Nowcasting Report of the US Economy

Macrosynergy Partners LLP

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Abstract

Report of the Nowcasting model at *Macrosynergy Partners LLP* that summarise the forecasts of the US Economy as of May 5, 2017. Furthermore the appendix gives a (currently incomplete) summary of the dynamic factor model used to generate these forecasts.

1 Current Status of the Nowcasting Project

1. The difference between a VW and a Rolls-Royce, the current model is in the VW category

2. In the key report we summarise the key predictions from the model based on the joint co-movement of the variables (and thereby the state of the US economy) as well as an explicit modelling of the idiosyncratic component for each of the series.

2 Main Forecasts

Figure 1: Key Now-Casts and Forecasts

Datasource: Macrobond Financial AB

Table 1: United States: Nowcasting Model output

	2016 Q4			2017 Q1				
	2016M10	2016M11	2016M12	2017M1	2017M2	2017M3	Units	Next Event
Surveys								
Manufacturing, PMI	51.9	53.2	54.7	54.0	53.79	54.64	Index	2017-2-1
Non-Manufacturing, PMI	54.8	57.2	57.2	57.17	54.86	57.25	Index	2017-2-3
Consumer Sentiment Index	87.2	93.8	98.2	95.2	89.48	95.15	Index	2017-1-13
Business Outlook Survey, Manufacturing	9.7	7.6	21.5	17.47	12.25	9.11	Index	2017-1-12
Business Surveys, ISM Chicago	50.6	57.6	54.6	54.58	50.91	57.87	Index	2017-1-31
Consumer Sentiment Index	100.8	109.4	113.7	109.99	104.87	112.04	Index	2017-1-31
Production and Trade								
Retail Sales	6.67	7.01	7.42	5.83	5.09	4.74	QoQ, AR (%)	2017-1-13
Capacity Utilization	75.38	75.0	75.16	75.12	75.23	74.9	%	2017-1-18
Industrial Production	-0.85	-2.28	0.81	1.49	1.84	2.02	QoQ, AR (%)	2017-1-18
Construction Started, Residential	38.18	-26.27	-4.94	0.61	1.97	2.22	QoQ, AR (%)	2017-1-19
Construction by Status	38.63	20.31	13.19	8.67	6.04	4.49	QoQ, AR (%)	2017-1-19
Unfilled Orders, Durable Goods	1.63	1.66	4.93	4.8	4.68	4.57	QoQ, AR (%)	2017-1-27
New Orders, Durable Goods	21.58	2.24	4.05	3.87	3.74	3.62	QoQ, AR (%)	2017-1-27
Manufacturers' Inventories	0.21	0.63	2.35	2.31	2.27	2.23	QoQ, AR (%)	2017-1-27
Labour Market								
Hours Worked, Average Weekly	-1.19	0.0	0.0	-0.01	-0.01	-0.02	QoQ, AR (%)	2017-2-3
Unemployment	4.8	4.6	4.7	4.75	4.7	4.52	%	2017-2-3
Employment, Nonfarm, Payroll	1.43	1.51	1.36	2.33	2.2	2.08	QoQ, AR (%)	2017-2-3
Consumption and Income								
Consumer Price Index	1.63	1.68	1.49	1.76	1.84	1.83	Annual rate (%)	2017-1-18
Personal Consumption Expenditures	2.13	3.1	4.64	3.92	3.5	3.25	QoQ, AR (%)	2017-1-30
Quarterly Series								
Real Gross Domestic Product			3.18			2.78	QoQ, SAAR (%)	2017-1-27

Red numbers indicates a Nowcast, green numbers are forecasts. Datasource: Macrobond Financial AB

Figure 2: Estimated Common Factors

Datasource: Macrobond Financial AB

3 Investigating US Retail Sales

Figure 3: Retail Sales

Datasource: Macrobond Financial AB

Table 2: United States: Retail Sales

Date	μ	x_t	λ	$\hat{\psi}_t = \lambda \hat{x}_t$	$\hat{u}_t$	$y_t^f = \mu + \hat{\psi}_t + \hat{u}_t$	$y_t = y_t^f + \varepsilon_t$
2016-10	4.15	0.14	2.27	0.31	1.83	6.3	6.67
2016-11	4.15	0.77	2.27	1.74	2.79	8.68	7.01
2016-12	4.15	0.49	2.27	1.11	2.57	7.82	NA
2017-1	4.15	0.43	2.27	0.98	1.14	6.27	NA
2017-2	4.15	0.39	2.27	0.87	0.51	5.53	NA
2017-3	4.15	0.34	2.27	0.78	0.23	5.15	NA
2017-4	4.15	0.3	2.27	0.69	0.1	4.94	NA
2017-5	4.15	0.27	2.27	0.61	0.04	4.8	NA
2017-6	4.15	0.24	2.27	0.54	0.02	4.71	NA
2017-7	4.15	0.21	2.27	0.48	0.01	4.64	NA
2017-8	4.15	0.19	2.27	0.43	0.0	4.58	NA
2017-9	4.15	0.17	2.27	0.38	0.0	4.53	NA

