Stochastic Stability in Monotone Economies

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July 2012

Problem Statement

Consider a stochastic model with dynamics

$$X_{t+1} = F(X_t, \xi_{t+1}) \quad \text{with} \quad X_0 \text{ given} \tag{1}$$

- $X_t \in$ state space S
- $\{\xi_t\}$ IID

Definitions

It will be more convenient to write

$$X_{t+1} = F(X_t, \xi_{t+1}) \tag{2}$$

as

$$X_{t+1} = F_{\xi_{t+1}}(X_t) \tag{3}$$

$$\therefore \quad X_t = F_{\xi_t} \circ \cdots \circ F_{\xi_2} \circ F_{\xi_1}(X_0) \tag{4}$$

Notation: $\psi_t \overset{\mathscr{D}}{=} X_t$

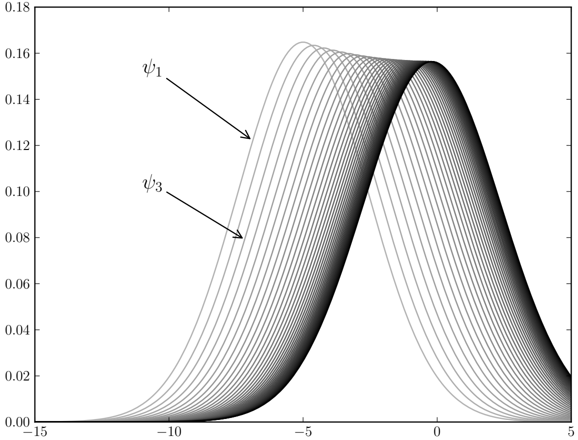


Figure: ψ_t = distribution of X_t

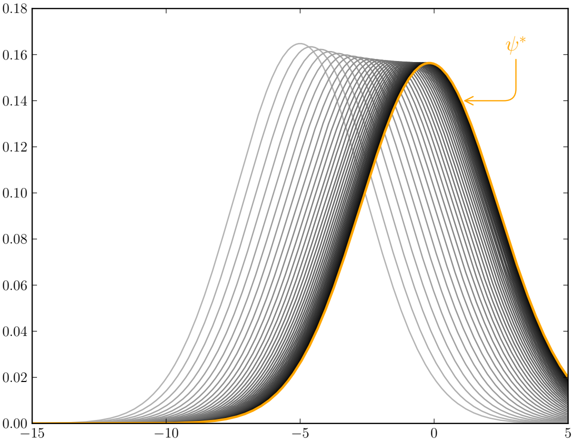
Problem: Is this process stable?

Distribution ψ^* called **stationary** (or **invariant**) for F_ξ if

$$\int \mathbb{P}\{F_\xi(x) \in B\} \psi^*(dx) = \psi^*(B) \qquad \forall\, B \in \text{ Borel sets} \tag{5}$$

Global stability:

- $\exists$ unique stationary ψ^*, and
- $\psi_t \to \psi^*$ independent of X_0



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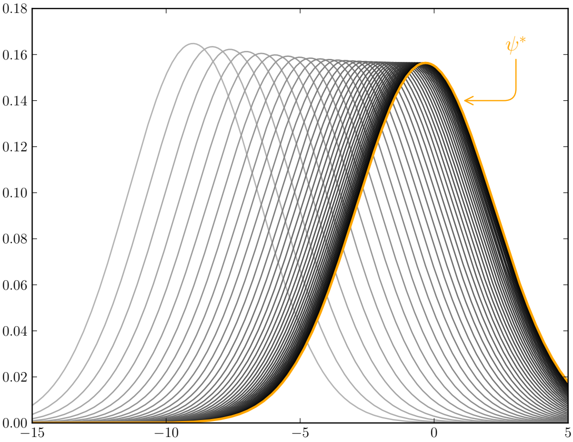
ψ^*



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ψ^*

How to Prove Global Stability?

One approach is Hopenhayn and Prescott (1992, Econometrica)

Used to prove global stability in

- Huggett (1993)
- Aghion and Bolton (1997)
- Cooley and Quadrini (2001)
- Piketty (1997)
- Owen and Weil (1998)
- ...
- ...

