1. Interpolate for the 6m and 12m discount factors, and then compute the 6m and 12m swap rates.
  2. Interpolate for the 18m swap rate from the 12m and the 2y swap rate, and compute the 18m discount factor.
  3. Interpolate for swap rates $2\frac{1}{2}$, $3\frac{1}{2}$,... at six-monthly intervals from the 2y, 3y, 4y ... swap rates and compute the discount factors.

Interpolating for 6m and 12m discount factors

- For computing the 6m (182 days) d.f., we use the d.f. for date 19/06 (160 days) and the d.f. for date 19/09 (251 days).
- The weight factor is then:

$$182 = w160 + (1 - w)\,251.$$

and we have:

$$w = \frac{69}{91} = 0.75824.$$

- The interpolated 6m discount factor is then:

$$0.997276^{0.75824} 0.995389^{1-0.75824} = 0.99682.$$

Interpolating for 12m discount factors

- For interpolating the 12m (368 days, because 11/01/97 is saturday) d.f., we use the d.f. for date 18/12/96 (69+91+91+91= 342.0) and 19/03/97 (69+91+91+91+91= 433.0).
- The weight factor is
$$w = \frac{433 - 368}{433 - 342} = .71429.$$
- The interpolated 12m discount factor is then:
$$0.993129^{0.71429} 0.990376^{1-0.71429} = 0.99234.$$

Calculating Discount Factors from Swap

- Yen swaps have cashflows at six-monthly intervals so discount factors are needed at six-monthly intervals.
- Proceeds as follows:
  - Interpolate for 6m and 12m discount factors and compute the corresponding swap rates.
  - Linearly interpolate between the 12m and 2 y swap rates to get the 18m swap rate and then the discount factor.
  - Continue linearly interpolating for swap rates 2.5, 3.5, 4.5, etc and then compute the discount factors.
  - The remaining swap rates and discount factors are obtained by recursion:

$$P(t, T_i) = \frac{1 - S(t, T_i) \sum_{j=1}^{i-1} P(t, T_j)\, \alpha_{T_{j-1}, Tj}}{1 + S(t, T_i)\, \alpha_{T_{i-1}, T_i}}$$

Computing the swap rates

- Par swap rate is calculated from discount factor using the formula:

$$S(t, T_n) = \frac{1 - P(t, T_n)}{\sum_{i=1}^{N} \alpha_{T_{i-1}, T_i} P(t, T_i)}.$$

  and the convention is $\alpha_{T_{i-1}, T_i} = \mathrm{ACT}/365$.
- The 6m discount factor is obtained as geometric average

$$P\left(t, \frac{182}{365}\right) = 0.997276^{\frac{79}{91}} 0.995389^{\frac{22}{91}} = 0.996819.$$

- The 12m discount factor is obtained as geometric average

$$P\left(t, \frac{368}{365}\right) = 0.993129^{\frac{65}{91}} 0.990375602^{\frac{26}{91}} = 0.992342.$$

Computing the 6m and the 12m swap rates

- For the 6m swap rate, we have:

$$
\begin{aligned}
S\left(t, t+\frac{182}{365}\right) &= \frac{1-P\left(t, \frac{182}{365}\right)}{\frac{182}{365}P\left(t, \frac{182}{365}\right)} \\
&= \frac{1-0.996819}{\frac{182}{365}0.996819} \\
&= 0.0063978.
\end{aligned}
$$

- For the 12m swap rate, we have:

$$
\begin{aligned}
S\left(t, t+\frac{368}{365}\right) &= \frac{1-P\left(t, \frac{368}{365}\right)}{\frac{182}{365}P\left(t, \frac{182}{365}\right)+\frac{186}{365}P\left(t, \frac{368}{365}\right)} \\
&= \frac{1-0.992342}{\frac{182}{365}0.996819+\frac{186}{365}0.992342} \\
&= 0.0076371.
\end{aligned}
$$

Computing the 18m swap rates

- For the 18M swap rate, we use linear interpolation, i.e.

$$\begin{aligned} S(t, t+18M) &= \frac{1}{2} S(t, t+1Y) + \frac{1}{2} S(t, t+2Y) \\ &= \frac{1}{2}(0.76371\% + 1.14\%) \\ &= 0.95186\%. \end{aligned}$$

- Then $P(t, t+18M)$ is computed using the formula:

$$P(t, T_i) = \frac{1 - S(t, T_i) \sum_{j=1}^{i-1} P(t, T_j) \alpha_{T_{j-1}, Tj}}{1 + S(t, T_i) \alpha_{T_{i-1}, T_i}}. \tag{1}$$

and, given 11/7/97-13/1/97=179 days, we have:

$$P(t, t+18M) = \frac{1 - 0.0095186 * 0.992342 \frac{368}{365}}{1 + \frac{179}{365} 0.0095186} = 0.98587.$$