Datasource: Macrobond Financial AB

4 The individual fit of the series

Figure 4: Surveys

Datasource: Macrobond Financial AB

Figure 5: Production and Trade

Datasource: Macrobond Financial AB

Figure 6: Labour Market

Datasource: Macrobond Financial AB

Figure 7: Consumption and Income

Datasource: Macrobond Financial AB

5 Extensions

The next steps for the model could be some of the following.

5.1 Work in progress

1. Robustness: Historical performance of the model and benchmarking against alternative forecasters and models
   (a) Survey of Professional Forecasters (SPF) from Philidelphia Fed, NY Fed Nowcast and Atlanta Fed GDPnow
   (b) Various Econometric models (such as the constant growth model: $y_t = \mu + u_t$, AR process: $y_t = \mu + \rho y_{t-1} + u_t$, Local model: $y_t = \mu_t + u_t, \quad \mu_t = \mu_{t-1} + \epsilon_t$)
2. Decompose the forecast into the predictable component (common component and predictable idiosyncratic component) and non-predictable
3. Scenario based forecasting: Condition on the forecast of the next release for a series being higher or lower than the mean forecast of it ($\pm 2st.d.$), how will that shift the forecast for our key variables?
4. Decompose the releases into news and noise, and weight the innovation of the other series in terms of deviation from forecast (similar to what nowcasting.com is doing)

5.2 The next step?

1. Improve on the estimation methodology (Maximum likelihood or Bayesian estimation)
2. Model the trend and Stochastic volatility for the series
3. KloFlow v2: Nowcasting flow of funds?
4. All the infrastructure is scalable for developed economies and (should be) possible to implement for emerging economies

5.3 Transformations of the data

Table 3: Transformation of the data

Code	Model Transformation	Final Transformation
0	x_t	x_t
1	$\log(x_t)$	x_t
2	$\Delta_3 x_t$	x_t
3	$400\Delta_3 \log(x_t)$	$400\Delta_3 \log(x_t)$
4	$100\Delta_3 (\Delta_{12} \log(x_t))$	$100\Delta_{12} \log(x_t)$

References

Juan Antolin-Diaz, Thomas Drechsel, and Ivan Petrella. *Review of Economics and Statistics*, 2016. Forthcoming.

Catherine Doz, Domenico Giannone, and Lucrezia Reichlin. A two-step estimator for large approximate dynamic factor models based on kalman filtering. *Journal of Econometrics*, 164:188–205, 2011.

J. Durbin and S. J. Koopman. *Time Series Analysis by State Space Methods*. Oxford University Press, second edition edition, 2012.

Andrew C. Harvey. *Forecasting, structural time series models and the Kalman filter*. Cambridge Press, 1989.

Roberto S. Mariano and Yasutomo Murasawa. A new coincident index of business cycles based on monthly and quarterly series. *Journal of Applied Econometrics*, 18(4):427–443, 2003.

Appendix A Description of the model

Following Antolin-Diaz et al. (2016) we model the observation (measurement) equation using the assumption of all monthly variables of the quarter are available:

$$y_t^+ = \mu + \Lambda f_t + u_t \tag{1}$$

The state-equation of the common factors are given by a VAR(p_f) model:

$$f_t = \phi_1 f_{t-1} + \phi_2 f_{t-2} + \cdots + \phi_p f_{t-p_f} + \epsilon_t, \quad \epsilon_t \sim N(0, Q). \tag{2}$$

Or re-written:

$$\begin{aligned} (I_n - \Theta(L))\, u_t &= \varepsilon_t, \qquad \varepsilon_t \sim N(0, H) \\ (I_r - \phi(L))\, f_t &= \eta_t, \qquad \eta_t \sim N(0, Q). \end{aligned} \tag{3}$$

The factor loadings (Λ) are not time-invariant and the idiosyncratic components u_t follows an $ma(p_u)$ process.