We extend their theory

Background to H and P's Result

We say that F_ξ is **increasing** if

$$x \leq x' \implies F_z(x) \leq F_z(x'), \qquad \forall\, z \in Z \tag{6}$$

Example: Random walk

$$x \leq x' \implies x + z \leq x' + z \tag{7}$$

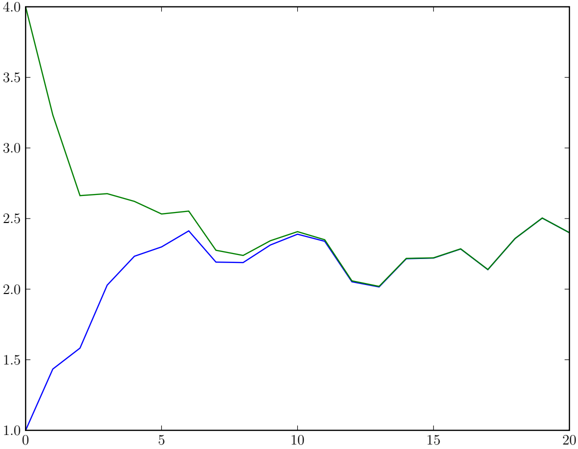
A key property of increasing processes is that for realizations receiving same shocks, initial orderings are preserved:

$$X_0 \leq X_0'$$

$$\therefore \quad F_{\xi_1}(X_0) \leq F_{\xi_1}(X_0')$$

$$\therefore \quad F_{\xi_2} \circ F_{\xi_1}(X_0) \leq F_{\xi_2} \circ F_{\xi_1}(X_0')$$

$$\vdots$$


4.0
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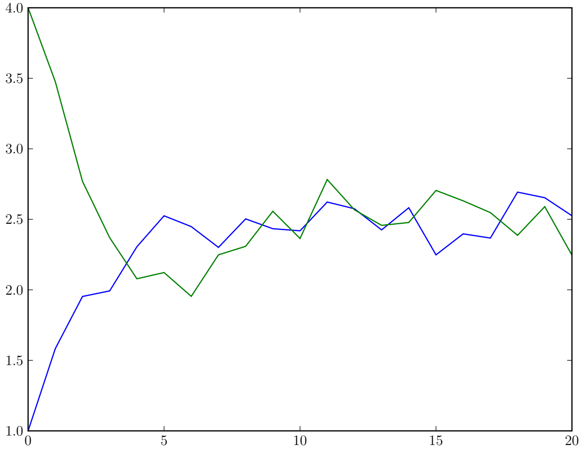
Of course, if they receive <u>different</u> shocks then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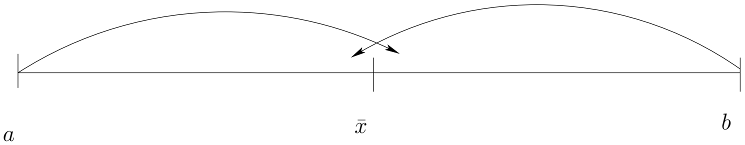
Suppose that $S = [a, b]$

Def. F_ξ satisfies the **MMC** if $\exists\ \bar{x} \in S$ and $k \in \mathbb{N}$ such that

$$\mathbb{P}\{X_k \geq \bar{x} \mid X_0 = a\} > 0 \quad \text{and} \quad \mathbb{P}\{X_k \leq \bar{x} \mid X_0 = b\} > 0$$



Hopenhayn and Prescott's Theorem

Theorem (HP, ECMA, 1992, theorem 2). Let S be a compact metric space with closed partial order $\leq$, least element a and greatest element b. If F_ξ is increasing and satisfies the MMC, then globally stable.

