VIX Futures pricing : Fair Value and Upper bound
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Abstract

We present a simple formula for computing the fair price of VIX futures by subtracting a correction term from the upper bound given by variance swap rate. The correction term is formulated using volatility of VIX Futures itself and hence is termed "VolVol" term. We present detailed empirical analysis and examples illustrating this formulation.

1 Introduction

The current price for VIX Futures maturing at time T_1 under the risk netural measure is given by

$$F_t^{VIX} = E_t^{\mathbb{Q}} \sqrt{E_{T_1}^{\mathbb{Q}} [\frac{1}{T_2 - T_1} V_{T_1,T_2}]}. \tag{1}$$

The Futures matures at T_1, and quotes the expected volatility for the next 30 days, so $T_2 = T_1 + \frac{30}{365}$. $V_{t,T}$ stands for overall variance (or quadratic variation) of the process between times t and T. For continuous processes $V_{t,T} = \int_t^T \sigma_t^2 dt$.

From concavity of square-root and law of iterated expectations, we have,

$$E_t^{\mathbb{Q}} \sqrt{x} \leq E_t^{\mathbb{Q}} \sqrt{E_{T_1}^{\mathbb{Q}} x} \leq \sqrt{E_t^{\mathbb{Q}} x} \tag{2}$$

for any stochastic variable x.

Hence, a model-free lower bound for VIX futures price is given by

$$LB = E_t^{\mathbb{Q}} \sqrt{\frac{1}{T_2 - T_1} V_{T_1,T_2}} \tag{3}$$

and a model-free upper bound is given by

$$UB = \sqrt{E_t^{\mathbb{Q}}[\frac{1}{T_2 - T_1} V_{T_1,T_2}]} \tag{4}$$

Expected variance of VIX futures price (this follows trivially from equation (2)) at maturity:

$$\begin{aligned}
Var_t^{\mathbb{Q}}(VIX_{T_1}) &\equiv Var_t^{\mathbb{Q}} \sqrt{E_{T_1}^{\mathbb{Q}}[\frac{V_{T_1,T_2}}{T_2 - T_1}]} \\
&= E_t^{\mathbb{Q}}[E_{T_1}^{\mathbb{Q}}[\frac{V_{T_1,T_2}}{T_2 - T_1}]] - (E_t^{\mathbb{Q}}[\sqrt{E_{T_1}^{\mathbb{Q}}[\frac{V_{T_1,T_2}}{T_2 - T_1}]}])^2 \\
&= E_t[\frac{V_{T_1,T_2}}{T_2 - T_1}] - (E_t^{\mathbb{Q}}[\sqrt{E_{T_1}^{\mathbb{Q}}[\frac{V_{T_1,T_2}}{T_2 - T_1}]}])^2 \\
&= UB^2 - (F_{t,T_1}^{VIX})^2
\end{aligned}$$

and the expected variance of the realized volatility between T_1 and T_2, lets denote $RV_{T_1,T_2} = \sqrt{\frac{V_{T_1,T_2}}{T_2 - T_1}}$:

$$Var_t(RV_{T_1,T_2}) = Var_t^{\mathbb{Q}} \sqrt{\frac{V_{T_1,T_2}}{T_2 - T_1}} = UB^2 - LB^2$$

rewriting, we have,

$$(F_{t,T_1}^{VIX})^2 = UB^2 - Var_t(VIX_{T_1})$$

and

$$LB^2 = UB^2 - Var_t(RV_{T_1,T_2})$$

This is the main result. In the next subsection, we present the result differently in terms of lognormal language for a better user interface. This is the way the API would be designed.

1.1 Volvol input

So, we can express both the fair value and the lower value as a correction to the upper bound with a variance term (we call this "volvol" as it stands for variance of VIX and for variance of realized volatility both of which are volatility variables).

1.1.1 Normal Volvol input

In particular, if we define a volatility parameter for VIX, σ_{VIX} as (not to be confused with instantaneous volatility of spot VIX):

$$\sigma_{VIX}(t, T_1) = \sqrt{\frac{Var_t(VIX_{T_1})}{T_1 - t}}$$

then

$$F_t^{VIX} = \sqrt{UB^2 - \sigma_{VIX}^2(t, T_1)(T_1 - t)} \tag{5}$$

Equivalently,

$$F_t^{VIX} = UB\sqrt{1 - \frac{\sigma_{VIX}^2(t, T_1)(T_1 - t)}{UB^2}}$$

1.1.2 Lognormal volvol input

Since the fair price calculation would require volvol as user input, the volvol quoted in relative percentage terms would be a convenient user input as compared to the absolute volvol defined in previous subsection.