Following Mariano and Murasawa (2003), we approximate the quarterly variables as unobserved variables with the measurement equation:

$$\underset{n_q.1}{y_t^Q} - \mu^Q = \lambda^Q \widetilde{f}_t + \widetilde{u}_t^Q \tag{4}$$

With the variables given as:

$$\begin{aligned}
\widetilde{f}_t &= \frac{1}{3}f_t + \frac{2}{3}f_{t-1} + f_{t-2} + \frac{2}{3}f_{t-3} + \frac{1}{3}f_{t-4} \\
\widetilde{u}_t^Q &= \frac{1}{3}u_t^Q + \frac{2}{3}u_{t-1}^Q + u_{t-2}^Q + \frac{2}{3}u_{t-3}^Q + \frac{1}{3}u_{t-4}^Q.
\end{aligned}$$

Following Durbin and Koopman (2012) and Harvey (1989) and inverting the idiosyncratic component we can re-write the model in terms of quasi-differences for the monthly variables:

$$\begin{aligned}
\widetilde{y}_t &= CX_t + \tilde{\epsilon}_t, \qquad & \tilde{\epsilon}_t &\sim N(0, \widetilde{H}), \\
X_t &= AX_{t-1} + \tilde{\eta}_t, \qquad & \tilde{\eta}_t &\sim N(0, \widetilde{Q}).
\end{aligned} \tag{5}$$

Where we have written out the monthly observations in terms of quasi-differences (again following Antolin-Diaz et al. (2016)):

$$\underset{n_y.1}{\widetilde{y}_t} = \begin{bmatrix} \left(I_{n_m} - \Theta^{(m)}(L)\right)\left(y_t^{(m)} - \mu^{(m)}\right) \\ y_t^{(q)} - \mu^{(q)} \end{bmatrix}.$$

The state-variable (X_t) is found by first defining the two auxiliary variables: $pp = \max(p_f, p_u + 1, 5)$ and $qq = \max(p_u, 5)$, where p_f and p_u is the lags for the dynamic process of the common components and the idiosyncratic components. Then noting that the dimension of X_t is $n_x = pp \times r + qq \times n_q$ we can set up the state-vector as:

$$\underset{n_x.1}{X_t'} = \begin{bmatrix} \underset{r.1}{f_t}' & \underset{r.1}{f_{t-1}}' & \cdots & \underset{r.1}{f_{t-pp+1}}' & \underset{n_q.1}{u_t^{(q)}}' & \cdots & \underset{n_q.1}{u_{t-qq+1}^{(q)}}' \end{bmatrix}$$

The measurement equation matrix is given by and the variance-covariance of the idiosyncratic component are given by:

$$\underset{n_y.n_x}{C(\Lambda)} = \begin{bmatrix} \underset{n_m.pp\times r}{c_m} & \underset{n_m.(qq\times n_q)}{\mathbf{0}} \\ \underset{n_q.r}{\lambda^q} \times \underset{r.pp\times r}{c_Q^f} & \underset{n_q.(qq\times n_q)}{c_Q^u} \end{bmatrix}$$

The loadings are determined by the matrices:

$$
\begin{aligned}
\underset{r.(p_f-1)r}{c_Q^f} &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{1}{3} & \underset{1.(pp-5)}{\mathbf{0}} \end{bmatrix} \otimes I_r \\
\underset{n_q.qq\times n_q}{c_Q^u} &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{1}{3} & \underset{1.(qq-5)}{\mathbf{0}} \end{bmatrix} \otimes I_{n_q}
\end{aligned} \tag{6}
$$

and for the monthly Variables:

$$
\underset{n_m.pp\times r}{c_m(\Theta^m, \lambda_m)} = \left[\begin{bmatrix} I_{n_m} & \underset{n_m.p_u\times n_m}{-\Theta^m} \end{bmatrix} \times \left(I_{(p_u+1)} \otimes \underset{n_m.r}{\lambda^m} \right) \quad \underset{n_m.(pp-p_u-1)r}{\mathbf{0}} \right] \tag{7}
$$

Where the lag coeffiicents are given by (all diagonal matrices):

$$
\Theta_m = \begin{bmatrix} diagm(\theta_1^m) & diagm(\theta_2^m) & \dots & diagm(\theta_{p_u}^m) \end{bmatrix}
$$

The idiosyncratic component and it's variance covariance in the measurement equation is defined as:

$$
\tilde{\varepsilon}_t' = \begin{bmatrix} \underset{n_m.1}{\varepsilon_t^{(m)'}} & \underset{n_q.1}{\mathbf{0}'} \end{bmatrix}, \quad \widetilde{H} = \begin{bmatrix} \underset{n_m.n_m}{H_m} & \underset{n_m.n_q}{\mathbf{0}} \\ \underset{n_q.n_m}{\mathbf{0}} & \underset{n_q.n_q}{\mathbf{0}} \end{bmatrix}
$$

where the variance-covariance matrix is a diagonal matrix. This could be exploited to reduce the number of parameters of the model.