Nice theorem:

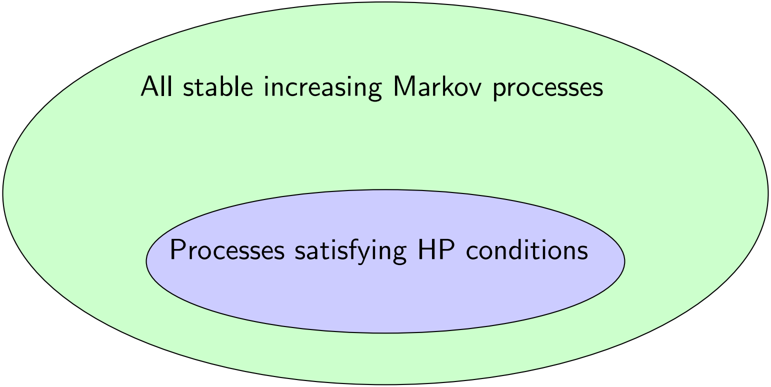
- Conditions are easy to check
- Applies to many models
- No continuity requirement

What's the intuition?

- Divergence ruled out by compactness
- Sufficient mixing guaranteed by MMC

Monotonicity is a side condition that makes the proof work

Room for Extension



Example: Linear Gaussian AR(1)

$$X_{t+1} = \rho X_t + \xi_{t+1} \quad \text{on} \quad S = \mathbb{R} \quad \text{with} \quad 0 \leq \rho < 1 \tag{8}$$

- globally stable
- increasing
- very simple, common process

. . . but does not satisfy HP's conditions

We extend Hopenhayn & Prescott's theorem

Step 1: How to guarantee sufficient mixing?

Mixing Condition 1: Order Mixing

Process called **order mixing** if, given independent copies

$$X_t = F_{\xi_t} \circ \cdots \circ F_{\xi_1}(x) \;\text{ and }\; X'_t = F_{\xi'_t} \circ \cdots \circ F_{\xi'_1}(x')$$

we have

$$\mathbb{P}\{X_t \leq X'_t \text{ for some } t\} = 1$$

- Weaker than MMC for increasing processes

Theorem 1. For increasing processes, order mixing implies global stability whenever a stationary distribution exists.

Sketch of Proof: Given arbitrary ψ_0, we claim that $\psi_t \to \psi^*$

Consider two versions of the process

$$X_t = F_{\xi_t} \circ \cdots \circ F_{\xi_1}(X_0), \qquad X_0 \stackrel{\mathscr{D}}{=} \psi_0 \tag{9}$$

and

$$X_t^* = F_{\xi_t^*} \circ \cdots \circ F_{\xi_1^*}(X_0^*), \qquad X_0^* \stackrel{\mathscr{D}}{=} \psi^* \tag{10}$$

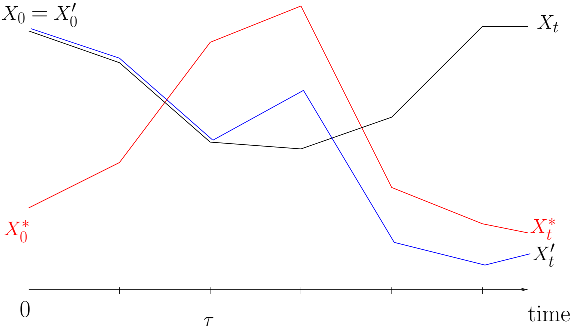
- Note that $X_t \stackrel{\mathscr{D}}{=} \psi_t$ and $X_t^* \stackrel{\mathscr{D}}{=} \psi^*$

Consider also a modified process

$$X'_t = F_{\xi'_t} \circ \cdots \circ F_{\xi'_1}(X_0), \tag{11}$$

where

- $\xi'_t = \xi_t$ until the first time τ such that $X_t \leq X^*_t$
- $\xi'_t = \xi^*_t$ for $t > \tau$



- $X'_t \overset{\mathscr{D}}{=} X_t$ for all t
- $X'_t \leq X^*_t$ with high prob when t large

Loosely speaking,

$$X_t' \leq X_t^* \quad \text{for large } t \text{ with high probability}$$

$$\therefore \quad \psi_t' \preceq \psi^* \text{ in the limit}$$

$$\therefore \quad \psi_t \preceq \psi^* \text{ in the limit}$$

Similar argument gives reverse inequality $\psi^* \preceq \psi_t$.