We now define an equivalent lognormal volvol parameter from the normal volvol using a conversion formula satisfying:

$$UB^2 - \sigma_{VIX}^2(t, T_1)(T_1 - t) = UB^2 e^{-\sigma_{VIX}^l(t, T_1)^2(T_1 - t)} \tag{6}$$

Rewriting,

$$(\sigma_{VIX}^l(t, T_1))^2(T_1 - t) = -log(1 - \frac{\sigma_{VIX}^2(t, T_1)(T_1 - t)}{UB^2}) \tag{7}$$

The rationale for using this mechanism to define lognormal volatility input is when the variable $V_t = E_t[VIX^2(T_1)]$ (Variance swap rate) is log-normally distributed with lognormal volatility $2\sigma^l_{VIX}$ (since lognormal volatility of VIX is intended to be σ^l_{VIX}) the variance of $VIX(T_1)$ at time t is given by

$$E_t[VIX(T_1)^2] - (E_t[VIX_(T_1)])^2 \tag{8}$$

The distribution of $VIX^2(T_1)$ is lognormal and since its mean is the variance swap rate (=UB), we can write this variable as

$$VIX(T_1)^2 = UB^2 e^{-\frac{(2*\sigma^l)^2}{2}(T_1-t)+2\sigma^l\sqrt{T_1-t}N(0,1)} = UB^2 e^{-(2\sigma^{l^2})(T_1-t)+2\sigma^l\sqrt{T_1-t}N(0,1)}$$

where $N(0,1)$ is the standard normal random variable.

Hence, taking square-roots

$$VIX(T_1) = UBe^{-\sigma^{l^2}(T_1-t)+\sigma^l\sqrt{T_1-t}N(0,1)}$$

So $E_t(VIX(T_1))$ is given by

$$\begin{aligned} E_t[VIX(T_1)] &= UBE_t[e^{-\sigma^{l^2}(T_1-t)+\sigma^l\sqrt{T_1-t}N(0,1)}] \\ &= UBe^{-\frac{\sigma^{l^2}(T_1-t)}{2}}E_t[e^{-\frac{(\sigma^{l^2})(T_1-t)}{2}+\sigma^l\sqrt{T_1-t}N(0,1)}] \\ &= UBe^{-\frac{\sigma^{l^2}(T_1-t)}{2}} \end{aligned}$$

Plugging these results into equation (8), we get

$$Var_t(VIX_{T_1}) = \sigma^2_{VIX}(t,T_1)(T_1-t) = UB^2(1-e^{-\sigma^2_l(T_1-t)})$$

which is the form of our lognormal volatility definition in equation (6)

Plugging the lognormal volatility definition in equation (6) into equation (5), we get,

$$F^{VIX}_{t,T_1} = UBe^{-\frac{\sigma^l_{VIX}(t,T_1)^2(T_1-t)}{2}}$$

Implied Volvol

The above formula can be inverted to give implied volvol:

$$\boxed{\sigma^l_{VIX}(t,T_1) = \sqrt{\frac{-2log(\frac{F^{VIX}_{t,T_1}}{UB})}{T_1-t}}}$$

whenever valid (i.e., $F^{VIX}_{t,T_1} <= UB$), else we return “N.A”.

1.2 Summary

To summarize, the fair value of the futures can be thought of as comprised of two components:

$$FV = IV - TV$$

where IV is the intrinsic value (same as upper bound) and TV is a positive time-value for the futures. We compute the time-value based on a input σ^l_{VIX}, which stands for lognormal volatility of VIX Futures price.

The "volvol" would in general have a term-structure, i.e, would have a different value for the different futures contracts.

The treatment is exactly the same for the lower bound, albeit with a different (higher) volvol value. Now in the next section, we show how to model the two volvol quantities and their relationship. We also present some examples based on model estimations done using historical data.

2 The Upper bound or Intrinsic value computation

The upper bound is given by the forward variance swap rate, which can be formed in a couple of different ways. Firstly, we can construct a synthetic variance swap term structure replicated from the market prices of vanilla options using prices of log contracts in exactly the same way spot VIX is calculated. Secondly, the term structure can be formed using the market prices from the active OTC variance swap market. Recently, CBOE added futures on realized variance whose prices risk-neutrally would equal the forward variance swap as well. The latter market is not active yet and the OTC quotes are not readily available so in our implementation we use the synthetic variance swap rates with an option for the user to input their own term structure.

In the next section illustrate how we form the term structure of variance swap rates.

2.1 Term structure of variance swap rates

Notation: We use $\sigma^2(T)$ to stand for expected variance rate between times 0 and T and $\sigma^2(T_1, T_2)$ to be the expected variance rate between times T_1 and T_2.

The theory of variance swap pricing implies that for any T:

$$\sigma^2(T)T = E_0^{\mathbb{Q}} V_{0,T} = \frac{2}{B_0(T)} \int_0^\infty \frac{Q_0(K,T)}{K^2} dK. \tag{9}$$

Here, $V(0,T)$ stands for the variance swap rate for maturity T, $B_0(T)$ is the price of a zero coupon bond of maturity T – in case of a constant interest rate $B_0(T) = e^{-rT}$. The term $Q_0(K,T)$ stands for out-of-the-money option price with strike K.

CBOE uses the above theoretical foundation for the computation of spot VIX using the strikes available in the market. Please refer to VIX white paper for complete detail, however, here we reproduce CBOE formula for computing expected variance for maturity T,

$$\sigma^2(T) = E_0^{\mathbb{Q}} \frac{V(0,T)}{T} = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}[\frac{F}{K_0} - 1]^2$$

Hence we have a discrete term structure of variances from option prices available in the market.

