Linear Algebra Review

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1. Vector Space

- Defined over a field K ($\mathbb{R}$ or $\mathbb{C}$)
- Vectors $\in V$
- scalars $\in K$
- Operations

$$a\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} a \cdot v_1 + w_1 \\ a\, v_2 + w_2 \\ a\, v_3 + w_3 \end{bmatrix}$$

0 vector

$$\forall \psi \quad |\psi\rangle + 0 = |\psi\rangle$$

$$|\psi\rangle = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\left.\begin{matrix} |\ \rangle \text{ ket} \\ \langle\ | \text{ Bra} \end{matrix}\right\} \begin{matrix} \text{Dirac Notation} \\ \text{Braket Notation.} \end{matrix}$$

2. Dirac Notation

Notation	Description
z^*	Complex conjugate of the complex number z.
	$(1+i)^* = 1 - i$ $\quad i = \sqrt{-1}$
$\vert\psi\rangle$ Column	Vector. Also known as a *ket*.
$\langle\psi\vert$ Row	Vector dual to $\vert\psi\rangle$. Also known as a *bra*.
$\langle\varphi\vert\psi\rangle$	Inner product between the vectors $\vert\varphi\rangle$ and $\vert\psi\rangle$.
$\vert\varphi\rangle \otimes \vert\psi\rangle$	Tensor product of $\vert\varphi\rangle$ and $\vert\psi\rangle$.
$\vert\varphi\psi\rangle$ $\vert\varphi\rangle\vert\psi\rangle$	Abbreviated notation for tensor product of $\vert\varphi\rangle$ and $\vert\psi\rangle$.
A^*	Complex conjugate of the A matrix.
A^T	Transpose of the A matrix.
$A^\dagger$	Hermitian conjugate or adjoint of the A matrix, $A^\dagger = (A^T)^*$.
	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}.$
$\langle\varphi\vert A\vert\psi\rangle$	Inner product between $\vert\varphi\rangle$ and $A\vert\psi\rangle$.
	Equivalently, inner product between $A^\dagger\vert\varphi\rangle$ and $\vert\psi\rangle$.

3. Bases and linear independence

- Spaning Set

$\{|v_1\rangle, \dots, |v_n\rangle\} \quad \forall |v\rangle \in V \qquad |v\rangle = \sum_i a_i |v_i\rangle$

$V = \mathbb{C}^2$

$|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1 |0\rangle + a_2 |1\rangle$

$|v_1\rangle \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad |v_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{a_1 + a_2}{\sqrt{2}} |v_1\rangle + \frac{a_1 - a_2}{\sqrt{2}} |v_2\rangle$

- Linear dependent set

$\{|v_1\rangle, \dots, |v_n\rangle\} \quad \exists\, a_1 \dots a_n \qquad a_1 |v_1\rangle + \dots + a_n |v_n\rangle = 0$

at least one $a_i \neq 0$

- Basis: a non-linear dependent set that spans V

\# elements of a basis = Dimension of the space

4. Linear operators and matrices

- Linear operator $\quad V, W$ vector spaces

$$A : V \longrightarrow W \qquad A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A(|v_i\rangle)$$

$$A|v_i\rangle := A(|v_i\rangle)$$

Identity operator $\quad I_V |v\rangle = |v\rangle$

- Composition $\quad A : V \longrightarrow W \quad B : W \longrightarrow X$

$$B(A(|v\rangle)) =: (BA)(|v\rangle) = BA|v\rangle$$

- Matrix representation

$A : V \longrightarrow W \quad \{|v_1\rangle, \dots, |v_n\rangle\}$ a basis for V

$\{|w_1\rangle, \dots, |w_m\rangle\}$ a basis for W

$$A(|v_i\rangle) = \sum_{j=1}^{m} A_{ij} |w_j\rangle \qquad |v\rangle = a_1|v_1\rangle + \dots + a_n|v_n\rangle$$

$$A(|v\rangle) = \sum_i A(a_i|v_i\rangle) = \sum_i \sum_j a_i A_{ij} |w_j\rangle$$

5. Pauli Matrices

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

6. Inner product

- Inner product $(\,,\,) : V \times V \longrightarrow \mathbb{C}$ $\qquad \langle v | w \rangle := (|v\rangle, |w\rangle)$

$$|v\rangle, |w\rangle \longmapsto (|v\rangle, |w\rangle)$$

1. Linear in the second argument $\left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right) = \sum_i \lambda_i \left(|v\rangle, |w_i\rangle\right)$
2. $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$
3. $(|v\rangle, |v\rangle) \geq 0 \qquad (|v\rangle, |v\rangle) = 0 \Longleftrightarrow |v\rangle = 0$