The system matrix is given by:

$$
\underset{n_x.n_x}{A(\Phi, \Theta^Q)} = \begin{bmatrix} \underset{pp\times r.pp\times r}{a_f(\Phi)} & \underset{pp\times r.qq\times n_q}{\mathbf{0}} \\ \underset{qq\times n_q.pp\times r}{\mathbf{0}} & \underset{qq\times n_q.qq\times n_q}{a_Q\left(\Theta^Q\right)} \end{bmatrix} \tag{8}
$$

With the individual elements given by:

$$
\underset{pp\times r.pp\times r}{a_f} = \begin{bmatrix} \underset{r.p_f\times r}{\Phi} & \underset{r.(pp-p_f)r}{\mathbf{0}} \\ I_{(pp-1)r} & \underset{(pp-1)r.r}{\mathbf{0}} \end{bmatrix}, \quad \underset{r.p_f r}{\Phi} = \begin{bmatrix} \underset{r.r}{\phi_1} & \underset{r.r}{\phi_2} & \dots & \underset{r.r}{\phi_{p_f}} \end{bmatrix}
$$

The quarterly idiosyncratic component:

$$
\underset{qq\times n_q.qq\times n_q}{a_Q} = \begin{bmatrix} \underset{n_q.p_u\times n_q}{\Theta_Q} & \underset{n_q.(qq-p_u)\times n_q}{\mathbf{0}} \\ I_{(qq-1)n_q} & \underset{(qq-1)n_q.n_q}{\mathbf{0}} \end{bmatrix}
$$

And lastly the state innovations are given by:

$$
\underset{n_x.1}{\tilde{\eta}_t}' = \begin{bmatrix} \underset{r.1}{\eta_t'} & \underset{(pp-1)r.1}{\mathbf{0}}' & \underset{n_q.1}{\varepsilon_t^{Q\,\prime}} & \underset{(qq-1)n_q.1}{\mathbf{0}}' \end{bmatrix},
$$

And the variance-covariance matrix of the state are given by:

$$
\underset{n_x.n_x}{\widetilde{Q}} = \begin{bmatrix} \underset{r.r}{Q} & \underset{r.(pp-1)r}{\mathbf{0}} & \underset{r.n_q}{\mathbf{0}} & \underset{r.(qq-1)n_q}{\mathbf{0}} \\ \underset{(pp-1)r.r}{\mathbf{0}} & \underset{(pp-1)r.(pp-1)r}{\mathbf{0}} & \underset{(pp-1)r.n_q}{\mathbf{0}} & \underset{(pp-1)r.(qq-1)n_q}{\mathbf{0}} \\ \underset{n_q.r}{\mathbf{0}} & \underset{n_q.(pp-1)r}{\mathbf{0}} & \underset{n_q.n_q}{H_Q} & \underset{n_q.(qq-1)n_q}{\mathbf{0}} \\ \underset{(qq-1)n_q.r}{\mathbf{0}} & \underset{(qq-1)n_q.(pp-1)r}{\mathbf{0}} & \underset{(qq-1)n_q.n_q}{\mathbf{0}} & \underset{(qq-1)n_q.(qq-1)n_q}{\mathbf{0}} \end{bmatrix}
$$

We check whether the system is observable and controllable following the rank conditions of Harvey (1989, pp.113–117).

Appendix B Kalman filter, Kalman Smoother and the log–likelihood

Firstly following Harvey (1989) we check whether the state-space of the model is controllable and observable.

Add the following derivations and use of:

1. Kalman filter
2. Kalman smoother
3. log–likelihood

First, recall the system:

$$
\begin{aligned}
\widetilde{y}_t &= CX_t + \tilde{\epsilon}_t, & \epsilon_t &\sim N(0, \widetilde{H}), \\
X_t &= AX_{t-1} + \tilde{\eta}_t & \tilde{\eta}_t &\sim N(0, \widetilde{Q}).
\end{aligned} \tag{9}
$$

B.1 The Kalman filter

The following is the algorithm for filtering out (recursively) the unobserved state.