2.2 Forming forward variances

Computing forward variances from the term structure of variances is basically a problem of interpolation of variance swap rates. An analogous problem is that of computing the forward rates from zero coupon bond prices of different maturities. For the purpose of this TREQ we are just going to do piece-wise linear interpolation in cumulative variance, i.e. $\sigma^2(T)T$. To compute the forward variance from term $[T_1, T_2]$, we first read-off the value $(\sigma^2 T)_1, (\sigma^2 T)_2$ from the cumulative variance piece-wise linear curve and then form the forward variance as,

$$\sigma^2(T_1, T_2) = \frac{(\sigma^2 T)_2 - (\sigma^2 T)_1}{T_2 - T_1}$$

In general for upper bound on VIX futures, we would require interpolating three observed variance swap rates. The VIX futures always matures on a wednesday and there would be an option expiring on a saturday (3 days after futures), and if its a 4 week cycle there would be an option expiring 31 days after futures expiry. The interpolation scheme would want us to also take into account the variance expectation from the 3 days after futures expiry which would need us to look at variance swap rates for maturity before Futures maturity. We first look at the 4 week cycle.

1. **4 week cycle:**

   Note: Since an option is settled on friday prior to saturday expiry, we use $N-1$ day maturity for a N day expiry option. So, we would use $T+30$ and $T+2$ to refer to the maturities of the two options discussed above.

   However, in order to keep the VIX futures upper bound simple, we decide not to use the previous maturity for two reasons.

   (a) There may not be a previous maturity available if we are close to Futures maturity. In this case the variance for those 3 days would need to be extrapolated from the longer maturity variance swap rates.

   (b) We'd like to minimize the number of variance swap rates used in upper bound calculation so as to minimize the inputs on Futures FVD screen.

   Away from the maturity:

   We would use two variance swap rates to find forward variance, in particular, we approximate

   $$\sigma^2(T, T{+}30) \approx \sigma^2(T{+}2, T{+}30) = \frac{(\sigma^2(T+30)(T+30)) - (\sigma^2(T+2)(T+2))}{28} \tag{10}$$

   This formula is valid away from the futures maturity and in particular when $T - t \geq 5$.

   Far Away from the maturity:

   If we are very very far from the futures maturity, and we don't have the next two options maturities of options after the given future then we just use the two relevant options maturities, one on either side, $T_1 < T < T_2$. In particular, denote T_1 as the option of longest existing maturity $\leq T{+}2$. Lets denote T_2 as option of shortest existing maturity $\geq T+30$.

$$\sigma^2(T, T+30) = \frac{\sigma^2(T_2)T_2 - \sigma^2(T_1)T_1}{T_2 - T_1}$$

Close to the maturity: When we are 4 days (or less) away from futures maturity, we actually use a different approximation so make sure at expiry we just get the $(T+30)$ day variance which the equation (10) doesn't give.

The reason we choose 4 days is to be consistent with VIX rules by CBOE to not using short maturity options when they are less than 7 days away (**supposed to be 8 days according to VIX whitepaper, but as confirmed on July 9th, its only 7 days or less when they actually switch**) and 4 days to futures maturity equal exactly 7 days to short option expiry.

We do a transition to patch the equation (10) with the correct value at expiry by using:

$$\sigma^2(T, T+30) \approx \frac{(\sigma^2(T+30)(f(T)+30)) - (\sigma^2(T+2)(f(T)+2))}{28} \tag{11}$$

where we define $f(T)$ in two different ways, denoting a smooth and a jump transition respectively:

(a) $f(T) = T, T-t \geq 4$ and $f(T)$ goes down to -2 as $T-t \to 0$. We can choose just a linear function $f(T) = \frac{3}{2}(T-t) - 2$ when $T - t \leq 4$.

(b) $f(T) = T, T-t \geq 4$ and $f(T) = -2, T-t < 4$. This choice just switches to upper bound as the $T+30$ day variance which infact is exactly true at the expiry.

We can also incorporate the maturity switching as spot VIX does by define the shorter variance using extrapolation from two longer ones. I.e., instead of just reading from available options we just replace $\sigma^2(T+2)$ with

$$\sigma^2(T+2) = \frac{2(\sigma^2(T+30)(T+30)) - (\sigma^2(T+58)(T+58))}{T+2} \tag{12}$$

If above value is returned negative, its set to 0.

If the third expiry is a five week cycle, we'd instead define

$$\sigma^2(T+2) = \frac{1.8(\sigma^2(T+30)(T+30)) - 0.8(\sigma^2(T+65)(T+65))}{T+2} \tag{13}$$

Again, if above value is returned negative, its set to 0.

So at close to the maturity, we'd use the formula from equation 11, with $\sigma^2(T+2)$ input replaced with quantitiy from equation 12 or 13. Hence we'd achieve the switching when spot VIX switches maturities while also making sure that at maturity the futures matures into the variance at $T+30$ maturity.

2. **5 week cycle:**

   In case of 5 week cycle, the spot VIX computation uses maturity strips. I.e., the VIX futures will mature into a linear combination of variances of two expiries, the T+38 day (i.e. T+37 day maturity) and the T+66 day expiry (T+65 day maturity) for options where T is the futures maturity.