$$\therefore \quad \psi^* = \psi_t \text{ in the limit}$$

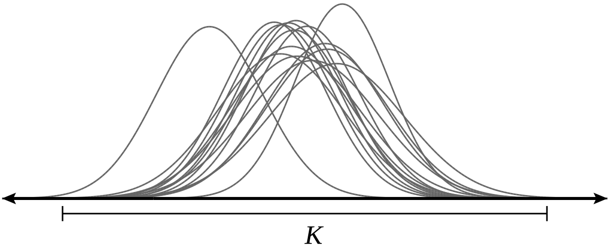
How to get existence?

- Assume process is bounded in probability

Bounded in probability means $\{\psi_t\}_{t\geq 0}$ tight for any X_0

Tightness means

$$\forall\, \epsilon > 0,\ \exists \text{ compact } K \text{ s.t. } \sup_t\ \psi_t(K) \geq 1 - \epsilon$$



The boundedness in probability assumption is:

- weaker than compactness of S
- <u>necessary</u> for global stability

Main Stability Result

Loosely speaking, some details omitted!

Theorem 2. For increasing processes that are bounded in probability, order mixing implies global stability

This result generalizes Hopenhayn and Prescott

In fact order mixing can be replaced by <u>any</u> of the following:

Order reversing: Given independent copies $\{X_t\}$ and $\{X'_t\}$ with $X_0 \geq X'_0$, exists a $k \in \mathbb{N}$ with

$$\mathbb{P}\{X_k \leq X'_k\} > 0$$

Downward reaching:

$$\text{for all } x_0 \text{ and } \bar{x} \in S, \quad \exists\, t \in \mathbb{N} \;\; \text{s.t.} \;\; \mathbb{P}\{X_t \leq \bar{x} \,|\, X_0 = x_0\} > 0$$

Upward reaching:

$$\text{for all } x_0 \text{ and } \bar{x} \in S, \quad \exists\, t \in \mathbb{N} \;\; \text{s.t.} \;\; \mathbb{P}\{X_t \geq \bar{x} \,|\, X_0 = x_0\} > 0$$

Sample Paths

Def. **SLLN** said to hold if

$$\frac{1}{n}\sum_{t=1}^{n} h(X_t) \to \int h\, d\psi^* \quad \text{as } n \to \infty \quad \mathbb{P}\text{-a.s.} \tag{12}$$

Theorem 3. Assume conditions of previous theorem. If h is sufficiently nice, then SLLN holds regardless of X_0.

CLT: On the to-do list

Application

Consider optimal exploitation process

$$\max_{c \in \mathscr{C}} \mathbb{E} \left\{ \sum_{t \geq 0} \beta^t u(c(X_t)) \right\} \quad \text{s.t.} \quad X_{t+1} = f(X_t - c(X_t))\, \xi_{t+1}$$

where

- $\mathscr{C}$ = set of feasible policies
- $\{\xi_t\}_{t \geq 1}$ is IID and lognormal

Let $c^* \in \mathscr{C}$ be optimal, define

- $F_\xi(x) = f(x - c^*(x))\, \xi$ and $S = (0, \infty)$

HP cannot be applied

But with Inada type conditions and concavity of u, process is

- Increasing
- Downward (and upward) reaching
- Bounded in probability

Therefore,

- Process is globally stable
- Stationary density can be computed by simulation (SLLN)

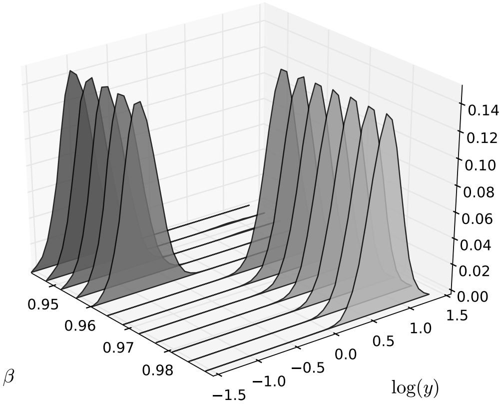


Figure: Stationary distributions