$V = \mathbb{C}^n$

$$((v_1 \ldots v_n), (w_1, \ldots, w_n)) = \sum_i v_i^* w_i$$

- Dual vector

$$|v\rangle \Longrightarrow \langle v| : V \to \mathbb{C}$$
$$|w\rangle \to (|v\rangle, |w\rangle)$$

$$\langle v| (|w\rangle) = \langle v | w \rangle$$

- Inner product space $\equiv$ Hilbert space if the dimension is finite
- orthogonality, norm $\qquad |v\rangle$ and $|w\rangle$ are orthogonal if $\langle v | w \rangle = 0$

$\dfrac{|v\rangle}{\| |v\rangle \|}$ is normal $\qquad |v\rangle$ is normal if $\| |v\rangle \| = \langle v | v \rangle = 1$

- Orthonormal Basis

$\{|v_1\rangle, \ldots, |v_n\rangle\}$ is a Basis, it is orthonormal if $\begin{matrix} \forall i \ \| |v_i\rangle \| = 1 \\ \forall i \neq j \ \langle v_i | v_j \rangle = 0 \end{matrix} \Longleftrightarrow \langle v_i | v_j \rangle = \delta_{ij}$

- Gram - schmidt

$\{|w_1\rangle, \dots, |w_d\rangle\}$ a basis

$$|v_1\rangle = \frac{|w_1\rangle}{\| |w_1\rangle \|} \qquad |v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle}{\left\| |w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle \right\|}$$

- $\{|v_1\rangle, \dots, |v_d\rangle\}$ is an orthonormal basis.

- Outer product

$|v\rangle \in V$, $|w\rangle \in W$ $\qquad |w\rangle\langle v| : V \longrightarrow W$

$$|v'\rangle \longmapsto \langle v|v'\rangle |w\rangle$$

$$|w\rangle\langle v| \left(|v'\rangle \right) = |w\rangle\langle v|v'\rangle = \langle v|v'\rangle |w\rangle$$

$$|v\rangle\langle w|y\rangle\langle w|x\rangle$$

$$= \langle w|y\rangle\langle w|x\rangle |v\rangle$$

$\{|1\rangle, |2\rangle \dots |d\rangle\}$ orthonormal basis

$$|v\rangle = v_1|1\rangle + \dots v_d|d\rangle$$

$$\langle i|v\rangle = v_1 \underbrace{\cancel{\langle i|1\rangle}}_{0} + \dots + v_i \underbrace{\langle i|i\rangle}_{1} + \dots + v_d \underbrace{\cancel{\langle i|d\rangle}}_{0}$$

$$= v_i$$

$$O = \sum_{i=1}^{d} |i\rangle\langle i| = I$$

$$O : V \longrightarrow V$$

$$|v\rangle \longmapsto O|v\rangle = \sum_{i=1}^{d} |i\rangle\langle i|v\rangle = \sum_{i=1}^{d} \overbrace{\langle i|v\rangle}^{v_i} |i\rangle$$

$$= |v\rangle$$

7. Eigenvectors and Eigenvalues (Sect. 2.1.5)

- Eigenvector
- characteristic function
- Diagonal representation

Homework [Nielsen10]

Exercises of Sections 2.1.1 to 2.1.5

For those exercises that apply solve them in numpy.

8. Adjoints and Hermitian operators

- Adjoint
- Hermitian
- Projector
- Normal
- Unitary

9. Tensor Product

- Tensor product of spaces

- Tensor product properties

(1) For an arbitrary scalar z and elements $|v\rangle$ of V and $|w\rangle$ of W,

$$z\left(|v\rangle \otimes |w\rangle\right) = \left(z|v\rangle\right) \otimes |w\rangle = |v\rangle \otimes \left(z|w\rangle\right). \tag{2.42}$$

(2) For arbitrary $|v_1\rangle$ and $|v_2\rangle$ in V and $|w\rangle$ in W,

$$\left(|v_1\rangle + |v_2\rangle\right) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle. \tag{2.43}$$

(3) For arbitrary $|v\rangle$ in V and $|w_1\rangle$ and $|w_2\rangle$ in W,

$$|v\rangle \otimes \left(|w_1\rangle + |w_2\rangle\right) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle. \tag{2.44}$$

- Linear operators on Tensor product spaces

- Kronecker Product

10. Qubits and the Bloch Sphere