   In this case, the formula remains the same, i.e.,

$$\sigma^2(T, T+30) = \frac{(\sigma^2(T+30)(f(T)+30)) - (\sigma^2(T+2)(f(T)+2))}{28}$$

   Away from the maturity: All of T+2, T+37 and T+65 maturities are assumed available (if not, then we would apply the formulae given in far away from maturity section). In the RHS of above formula, $\sigma^2(T+30)$ is computed via the CBOE weighted average

$$\begin{aligned}(T+30)\sigma^2(T+30) &= (T+37)\sigma^2(T+37)\frac{(T+65)-(T+30)}{(T+65)-(T+37)} \\ &+ (T+65)\sigma^2(T+65)\frac{(T+30)-(T+37)}{(T+65)-(T+37)}\end{aligned}$$

   This can be simplified to

$$\sigma^2(T+30) = \frac{(T+37)\sigma^2(T+37)1.25 + (T+65)\sigma^2(T+65)(-0.25)}{T+30}$$

Close to the maturity:

Close to the maturity, we just patch the above equations by the exact formulation as in 11, with exactly the same definition of function $f(T)$.

In particular For computing $\sigma^2(T+2)$ we implement maturity switching as well similar to 4-week cycle. In this case we replace $\sigma^2(T+2)$ input in equation 11 with

$$\sigma^2(T+2) = \frac{2.25(\sigma^2(T+37)(T+37)) - 1.25(\sigma^2(T+65)(T+65))}{T+2} \tag{14}$$

If above value is returned negative, its set to 0.

Far away from the maturity:

This is the tricky case when one of T+37 or T+65 option doesn't exist.

Denote T_1 as the option of longest existing maturity $\leq T+2$. Lets denote T_2 as option of shortest existing maturity $\geq T+37$. Then we compute the upper bound as:

$$\sigma^2(T, T+30) = \frac{\sigma^2(T_2)T_2 - \sigma^2(T_1)T_1}{T_2 - T_1}$$

Calculating Maturities: Note that in the above formulae, the maturity themselves are computed in terms of minutes remaining till expiry – refer to CBOE VIX white paper for the exact formulae.

Note: We have requested sales to enter a new TREQ for term structure of variance swap rates in order to investigate different interpolation schemes and additionally to provide a different screen for variance swap term structure users who might want to view the term structure in a graphic form and might want to compute the forward variances using their choice of interpolation scheme.

The CBOE have just introduced new volatility derivatives. The variance futures which started trading on May 18th, 2004. The trading volume is still low on these products. In future, we'd consider incorporating these into VIX futures FVD.

3 Modeling the vol of VIX Futures : the volvol input for Fair price

Lets make the simple but realistic assumption that spot VIX price (observable) follows a mean-reverting process.

$$dVIX_t = \kappa_s(\theta_s - VIX_t)dt + \sigma_s dW \tag{15}$$

The estimation for the above process would be done using historical time-series data, i.e. in the real-world P measure. However, for pricing VIX Futures we need the Q-dynamics. However, the volatility of the VIX would remain the same under the two measures and that is the main input so we would just use the estimated P-dynamics as a proxy for the risk-neutral dynamics.

Issue: The VIX is allowed to be negative in the OU-dynamics. We could estimate a more complex time-series (e.g. a square-root mean reverting) model using GARCH/ARCH which doesn't allow negative VIX values. However, keep in mind that we are using this model just to estimate the approximate volvol parameters and not doing any pricing directly based on the model, so in the interest of simplicity of inputs, we would stick with mean reverting OU dynamics

We favor this simple approach due to its ease in terms of usability, i.e. instead of solving for multi-parameter model with complex meaning for parameters, we just use a volvol input which is specified in terms of percentage and can be input easily. Another approach is to identify both the objective and risk neutral dynamics, the former from VIX time-series and the latter from the option prices by fitting a stochastic volatility model. When we fit an AR model to the above model with a CEV term $\sigma_s VIX_t^\beta$ for volatility, we find $\beta \sim 0.65$ which is somewhere between square-root and lognormal model.

From the vasicek interest rate model, we know the variance of the VIX process at T_1 at time t is given by

$$Var_t(VIX_{T_1}) = \sigma_s^2 \frac{1 - e^{-2\kappa_s(T_1 - t)}}{2\kappa_s} \tag{16}$$

3.1 VolVol input computation

Recall the lognormal volatility from equation 7 (volvol input for our computation) is given by

$$(\sigma^l_{VIX}(t,T_1))^2(T_1-t) = -log(1-\frac{Var_t(VIX_{T_1})}{UB^2}) = -log(1-\frac{\sigma_s^2}{UB^2}\frac{1-e^{-2\kappa_s(T_1-t)}}{2\kappa_s})$$

This definition is invalid when $Var_t(VIX_{T_1}) = \sigma^2_{VIX}(t,T_1)(T_1-t) > UB^2$ so we use the approximation instead which is valid everywhere, i.e.,

$$\boxed{(\sigma^l_{VIX}(t,T_1))^2(T_1-t) = \frac{\sigma^2_{VIX}(t,T_1)(T_1-t)}{UB^2}(17)}$$

Rewriting,

$$(\sigma^l_{VIX}(t,T_1))^2(T_1-t) = \frac{\sigma_s^2}{UB^2}\frac{1-e^{-2\kappa_s(T_1-t)}}{2\kappa_s}$$

The volvol requires σ_s, κ_s as inputs and given these parameters would clearly have a term structure as specified by above formula. We now provide two different mechanisms to compute the process parameters κ_s, σ_s for the volvol input in the next two sections.

