Linear Classification Models

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Content

Outline

Classification problems

- $predict(x) = \begin{cases} C_1, & y(x) \geq \text{threshold} \\ C_2, & y(x) < \text{threshold} \end{cases},$
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  2. Generative model:
  $$y(x) = P(C_k|x) = \frac{P(x|C_k)P(C_k)}{P(x)}$$
  3. Discriminative model:
  $$y(x) = P(C_k|x) = f(x),$$
  with f an arbitrary function.

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- Also called *generalized linear models*
- Applicable if instead of x we use a vector of basis functions $\phi(x)$, corresponding to features in a feature space

Using regression for classification

- We can use a regression model, such as least squares to fit a linear classification model with a linear activation function

$$\min_{w,w_o} \sum_{i=1}^{\ell} (t_i - w^T x_i + w_0)^2,$$

where $t_i \in \{-1, 1\}$ is the label of the i-th training sample,

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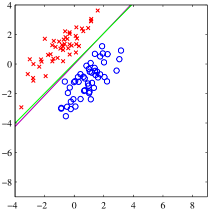
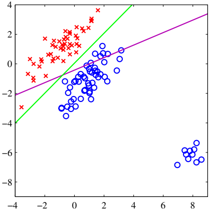
- but this strategy does not work well:





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- The precursor of neural networks
- Criticized by Marvin Minsky, producing a decline in research funding

Perceptron learning

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where $\phi_n = \phi(x_n)$.

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- Learning rule:

$$w^{(\tau+1)} = w^{(\tau)} + \eta \phi_n t_n$$

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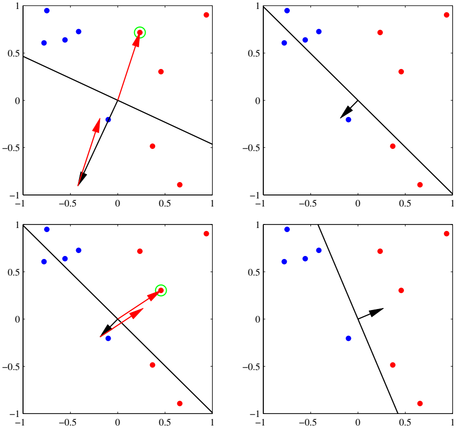

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- Non-convex optimization problem
- No convergence guarantee if samples are not linearly separable
- Its power may be increased by stacking several perceptron layers (multilayer perceptrons) and using smooth activation functions

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Parametric discrimination

- These three conditions are equivalent:
  - $P(C_1|x) \geq 0.5$
  - $\frac{P(C_1|x)}{1-P(C_1|x)} \geq 1$
  - $\text{logit}(P(C_1|x)) = \log \frac{P(C_1|x)}{1-P(C_1|x)} \geq 0$

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- If we assume that $P(x|C_1)$ and $P(x|C_2)$ are normally distributed sharing the same covariance matrix:

$$\text{logit}(P(C_1|x)) = \log \frac{P(C_1|x)}{P(C_2|x)} = w^T x + w_0,$$

where

$$w = \Sigma^{-1}(\mu_1 + \mu_2)$$
$$w_0 = -\frac{1}{2}(\mu_1 + \mu_2)^T \Sigma^{-1}(\mu_1 + \mu_2) + \log \frac{P(C_1)}{P(C_2)}$$

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$$P(C_1|x) = \sigma(w^T x + w_0) = \frac{1}{1 + e^{-(w^T x + w_0)}}$$

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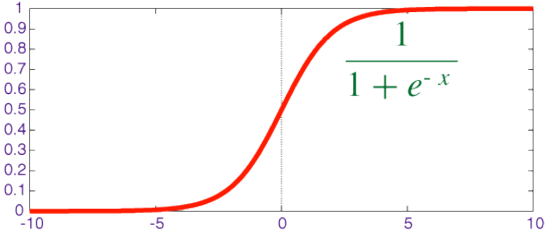
- σ is called the logistic or sigmoid function.



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where $\mathbf{t} = \{t_1, \ldots, t_\ell\}$ and $y_n = y(x_n)$.

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- Cross-entropy error:

$$E(w) = -\ln p(\mathbf{t}|w) = -\sum_{n=1}^{\ell} [t_n \ln y_n + (1 - t_n)\ln(1 - y_n)]$$

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where $y_{nk} = y_k(x_n)$ and $\mathbf{T} \in \mathbb{R}^{\ell \times K}$ is a matrix of target variables with elements t_{nk}.

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- Multiclass cross-entropy error:

$$E(w_1, \ldots, w_K) = -\ln p(\mathbf{T}|w_1 \ldots w_K) = -\sum_{n=1}^{\ell} \sum_{k=1}^{K} t_{nk} \ln y_{nk}$$

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- $$w^{(\tau+1)} = w^{(\tau)} - \eta \sum_{n=1}^{\ell} (y_n - t_n)\phi_n$$

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with $\mathbf{R}$ a diagonal matrix with $R_{nn} = y_n(1 - y_n)$.

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  with $\mathbf{R}$ a diagonal matrix with $R_{nn} = y_n(1 - y_n)$.
- The resulting algorithm is called *iterative reweighted least squares.*

Regularization

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- Equivalent to the inclusion of a prior and finding a MAP solution for W.

Stochastic gradient descent

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- Alpaydin, E. 2010 Introduction to Machine Learning, 2Ed. The MIT Press. (Chap 10)
- Russell, S and Norvig, P. 2010 Artificial Intelligence: a Modern Approach, 3rd Ed, Prentice-Hall (Sect 18.6)