Equilibrium Storage with Multiple Commodities

Kazuo Nishimura and John Stachurski

Institute of Economic Research
Kyoto University

October 2007

Introduction

One-sector commodity pricing model developed by Samuelson (1971) and Schectman and Scheinkman (1983).

Attractive features:

- Equilibrium prices and quantities endogenously determined.
- Intuitive: based on arbitrage and market clearing conditions.
- Elegant representation as a dynamic programming problem.

One limitation: in real world, prices and quantities of given commodity often influenced by prices and quantities of other related commodities.

In fact these prices and quantities are jointly determined.

We wish to model this, by extending the one-sector commodity pricing model to multiple commodities.

Our Model

Market has M commodities.

Demand for commodities comes from

- firms (final producers), who use the commodities as inputs, and
- speculators, who purchase the commodities for future sale.

Vector of spot prices is t by $p_t = (p_t^m)_{m=1}^M \in \mathbb{R}_+^M$.

Risk-free interest rate is r.

Define $\rho := (1+r)^{-1}$.

Firms demand vector C_t according to profit maximization:

$$C_t = \operatorname*{argmax}_{x \geqslant 0} \Pi_t(x), \quad \Pi_t(x) := F(x) - \langle p_t, x \rangle$$

(Price of the final good normalized to one.)

Assumption. Production function $F \colon \mathbb{R}^M_+ \to \mathbb{R}_+$ is strictly concave, strictly increasing, continuous and differentiable on $\mathbb{R}^M_{++}$.

Also use interiority condition for FOCs (see paper).

Solution: $\nabla F(C_t) = p_t$.

Unit mass of identical (risk-neutral) speculators store commodities between periods.

Purchasing I^m units of good m yields $\alpha^m I^m$ units next period, where $\alpha^m \in (0, 1)$.

Letting

$$\Lambda := \mathrm{diag}(\alpha^1, \ldots, \alpha^M) = \begin{pmatrix} \alpha^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha^M \end{pmatrix}$$

we have $(\alpha^m I^m)_{m=1}^M = \Lambda I$.

Supply X_t in the market at time t is the sum of ΛI_{t-1} and a "harvest" W_t:

$$X_t = \Lambda I_{t-1} + W_t$$

Assumption The shocks $(W_t)_{t \geqslant 1}$ are independent, identically distributed $\mathbb{R}^M_+$-valued random vectors with common distribution ϕ.