3.2 Approach 1 : VIX Time-series estimation

Now, to form a volvol estimate of σ_s and of κ_s, we fit an AR (auto-regressive) model to spot VIX The above is just a linear regression, in particular, we solve for,

$$VIX_i = aVIX_{i-1} + b + \epsilon_i, \quad i = 1, \ldots, n$$

where ϵ is a normally distributed independent white noise. The above is a regression problem, which would output a and b parameters, given VIX on $n+1$ days, $VIX_0, \ldots, VIX_n$. In particular, the regression problem would be solve for

$$Cx = y$$

where C is a 2×2 matrix, x is the unknown vector $x = [a \quad b]$ and y is a 2×1 vector with

$$
\begin{aligned}
C(1,1) &= 1 \\
C(1,2) &= C(2,1) = \sum_{i=0}^{i=n-1} VIX_i \\
C(2,2) &= \sum_{i=0}^{i=n-1} VIX_i^2 \\
y(1) &= \sum_{i=1}^{i=n} VIX_i \\
y(2) &= \sum_{i=0}^{i=n-1} VIX_i VIX_{i+1}
\end{aligned}
$$

Also the ϵ_i time-series can now be defined as, $\epsilon_i = VIX_i - aVIX_{i-1} - b$

We can then compute, $\kappa_s, \sigma_s, \theta_s$ by noting from basic properties of the OU (vasicek) model,

$$
\begin{aligned}
a &= e^{-\kappa_s \Delta t} \\
(1 - e^{-\kappa_s \theta_s \Delta t}) &= b \\
Var(\epsilon) &= \sigma_s^2 \frac{1 - e^{-2\kappa_s \Delta t}}{2\kappa_s}
\end{aligned}
$$

Δt is equal to 1 day time-series interval in this case. We compute the three parameters by solving the above equations in the given sequence for κ_s, θ_s and σ_s respectively as follows,

$$
\begin{aligned}
\kappa_s &= -\frac{log(a)}{\Delta t} \\
\theta_s &= -\frac{log(1-b)}{\kappa_s \Delta t} \\
\sigma_s^2 &= \frac{2\kappa_s Var(\epsilon)}{1 - e^{-2\kappa_s \Delta t}}
\end{aligned}
$$

where $Var(\epsilon) = \frac{\sum_{i=1}^{n} \epsilon_i^2}{n-1}$.

The numerical result on last 10 years of VIX data is $\kappa_s = 5.03$, $\theta_s = 21.18$ and $\sigma_s = 20.5$.

On the last 2 years of data, we get $\kappa_s = 5.18, \theta_s = 23.97, \sigma_s = 21.72$

3.2.1 benefits of this approach

The benefit of this approach is that it requires only dealing with spot VIX data. However, it has many drawbacks that it doesn't use only recent data but stale historical data and also it doesn't make use of the information of price time-series available from VIX Futures market itself. In the next method, we rectify these shortcomings.

3.3 Approach 2 : Robust estimation using VIX Futures time-series

In this approach, we also incorporate the observed VIX Futures price time series as well to get a better estimate of the variance.

From the property of the OU model, the VIX futures price $F_{t,T} = E_t[VIX_T]$ using equation (15) can be written explicitly as

$$F_{t,T} = (1 - e^{-\kappa(T-t)})\theta + e^{-\kappa(T-t)}(VIX_t) \tag{18}$$

Taking differential of both sides of equation (18) (use Ito's lemma), it can be shown that the Futures price , which is a martingale, follows the SDE,

$$dF_{t,T} = e^{-\kappa_s(T-t)}\sigma_s dW$$

Lets denote the instantaneous volatility of the VIX Futures as $\sigma(t,T) = e^{-\kappa_s(T-t)}\sigma_s$.

We can compute the instantaneous vol from the time series of the spot VIX and of the current VIX futures prices of different maturities. This turns out to be a robust way of estimating the κ as it doesn't rely only on spot VIX data.

We use the four currently available futures, viz for the June, July, August and November expiry, along with spot VIX data from March 26th, 2004 to current date.

The σ_s parameter can still be computed using spot VIX time-series, however, with this new approach we don't require a long time-series for this estimate anymore. We now describe the estimation procedure.

3.3.1 Computing σ_s

We don't use the time-series estimation method presented in the last section for estimation of σ_s as that requires a large time-series. Since in the current method we use less data, our recommendation is to use just the sample standard deviation of spot VIX time series for the last τ time-period.

We want to look at the sample standard deviation of the VIX time-series. The VIX dynamics from equation (15) reproduced here are:

$$dVIX_t = \kappa_s(\theta_s - VIX_t)dt + \sigma_s dW$$

So $\sigma_s\sqrt{\Delta t}$ is approximately the standard deviation of the VIX over small time Δt (The exact result for OU model is that standard deviation equals $\sigma_s\sqrt{\frac{1-e^{-2\kappa_s\Delta t}}{2\kappa_s}}$ which is very close to $\sigma_s\sqrt{\Delta t}$, especially for small Δt).