First we define the forecast error of the model as (conditional of the parameters for iteration i):

$$
\widehat{v}_t^{(i)} = \widetilde{y}_t - \widehat{C}^{(i)} \widehat{X}_{t|t-1}^{(i)} \tag{10}
$$

The equations for updating the Kalman filter are given by (need to add missing observation adjustment!):

$$
\begin{aligned}
\underset{n_y.1}{\widehat{v}_t^{(i)}} &= \widetilde{y}_t - \widehat{C}^{(i)} \widehat{X}_{t|t-1}^{(i)} \\
\underset{n_y.n_y}{\widehat{F}_t^{(i)}} &= \widehat{C}^{(i)} P_{t|t-1}^{(i)} \widehat{C}^{(i)\prime} + \widetilde{H}^{(i)} \\
\underset{n_x.1}{\widehat{X}_{t|t}^{(i)}} &= \widehat{X}_{t|t-1}^{(i)} + \widehat{P}_{t|t-1}^{(i)} \widehat{C}^{(i)} \left(\widehat{F}_t^{(i)}\right)^{-1} \widehat{v}_t^{(i)} \\
\underset{n_x.1}{\widehat{X}_{t+1|t}^{(i)}} &= \widehat{A}^{(i)} \widehat{X}_{t|t}^{(i)} \\
\underset{n_x.n_x}{\widehat{P}_{t|t}^{(i)}} &= \widehat{P}_{t|t-1}^{(i)} - \widehat{P}_{t|t-1}^{(i)} \widehat{C}_t^{(i)\prime} \left(\widehat{F}_t^{(i)}\right)^{-1} \widehat{C}^{(i)} \widehat{P}_{t|t-1}^{(i)} \\
\underset{n_x.n_x}{\widehat{P}_{t+1|t}^{(i)}} &= \widehat{A} \widehat{P}_{t|t}^{(i)} \widehat{A}' + \widetilde{Q}
\end{aligned}
$$

We deal with missing values by following Durbin and Koopman (2012) for how to deal with missing values.

B.2 Log-likelihood

B.3 Kalman Smoother

Appendix C Estimation

Following ... we use a two stage-procedure for the estimation, by first estimating the factors using principle components on a subset of the dataset, with a balanced panel and then using projections get the initial parameters of the model.

Then in the second stage we using the EM algorithm and quasi-maximum likelihood to estimate the model.

The basic steps of the estimation procedure is:

1. PCA estimates of initial states and parameters
2. Set up the modle with the initial parameter and starting values
3. Compute an estimate of the unobserved states (conditional of the parameters)
4. Re-estimate the parameters of the model (conditional of the estimates of the states)
5. Repeat step (3) and (4) until convergence – can add additional numerical maximisation step after EM-algorithm has converged to check if it has truly converged.

C.1 Stage (1) Initial conditions and parameters

WE follow Doz et al. (2011) in using a two-stage procedure to estimate the model and augment the last stage with an additional update of the log-likelihood. Using PCA we estimate the factors on a balanced panel of the data, without any missing values.

The balanced panel is chosen from the monthly variable with the following standardisation:

$$\left(\frac{y^m_{t,i} - \mu^m_i}{\sigma^m_i}\right) \quad \forall i = 1, \ldots, n_m$$

Using principal components And the initial parameters are then estimated on the demeaned variables:

$$\begin{aligned}
\tilde{y}^m_t = y^m_t - \mu_m &= \lambda_Q f_t + u^m_t, & u^m_t &= \Theta^m(L)u_t + \varepsilon^m_t, & \varepsilon^m_t &\sim N(\mathbf{0}_{n_m}, H^m) \\
\tilde{y}^Q_t = y^Q_t - \mu_Q &= \lambda_Q f_t + u^Q_t, & u^Q_t &= \Theta^Q(L)u_t + \varepsilon^Q_t, & \varepsilon^Q_t &\sim N(\mathbf{0}_{n_Q}, H^Q)
\end{aligned} \tag{11}$$

And using the OLS projection to get the initial parameters for the monthly variables:

$$\widehat{\lambda}^{(0)}_m = \widetilde{y}^m \widehat{\mathbf{f}}_{PCA} \left(\widehat{\mathbf{f}}'_{PCA} \widehat{\mathbf{f}}_{PCA}\right)^{-1}$$

C.2 Stage (2) Estimation: Maximum Likelihood

Outlining the EM-algorithm for estimating the Maximum Likelihood parameters of the model.