In addition, $\phi(\partial \mathbb{R}^M_+) = 0$ and

$$\mu := \mathbb{E}\|W_t\| = \int \|z\| \phi(dz) < \infty \tag{1}$$

Since $\phi(\partial \mathbb{R}^M_+) = 0$ and

$$0 \ll W_t \leqslant \Lambda I_{t-1} + W_t = X_t$$

we have

$$\mathbb{P}\{X_t \in \mathbb{R}^M_{++} \text{ for all } t \geqslant 0\} = 1$$

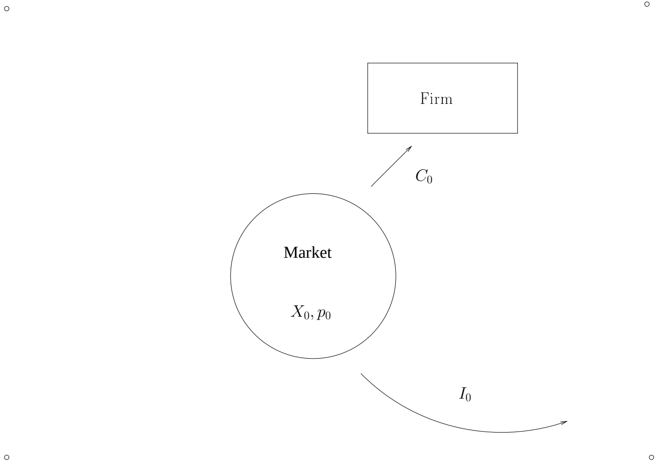

Figure: Market, $t = 0$

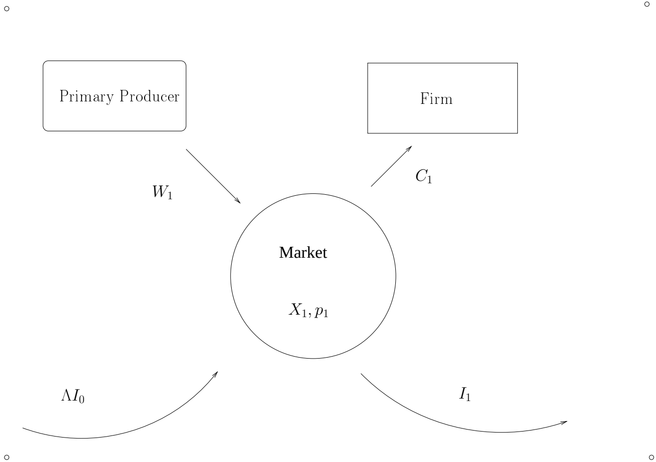

Figure: Market, $t = 1$

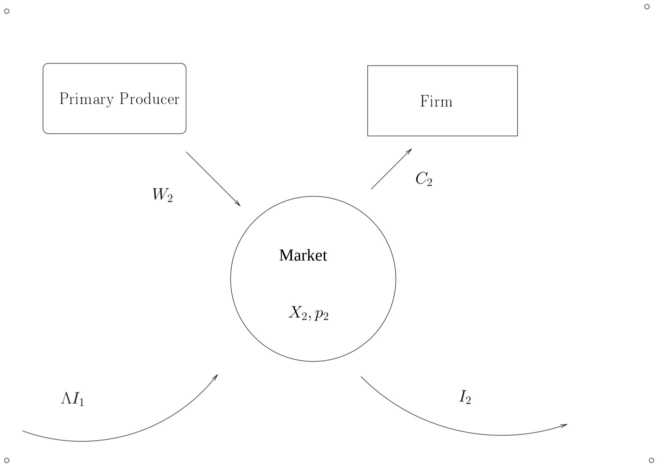

Figure: Market, $t = 2$

Competitive equilibrium

We consider functions c, i and p which map current state into firm demand, speculative investment and spot price respectively.

Given c, i and p, quantities and prices evolve according to

- $C_t = c(X_t)$
- $I_t = i(X_t)$
- $p_t = p(X_t)$
- $X_{t+1} = \Lambda I_t + W_{t+1}$

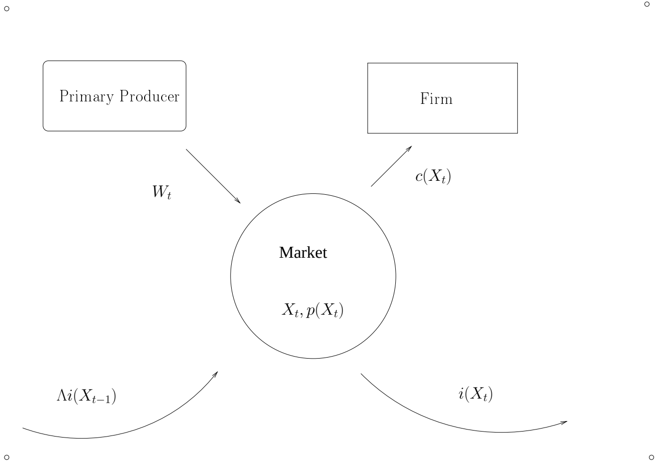

Figure: Market, time t

Defining the equilibrium

Equilibrium defined by three conditions:

- market clearing,
- profit maximization, and
- nonexistence of arbitrage

Market clearing and **profit maximization**:

$$c(x) + i(x) = x \quad \text{and} \quad \nabla F(c(x)) = p(x) \tag{2}$$

Arbitrage condition:

Given $x \in \mathbb{R}^M_+$ and investment policy i, we say that $h \in \mathbb{R}^M$ is a *feasible variation* at x if $i(x) + h \in [0, x]$.

Set of all feasible variations at x is denoted $F^i_\nu(x)$.

Nonexistence of arbitrage requires that i and p satisfy

$$\rho \int \langle p(\Lambda i(x) + z), \Lambda h \rangle \phi(dz) \leqslant \langle p(x), h \rangle \tag{3}$$

for all feasible variations $h \in F^i_\nu(x)$ and all $x \in \mathbb{R}^M_{++}$.