Hence we can estimate from the time-series,

$$\sigma_s^2 = \frac{1}{(n-1)\Delta t}\sum_{i=1}^{n}(VIX_{i+1} - VIX_i)^2$$

Here $n+1$ is the number of daily samples in the τ time-period.

Note that we recommend using the bias-corrected sample standard deviation for this estimation since the mean (of daily VIX difference) is unknown (we haven't estimated Kappa yet). Also, since this is daily data, we use $\Delta t = \frac{1}{252}$.

3.4 Data Issues and our recommendations

The Futures being recently introduced don't provide us with a luxury of a large historical time-series.

Our recommendation is to use last 3 months of data, when available. We recommend only to use the three active futures from the expiry month set of Feb, May, Aug and Nov futures (whichever available) as this is the Futures cycle from CBOE. We'd have large time-series of these available in general.Also, since futures products are introduced at different times into the market, we may not have equal amount of data available historically for all futures instruments and this may be a problem.

3.4.1 Computing κ_s

Recall the VIX Futures dynamics,

$$dF_{t,T} = e^{-\kappa_s(T-t)}\sigma_s dW$$

Lets denote by $\sigma_{t,T;\tau}$ the standard deviation of time-series of length τ of the VIX futures of maturity T.We would compute it from the time-series of futures as

$$\sigma_{t,T;\tau}^2 = \frac{1}{n\Delta t}\sum_{i=1}^{n}(VIXF_{i+1;T} - VIXF_{i;T})^2 \tag{19}$$

From the futures dynamics, we know that

$$E[(VIXF_{i+1;T} - VIXF_{i;T})^2] = e^{-2\kappa_s(T-t_i)}\sigma_s^2\Delta t$$

Hence the volatility in equation (19) would be given by RMS (Root mean squared) of instantaneous vol, i.e,

$$\sigma_{t,T;\tau} = \sigma_s\sqrt{\frac{\int_t^{t+\tau} e^{-2\kappa_s(T-u)}du}{\tau}} = \sigma_s e^{-\kappa_s(T-t)}\sqrt{\frac{1-e^{-2\kappa_s\tau}}{2\kappa_s\tau}}$$

Here, τ is the length of sample size in years and n is the number of daily samples corresponding to this period.

Note that we recommend using the sample standard deviation for this estimation since the mean is known (Futures is a martingale, hence daily Futures difference has mean of 0).

Hence the ratio of two standard deviations ratios for futures of maturity T_2 and T_1 is is

$$\frac{\sigma_{t,T_2;\tau}}{\sigma_{t,T_1;\tau}} = e^{-\kappa_s(T_2-T_1)}$$

Hence κ can be estimated from successive ratios. Since we have multiple of these future pairs, we could do a least squares fit in general to find κ_s.

The least-squares set up would be as follows, suppose we have n futures trading in the market, for maturities, $T_1, \ldots, T_n$.

Then we form κ_s simply as (this is the least squares solution)

$$\kappa_s = \frac{1}{n-1}\sum_{i=1}^{i=n-1}\frac{log(\sigma_{t,T_i;\tau}) - log(\sigma_{t,T_{i+1};\tau})}{T_{i+1}-T_i}$$

where σ_i's are formed as sample standard deviations from the Futures time series.

3.4.2 A more robust approach for κ_s computation

In general, the futures have different length of historical time-series available so it makes sense to do a weighting of the κ_s value to give more importance to the futures which have longer time-series available. We suggest an alternate scheme which accounts for every pair and gives the weights according to number of common data points available.

$$\kappa_s = \frac{\sum_{i=1}^{i=n-1} \sum_{j=i+1}^{n-1} N(i,j) \frac{log(\sigma_{t,T_i;\tau}) - log(\sigma_{t,T_j;\tau})}{T_j - T_i}}{\sum_{i=1}^{i=n-1} \sum_{j=i+1}^{n-1} N(i,j)}$$

where $N(i,j)$ is the number of common days for which the futures i and j have data available. The standard deviation calculation for the pair (i,j), i.e., $\sigma_{t,T_i;\tau}, \sigma_{t,T_j;\tau}$ is done using the data from those common days $N(i,j)$. In particular, the computation is:

$$\sigma^2_{t,T_i;\tau} = \frac{1}{(N(i,j)-1)\Delta t} \sum_{k=1}^{N(i,j)-1} (VIXF_{k+1;T_i} - VIXF_{k;T_i})^2$$

3.4.3 Generating Historical volvol for VIX futures pricing

Once we have estimates of κ_s and σ_s, we know that the variance of VIX at a future time T could be easily computed and the volvol input can be formed – please refer to formulae in equations 7 and 16 for the exact computations.

3.4.4 Benefits of this approach

Firstly, we use more of recent data and actually make use of historical Futures prices which are a valuable information for pricing Futures. Secondly, this estimation is more robust since it is more of a cross-sectional analysis compared to just looking at historical spot VIX series only.

We recommend using past 3 months of Futures time series for this analysis.

