lec 5

→ Integrating of complex functions

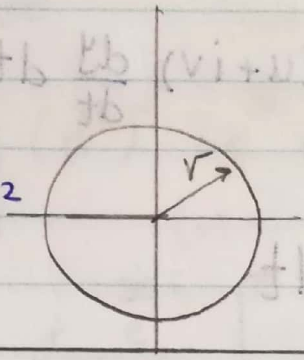
$|z| = r$

$|x+iy| = r$

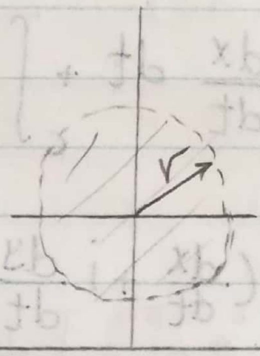
$\sqrt{x^2+y^2} = r$

$x^2+y^2 = r$

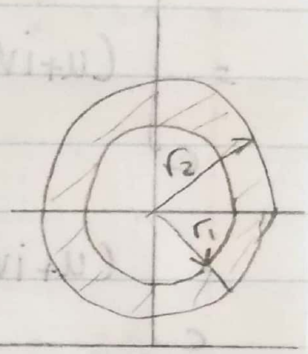
$(x-a)^2+(y-b)^2 = r^2$



$|z| < r$



$r_1 \leq |z| \leq r_2$

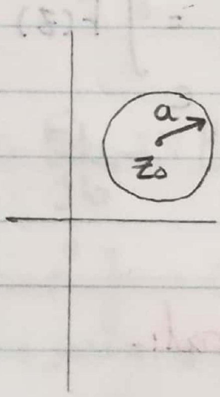


$|z - z_0| = a$

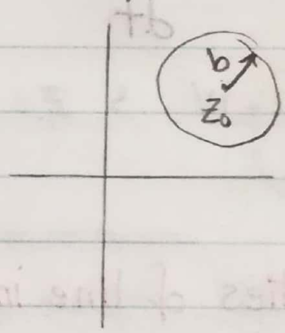
$|x+iy-(x_0+iy_0)| = a$

$\sqrt{(x-x_0)^2+(y-y_0)^2} = a$

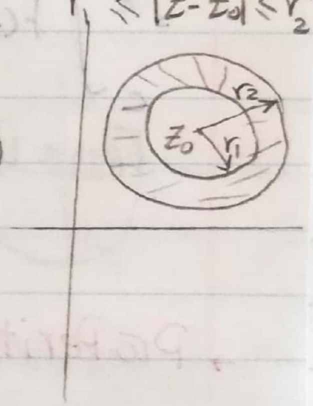
$(x-x_0)^2+(y-y_0)^2 = a^2$



$|z - z_0| \leq b$



$r_1 \leq |z-z_0| \leq r_2$



→ Line integral :-

$$\int_C f(z)\, dz$$

$f(z) = u+iv \quad , \quad z = x+iy$

$dz = dx + i\,dy$

$$\int_C f(z)\, dz = \int_C (u+iv)(dx+i\,dy) = \int_C (u\,dx - v\,dy) + i(v\,dx + u\,dy)$$

$$= \int_C \left(u\frac{dx}{dt} - v\frac{dy}{dt}\right) dt + i\int_C \left(v\frac{dx}{dt} + u\frac{dy}{dt}\right) dt$$

$$\int_C (u+iv)\frac{dx}{dt}\,dt + \int_C (-v+iu)\frac{dy}{dt}\,dt$$

$$= \int_C (u+iv)\frac{dx}{dt}\,dt + \int_C (i^2 v+iu)\frac{dy}{dt}\,dt$$

$$= \int_C (u+iv)\frac{dx}{dt}\,dt + \int_C i(u+iv)\frac{dy}{dt}\,dt$$

$$= \int_C (u+iv)\left(\frac{dx}{dt}+i\frac{dy}{dt}\right)dt$$

$$= \int_C F(z)\frac{dz}{dt}\,dt = \int_C F(z)\,z'(t)\,dt$$

$F(z) = u+iv$ ، $z = x+iy \rightarrow \frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$

→ Properities of line integral:-

$\boxed{1}$ $\int_C [F_1(z)+F_2(z)]\,dz = \int_C F_1(z)\,dz + \int_C F_2(z)\,dz$

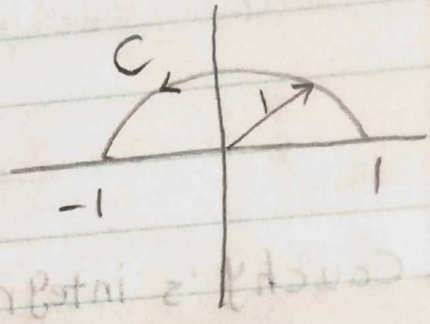
$\boxed{2}$ $\int_C k\,F(z)\,dz = k\int_C F(z)\,dz$