Key questions:

- Do there exist functions i, c and p which satisfy these conditions?
- Are they unique?
- How can we compute them?

To answer these questions we introduce a planning problem.

The Planner's Problem

Planning problem can be stated as

$$\max_{i \in \mathcal{I}} \mathbb{E} \sum_{t \geqslant 0} \rho^t F(X_t - i(X_t)) \tag{4}$$

$$\text{subject to} \quad X_{t+1} = \Lambda i(X_t) + W_{t+1}, \quad X_0 \sim \psi_0 \tag{5}$$

where $\mathcal{I}$ the set of investment policies $i \colon \mathbb{R}_+^M \to \mathbb{R}_+^M$ which are

- Borel measurable and
- satisfy the feasibility constraint $i(x) \leqslant x$.

Next we

- show how to solve this problem for optimal $\mathtt{i}$, and
- prove that this $\mathtt{i}$ yields a competitive equilibrium.

Let v be the value function:

$$v(x) := \sup_{i \in \mathcal{I}} v_i(x), \tag{6}$$

where

$$v_i(x) := \mathbb{E} \sum_{t \geqslant 0} \rho^t F(X_t - i(X_t)) \quad (X_0 = x)$$

Call $i \in \mathcal{I}$ optimal if attains sup in (6) for every $x \in \mathbb{R}^M_+$.

The Bellman operator T is defined by

$$Tw(x) = \sup_{0 \leqslant \xi \leqslant x} \left\{ F(x - \xi) + \rho \int w(\Lambda \xi + z) \phi(dz) \right\} \quad x \in \mathbb{R}^M_+$$

Problems: v is potentially unbounded, classical dynamic programming arguments do not apply.

We use a weighted norm approach, defining $\kappa \colon \mathbb{R}_+^M \to \mathbb{R}$ by

$$\kappa(x) := \text{ details omitted}$$

Say that function w is κ-bounded if

$$\|w\|_\kappa := \|w/\kappa\|_\infty := \sup_{x \in \mathbb{R}_+^M} |w(x)/\kappa(x)| < \infty$$

The function $w \mapsto \|w\|_\kappa$ is a norm on the set of all Borel measurable κ-bounded functions on $\mathbb{R}_+^M$.

This space is a Banach space, T is a contraction in this space.

Theorem

The value function v *is the unique fixed point of* T *in space of B-measurable* κ*-bounded functions, and for each* w *in this space we have* $\|T^n w - v\|_\kappa \to 0$ *as* $n \to \infty$.

In addition, v *is continuous, strictly increasing and strictly concave on* $\mathbb{R}_+^M$. *A unique optimal policy* $I \in \mathcal{I}$ *exists. It is continuous, and, for all* $x \in \mathbb{R}_+^M$,

$$I(x) = \operatorname*{argmax}_{0 \leqslant \xi \leqslant x} \left\{ F(x - \xi) + \rho \int v(\Lambda\xi + z)\phi(dz) \right\}$$