3.5 Example

Reproducing the old results, we have:

$$F^{VIX}_{t,T_1} = UB\sqrt{(1 - \sigma^l_{VIX}(t,T_1)^2(T_1 - t))}$$

where

$$(\sigma^l_{VIX}(t,T_1))^2(T_1-t) = -log(1-\frac{Var_t(VIX_{T_1})}{UB^2}) = -log(1-\frac{\sigma_s^2}{UB^2}\frac{1-e^{-2\kappa_s(T_1-t)}}{2\kappa_s})$$

Lets say the maturity of the futures is 2 months away, i.e. $T_1 - t = \frac{2}{12}$. Also, assume that the variance swap rate is at 22%. Then from equation 16 and 7, we get

$$\sigma^l_{VIX} = \sqrt{log(1-\frac{20.5^2}{22^2}\frac{1-e^{\frac{-2*5.03*2}{12}}}{2*5.03})} = 0.6607$$

Hence,

$$F^{VIX} = UB\sqrt{(1-\sigma^l_{VIX}(t,T_1)^2(T_1-t)))} = 22\sqrt{(1-\frac{0.6607^2*2}{12})} = 21.18$$

Now lets look at Futures maturing 6 months away. Assume that variance swap rate is 23% for that maturity. We get

$$\sigma^l_{VIX} = \sqrt{log(1-\frac{20.5^2}{23^2}\frac{1-e^{\frac{-2*5.03*6}{12}}}{2*5.03})} = 0.404$$

Hence,

$$F^{VIX} = UB\sqrt{(1-\sigma^l_{VIX}(t,T_1)^2(T_1-t)))} = 23\sqrt{(1-\frac{0.404^2*6}{12})} = 22.04$$

As you note, the variances have a term structure and we had different volvol for the two maturities. Consistent with mean-reverting process, the term structure would be decreasing with maturity as in above example.

4 Acknowledgements

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A Appendix - Arbitrage trading on upper bound

(The following is based on document "VIX Futures and Options" by Carr, Wu, Wurzburger. The Note at the end of section illustrates why this bound may not be easily arbitrageurable).

First, suppose the upper bound is violated such that $F_0^{vix} > UB$. Recall that UB is the square root of the forward variance swap rate. Then, we propose the following trading strategy:

- Long one unit of the forward variance swap contract,
- Short $2F_0^{vix}$ units of the VIX futures contract.

By the rule of "buying low and selling high," it is obvious that we need long the relatively undervalued forward swap contract (UB) and short the relatively overvalued VIX futures (F_0^{vix}). However, it is not clear how many units of each contract that we need to long or short.

To understand the problem, we inspect the payoff from the two contracts: the VIX futures and forward variance swap. At the future maturity T_1, the VIX futures value converges to the spot VIX level, VIX_{T_1}. The forward variance swap rate in volatility terms also converges to the square root of the spot variance swap rate, which is also VIX_{T_1}. Nevertheless, although the cumulative payoff from the futures position is defined on volatility units: $(VIX_{T_1} - F_0^{vix})$, the payoff from the forward variance swap contract is defined on variance: $\left(UB^2 - VIX_{T_1}^2\right)$. Thus, being long in one and short in the other does not lead to direct cancelation. By using $2F_0^{vix}$ shares of the VIX futures contract, we achieve the effect of cancelation.

We can enter both contracts at zero cost at time 0. At time T_1, the payoff from the two positions becomes

$$\begin{aligned}
P\&L &= \left[VIX_{T_1}^2 - UB^2\right] - 2F_0^{vix}\left[VIX_{T_1} - F_0^{vix}\right] \\
&= VIX_{T_1}^2 - 2F_0^{vix}VIX_{T_1} + 2(F_0^{vix})^2 - UB^2 \\
&\geq VIX_{T_1}^2 - 2F_0^{vix}VIX_{T_1} + (F_0^{vix})^2 \qquad (20) \\
&= \left(VIX_{T_1} - F_0^{vix}\right)^2 \\
&\geq 0. \qquad (21)
\end{aligned}$$

The inequality in (20) is due to the violation of the upper bound: $F_0^{vix} > UB$. The inequality in (21) essentially says that our strategy will never lose money and have the chance of making money if the upper bound on the VIX futures is violated.

An alternative strategy that also guarantees nonnegative payoff is to long one unit of the forward variance swap and short $2UB$ shares of the VIX futures. The payoff becomes

$$\begin{aligned}
P\&L &= \left[VIX_{T_1}^2 - UB^2\right] - 2UB\left[VIX_{T_1} - F_0^{vix}\right] \\
&= VIX_{T_1}^2 - 2UBVIX_{T_1} + 2UBF_0^{vix} - UB^2 \\
&\geq VIX_{T_1}^2 - 2UBVIX_{T_1} + (UB)^2 \qquad (22) \\
&= (VIX_{T_1} - UB)^2 \\
&\geq 0. \qquad (23)
\end{aligned}$$

Both strategies, as well as any convex combinations of the two strategies, generate nonnegative profits in the case of upper bound violation. We cannot run these two strategies unconditionally. However, when you have a view on the future VIX movement, you can choose one over the other. Using the second strategy is more beneficial if you expect the VIX level to increase so that $VIX_{T_1} > F_0^{vix}$. Otherwise, using the first strategy is more profitable.