C.3 Step (2.1) Quasi-difference of the series

Quasi-difference of the series conditional on parameters from iteration i

$$\underset{n_y .1}{\widetilde{y}^{(i)}_t} = \begin{bmatrix} \left(I_{n_m} - \Theta^{(i-1)}_m(L)\right)\left(y^{(m)}_t - \mu_m\right) \\ y^{(q)}_t - \mu_q \end{bmatrix}$$

C.4 Step (2.2) Re-estimate the factors ($X^{(i)}_t$)

Estimate states (factors) for iteration i conditional on the parameters from iteration i

$$X^{(i)}_t(\widetilde{y}^{(i)}_t)$$

C.5 Step (2.3) Re-estimate the parameters ($i+1$)

Conditional on the states from iteration $i+1$ re-estimate the parameters

The observations equation for the monthly variables (**OBS: Have to modify**):

$$vec(\lambda^{(i+1)}_m) = \left(\sum_{t=1}^{T} f^{(i)}_t f^{(i)\prime}_t \otimes W^m_t\right)^{-1} vec\left(\sum_{t=1}^{T} W^m_t y^{(m)*}_t f^{(i)\prime}_t\right) \tag{12}$$

Where i refers to the iterations of the EM algorithm.

The updates for the parameters of the quarterly observations are similarly given by:

$$vec(\lambda_q^{(i+1)}) = \left(\sum_{t=1}^{T} \widetilde{f}_t^{(i)} \widetilde{f}_t^{(i)\intercal} \otimes W_t^q \right)^{-1} vec\left(\sum_{t=1}^{T} W_t^q y_t^{(q)*} \widetilde{f}_t^{(i)\intercal} \right) \tag{13}$$

For the transition dynamics of the unobserved factors:

$$\widehat{\Phi}^{(i+1)} = \sum_{t=1}^{T} \left(f_t^{(i)} F_{t-1}^{(i)'} \right) \sum_{t=1}^{T} \left(F_{t-1} F_{t-1}' \right) \tag{14}$$

Where F_t is given by:

$$F_t = \begin{bmatrix} f_t & f_{t-1} & \dots & f_{t-p_f} \end{bmatrix} \tag{15}$$

The residuals are then estimated as:

$$\widehat{\eta}_t^{(i+1)} = f_t^{(i)} - \widehat{\Phi}^{(i+1)} F_{t-1}^{(i)}$$

Which means we can update the estimate of the variance-covariance matrix as (adjusting for the degree of freedoms):

$$\widehat{Q}^{(i+1)} = \frac{1}{T - p_f r} \sum_{t=1}^{T} \left(\eta_t^{(i+1)} \eta_t^{(i+1)'} \right)$$

The parameters for the moving average component of the quarterly idiosyncratic part can be estimated as:

$$\widehat{\Theta}_{Q,j}^{(i+1)} = \sum_{t=1}^{T} \left(u_{t,j}^{Q} U_{t-1,j}^{Q'} \right) \sum_{t=1}^{T} \left(U_{t-1,j}^{Q} U_{t-1,j}^{Q'} \right)^{-1}, \quad \forall j = 1, \dots, n_Q$$

Where the $U_{t-1,j}^{Q}$ variables is defined as:

$$U_{t-1,j}^{Q} = \begin{bmatrix} u_{t-1,j}^{Q} & \dots & u_{t-p_u,j}^{Q} \end{bmatrix}$$

The residuals can then be estimated as:

$$\varepsilon_{t,j}^{Q,(i)} = \widehat{u}_{t,j}^{Q,(i)} - \widehat{\Theta}_{Q,j}^{(i+1)} \widehat{U}_{t,j}^{Q,(i)}$$

And the individual elements of the diagonal variance-covariance matrix can then be estimated as:

$$\sigma_{Q,j}^{(i+1)} = \frac{1}{T-1} \sum_{t=1}^{T} \left(\varepsilon_{t,j}^{Q,(i)} \varepsilon_{t,j}^{Q,(i)'} \right), \quad \forall j = 1, \dots, n_Q$$

C.6 Step (2.4) Check convergence

Check convergence using the following criteria:

$$\bar{L} = \frac{L - L}{L}$$

C.7 Stage (3) Numerical optimisation of log–lik

Using Newtons methods we re-optimise the parameters and states conditional on the previous EM algorithm solution.