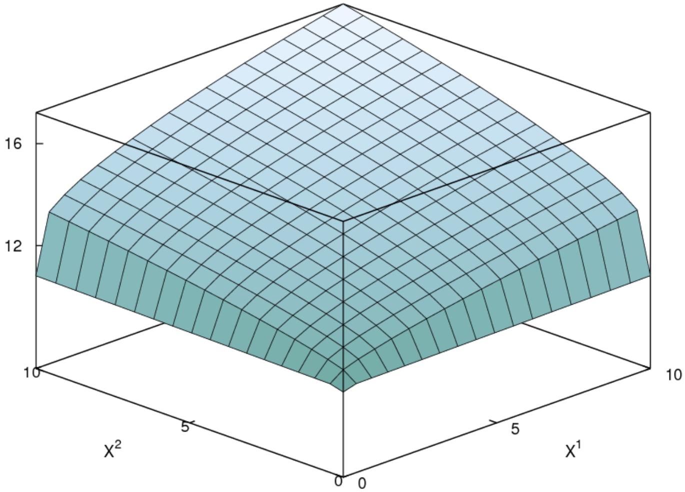

Figure: Value Function

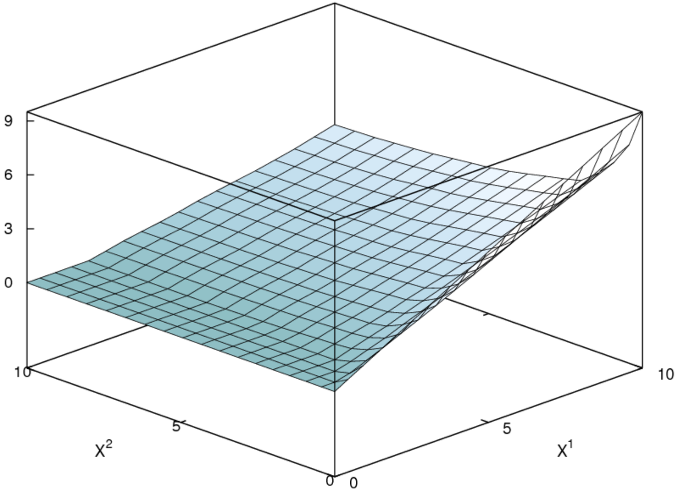

Figure: Investment in Commodity 1

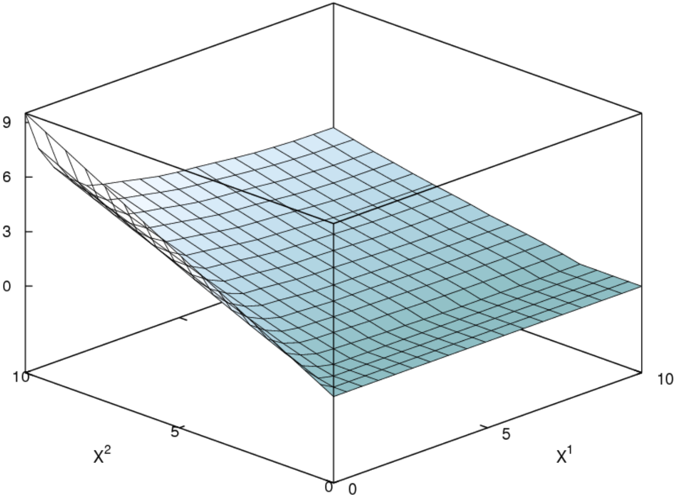

Figure: Investment in Commodity 2

Back to Competitive Equilibrium

Theorem

The policy function I *defined by*

$$\mathrm{I}(x) = \underset{0 \leqslant \xi \leqslant x}{\operatorname{argmax}} \left\{ F(x-\xi) + \rho \int v(\Lambda\xi + z)\phi(dz) \right\}$$

yields a (unique) competitive equilibrium for commodity market.

For proof, first recall the definition of equilibrium:

Market clearing (MC) and profit maximization (PM):

$$c(x) + i(x) = x \quad \text{and} \quad \nabla F(c(x)) = p(x)$$

Arbitrage condition (AC):

$$\rho \int \langle p(\Lambda i(x) + z), \Lambda h \rangle \phi(dz) \leqslant \langle p(x), h \rangle \quad \forall h \in F_{\nu}^{i}(x), \ \forall x$$

Note that if i is given then (MC) and (PM) determine c and p:

$$c(x) = x - i(x) \quad \text{and} \quad p(x) = \nabla F(x - i(x))$$

In summary:

Let I be the optimal policy for the dynamic program.

Let P be defined by $P(x) = \nabla F(x - I(x))$.

We claim that

$$\rho \int \langle P(\Lambda I(x) + z), \Lambda h \rangle \phi(dz) \leqslant \langle P(x), h \rangle \quad \forall h \in F_\nu^I(x), \ \forall x$$

Sketch of proof...