Note: The above strategy is actually based on the assumption that the VIX Futures and the variance swap rate have the same value at the maturity of the VIX Futures (viz, VIX_{T_1}). This may not hold in practice since the the VIX futures contract matures into synthetic variance swap rate from the standard options market but the variance swap product may be trading at a different value. Hence this arbitrage is actually much weaker than what the above strategy suggests. We could instead use synthetic variance swap as our long instrument for the above strategy (to make sure VIX Futures and variance swap equal at maturity) but this would require buying and selling a lot of options at t and then at T_1 that we would spend a lot in transaction costs. Also, since new options may get introduced with time which would additionally complicate this strategy.

B Formulation of Lower bound

We can also formulate the lower bound using a volvol parameter

$$LB \approx UB(1 - \frac{{\sigma^l_{LB}}^2(T_1 - t)}{2})$$

where $\sigma^l_{LB} \geq \sigma^l_{VIX}$ is the appropriate lognormal volatility for the realized variance.

The above formulas are intended for final screen calculations with the $\sigma^l_{LB}, \sigma^l_{VIX}$ quoted in percentage terms as the inputs.

B.1 Modelling the ratio of the two volvols

Earlier, we had illustrated a simple way to get the fair price and lower bound given volvol parameters. In this section, we show how to form these volvol parameters using a simple but realistic assumption.

We'd proxy the variance of VIX Futures price by volatility of the spot VIX and the variance of future realized volatility by volatility of current realized volatility over the next 30 days.

Lets assume that the instantaneous variance follows a mean-reverting CIR (so it doesn't go negative) diffusion process, which is a very reasonable and simple assumption(confirmed by market behavior). I.e.,

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW \tag{24}$$

Lets denote τ as the time-period over which the variance calculations are done (i.e., 30 days for VIX applications).

From the above model, its easy to show that the VIX spot price which is given by

$$VIX_t^2 = \frac{E_t[\int_t^{t+\tau} v_s ds]}{\tau}$$

can be explicitly written down as

$$VIX_t^2 = \theta + \frac{1 - e^{-\kappa\tau}}{\kappa\tau}(v_t - \theta)$$

From applying Ito's lemma to equation (24) the SDE for VIX_t^2 is given by

$$dVIX_t^2 = \kappa(\theta - VIX_t^2)dt + F\sigma\sqrt{v_t}dW$$

where we denote $\frac{1-e^{-\kappa\tau}}{\kappa\tau}$ by F and $\sigma_v = F\sigma$.

The above is the **key idea**, i.e., we can relate the volatility of VIX^2 to volatility of the instantaneous variance in this simple model. Now, we assume that future realized variance process is similar to instantaneous variance and use the latter as a proxy for former.

Clearly $F < 1$ and this shows that volatility of the VIX^2 is less than volatility of instantaneous variance.

Since we need a way to compute vol of VIX Futures (for fair value) and vol for realized vol (for lower bound), the ratio of these two volvols would be approximately $\sqrt{F}$, since F is the ratio of the vols of the squared variables.

Hence, we'd approximately have

$$\sigma^l_{LB}\sqrt{F} = \sigma^l_{VIX}$$

This is the final relationship between the two volvols.

To estimate F we would do a time-series estimation. Now we numerically estimate these parameters and hence we will devise techniques to compute the volvol parameters.

To estimate F more robustly, we can also use the variance swap rates (observable in the market via the log profile) in this analysis. It can be shown that the variance swap rate

$$V_{t,T} = \frac{E_t[\int_t^T v_s ds]}{T-t}$$

can be explicitly written down as

$$V_{t,T} = \theta + \frac{1-e^{-\kappa(T-t)}}{\kappa(T-t)}(v_t - \theta)$$

and hence the ratio of volatility of spot VIX_t^2 and various variance swap rates (for different terms T) are related by κ. We can solve for κ using a joint time-series estimation and hence form $F = \frac{1-e^{-\kappa\tau}}{\kappa\tau}$. In the numerical section of estimation 1, we use only the spot VIX data for now. We currently don't have the time-series of variance swap rates handy (Note: as of May 18th, 2004 the variance futures have started trading, but the volume is almost zero hence we haven't investigated this approach.

B.2 AR Estimation for the ratio

To form the factor F we first fit an AR (auto-regressive) model to spot VIX^2 data result in $\kappa = 7.0$, $\theta = 490.5$ and $\sigma_v = 1162.0$. The above is just a linear regression, in particular, we solve for,

$$v_t = Av_{t-1} + b + \epsilon_t$$

where ϵ is a normally distributed independent white noise.

We can then compute, κ, σ_v, θ by noting that

$$\begin{aligned} A &= e^{-\kappa\Delta t} \\ \kappa\theta\Delta t &= b \\ Var(\epsilon) &= \sigma_v^2 \frac{1 - e^{-2\kappa\Delta t}}{2\kappa} \end{aligned}$$

Δt is equal to 1 day time-series interval in this case. We compute the three parameters by solving the above equations in the given sequence for κ, θ and σ_v respectively.

Also, we get $F = \frac{1-e^{-\kappa\tau}}{\kappa\tau} = 0.6905$