Let f_x be the concave function

$$f_x(\xi) := F(x-\xi) + \rho \int v(\Lambda\xi + z)\phi(dz), \quad \xi \in [0, x]$$

Note: ξ^* maximizes f_x over $[0, x]$ if and only if

$$D_h f_x(\xi^*) \leqslant 0 \text{ for all } h \text{ with } \xi^* + h \in [0, x] \tag{7}$$

The directional derivative can be computed as

$$D_h f_x(\xi^*) = -\langle \nabla F(x-\xi^*), h\rangle + \rho \int \langle \nabla v(\Lambda\xi^* + z), \Lambda h\rangle \phi(dz)$$

Since $I(x)$ maximizes f_x over $[0, x]$ and $I(x) + h \in [0, x]$,

$$\rho \int \langle \nabla v(\Lambda I(x) + z), \Lambda h\rangle \phi(dz) \leqslant \langle \nabla F(x - I(x)), h\rangle$$

Envelope condition $P(x) = \nabla F(x - I(x)) = \nabla v(x)$ gives AC.

Therefore I and its induced firm demand and pricing functions satisfy (MC), (PM) and (AC), so we have a decentralized equilibrium.

Understanding the Planning Problem

The planning problem involves maximization of the firm's discounted revenue stream.

Why does maximizing the firm's discounted revenue stream yield the decentralized equilibrium?

In fact the planner's problem is equivalent to maximizing total surplus in the market.

Total surplus is given by

$$\mathrm{TS} = \mathbb{E} \sum_{t \geqslant 0} \rho^t \Pi_t(C_t) + \mathbb{E} \sum_{t \geqslant 0} \rho^t \langle p_t, W_t \rangle + \mathbb{E} \sum_{t \geqslant 1} \rho^t [\langle p_t, \Lambda I_{t-1} \rangle - \langle p_{t-1}, I_{t-1} \rangle]$$

Consider maximizing TS subject to the constraints

$$X_{t+1} = \Lambda I_t + W_{t+1}, \quad I_t + C_t \leqslant X_t$$

(Assume $W_0 := X_0$ and $\lim_{t \to \infty} \rho^t \langle p_t, I_t \rangle \to 0$.)

Note optimal paths satisfy $X_t = I_t + C_t$ for all $t \geqslant 0$, from which we obtain $I_t + C_t = \Lambda I_{t-1} + W_t$, and hence

$$\langle p_t, I_t \rangle + \langle p_t, C_t \rangle = \langle p_t, \Lambda I_{t-1} \rangle + \langle p_t, W_t \rangle$$

For each $T \in \mathbb{N}$ the sum $\sum_{t=1}^{T} \rho^t [\langle p_t, \Lambda I_{t-1} \rangle - \langle p_{t-1}, I_{t-1} \rangle]$ becomes

$$\sum_{t=1}^{T} \rho^t [\langle p_t, I_t \rangle + \langle p_t, C_t \rangle - \langle p_t, W_t \rangle - \langle p_{t-1}, I_{t-1} \rangle]$$

$$= \sum_{t=1}^{T} \rho^t [\langle p_t, C_t \rangle - \langle p_t, W_t \rangle] + \rho^T \langle p_t, I_t \rangle - \langle p_0, I_0 \rangle$$

Returning to our total surplus, then, for each $T \in \mathbb{N}$ the sum is

$$\sum_{t=0}^{T} \rho^t \Pi_t(C_t) + \sum_{t=0}^{T} \rho^t \langle p_t, W_t \rangle + \sum_{t=1}^{T} \rho^t [\langle p_t, \Lambda I_{t-1} \rangle - \langle p_{t-1}, I_{t-1} \rangle]$$

$$= \sum_{t=0}^{T} \rho^t F(C_t) - \langle p_0, C_0 \rangle + \langle p_0, W_0 \rangle + \rho^T \langle p_T, I_T \rangle - \langle p_0, I_0 \rangle$$

Since $X_0 = W_0$ we have $C_0 + I_0 = W_0$, and hence

$$\sum_{t=0}^{T} \rho^t \Pi_t(C_t) + \sum_{t=0}^{T} \rho^t \langle p_t, W_t \rangle + \sum_{t=1}^{T} \rho^t [\langle p_t, \Lambda I_{t-1} \rangle - \langle p_{t-1}, I_{t-1} \rangle]$$

$$= \sum_{t=0}^{T} \rho^t F(C_t) + \rho^T \langle p_T, I_T \rangle$$

Thus maximizing TS is equivalent to solving

$$\max_{(I_t)_{t \geqslant 0}} \mathbb{E} \sum_{t \geqslant 0} \rho^t F(X_t - I_t)$$

subject to $\quad X_{t+1} = \Lambda I_t + W_{t+1}$

Dynamics

The equilibrium state process obeys

$$X_{t+1} = \Lambda I(X_t) + W_{t+1}, \quad (W_t)_{t \geqslant 0} \overset{\text{IID}}{\sim} \phi$$

What happens to $\mathscr{D} X_t$ as $t \to \infty$?

Assumption. The distribution ϕ of the shock W can be represented by a density, which we again denote by ϕ. The density ϕ is continuous everywhere on $\mathbb{R}_+^M$ and positive on its interior.

Can show that $\mathscr{D}X_t$ represented by density ψ_t, where

$$\psi_{t+1}(y) = \int q(x, y)\psi_t(dx) \qquad y \in \mathbb{R}_+^M, \quad t \in \mathbb{N}$$

where $q(x, y)$ is the conditional density of X_{t+1} when $X_t = x$.

Density ψ^* called stationary if

$$\psi^*(y) = \int q(x, y)\psi^*(dx) \qquad y \in \mathbb{R}_+^M$$

Theorem

The following statements are true:

1. *The process* $(X_t)_{t \geqslant 0}$ *has a unique stationary density* ψ^**.*
2. *If* ψ_0 *is any distribution with* $\int \|x\| \psi_0(dx) < \infty$*, then there is an* $M < \infty$ *and a* $\beta < 1$ *such that,*

$$\forall t \in \mathbb{N}, \quad \sup_{h \in \mathcal{H}_1} \left| \int h(x) \psi_t(x) dx - \int h(x) \psi^*(x) dx \right| \leqslant \beta^t M$$

Now define

$$m_h^* := \int h(x)\psi^*(x)\,dx = \mathbb{E}h(X_0^*)$$

$$v_h^* := \mathbb{E}[h(X_0^*) - m_h^*]^2 + 2\sum_{t\geqslant 1}\mathbb{E}[h(X_0^*) - m_h^*][h(X_t^*) - m_h^*]$$

Corollary

Let ψ_0 *be an arbitrary in initial condition and let* $(X_t)_{t \geqslant 0}$ *be the process starting at* $X_0 \sim \psi_0$. *If* $h \in \mathcal{H}_1$, *then* m_h^* *is finite, and*

$$\frac{1}{n}\sum_{t=1}^{n} h(X_t) \to m_h^* \quad \mathbb{P}\text{-a.s. as } n \to \infty \qquad \text{(LLN)}$$

If, in addition, $h \in \mathcal{H}_2$, *then* v_h^* *is finite, and*

$$\frac{1}{n}\sum_{t=1}^{n} h(X_t) \to N(m_h^*, v_h^*) \quad \text{in distribution as } n \to \infty \qquad \text{(CLT)}$$

We can approximate the stationary density ψ^* by MC

$$\psi_n^*(y) := \frac{1}{n}\sum_{t=1}^{n} q(X_t, y) \to \int q(x, y)\psi^*(x)dx$$

with prob one as $n \to \infty$.

But recall that

$$\psi^*(y) = \int q(x, y)\psi^*(dx) \qquad y \in \mathbb{R}_+^M$$

$$\therefore \quad \psi_n^*(y) \to \psi^*(y)$$

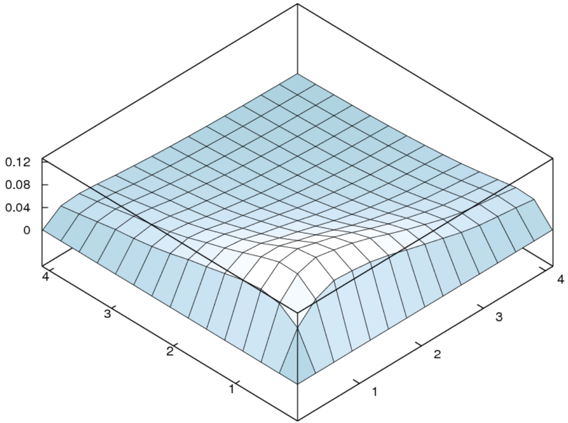

Figure: Approximation ψ_n^* of ψ^*