$\boxed{3}$ $\int_{-C} F(z)\,dz = -\int_C F(z)\,dz$

$-C$ ↓ في اتجاه عقارب الساعة

C ↓ على عقارب الساعة

EX: $\int_C z^2\, dz$



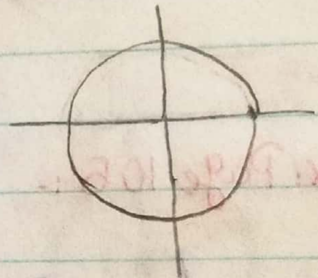
$C: \quad z = e^{i\theta} \qquad \theta \in [0, \pi]$

$\frac{dz}{d\theta} = i e^{i\theta}\, d\theta$

$\int_C z^2\, dz = \int_0^{\pi} (e^{i\theta})^2\, i e^{i\theta}\, d\theta$

$= i \int_0^{\pi} e^{3i\theta}\, d\theta = \frac{i\, e^{3i\theta}}{3i} \Big|_0^{\pi} = \frac{1}{3}\left[e^{3i\pi} - e^{0} \right] = \frac{-2}{3}$

if $C: \quad z = e^{i\theta} \qquad \theta \in [0, 2\pi]$

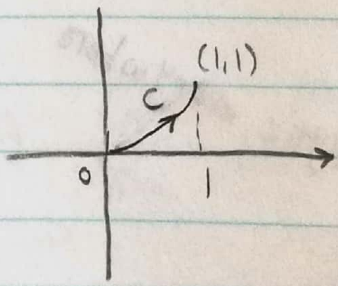


$= \frac{e^{3i\theta}}{3} \Big|_0^{2\pi} = \frac{1}{3}\left[e^{6i\pi} - e^{0} \right] = 0$

«المسار مغلق والدالة analytic ← تكاملها = صفر»

$y = x^2 \qquad x \in [0, 1]$

$\int_C z^2\, dz$



another sol.

$\int_0^{1+i} z^2\, dz$ z analytic

$\frac{z^3}{3} \Big|_0^{1+i} = \frac{(1+i)^3}{3}$

$= \frac{-2}{3} + \frac{2}{3} i$ #

$z = x + iy$

$dz = dx + i\, dy$

$\int_C (x+iy)^2 (dx + i\, dy)$

let $x = t \rightarrow dx = dt$

$y = t^2 \qquad t \in [0, 1]$

$dy = 2t\, dt$

$z = t + it^2 \rightarrow dz = (1 + 2it)\, dt$

$= \int_C (t + it^2)^2 (1 + 2it)\, dt$

$= \int_0^1 (t^2 + 2it^3 - t^4)(1 + 2it)\, dt = \frac{t^3}{3} - t^5 + it^4 - \frac{i}{3} t^6 \Big|_0^1$

$= \frac{-2}{3} + \frac{2i}{3}$

• لو الدالة analytic ← لا تعتمد على المسار

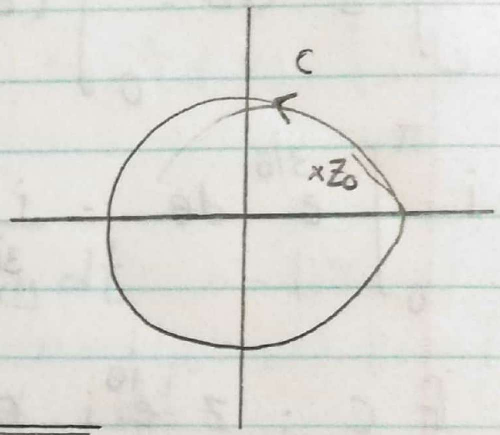
• " " " not ← تعتمد على المسار

→ Cauchy's integral Formula :

$$\oint_C \frac{F(z)}{z - z_0}\, dz = 2\pi i\, F(z_0)$$

$F(z)$ is analytic

z_0 in region

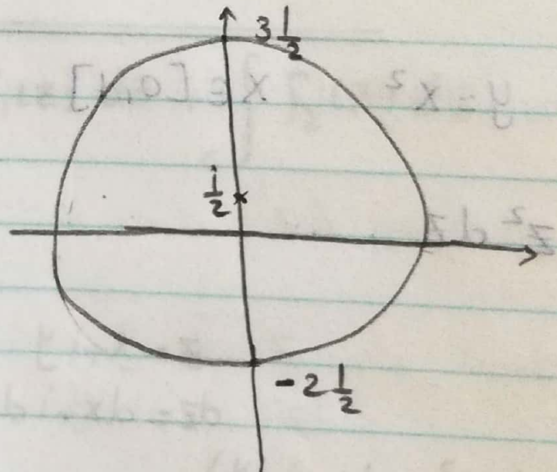


ex: Page 105 :-

$$\oint_C \frac{z^3}{2z - i}\, dz \qquad C: |2z - i| = 3$$

$2z - i = 0$

$z = z_0 = \frac{i}{2}$



analytic everywhere

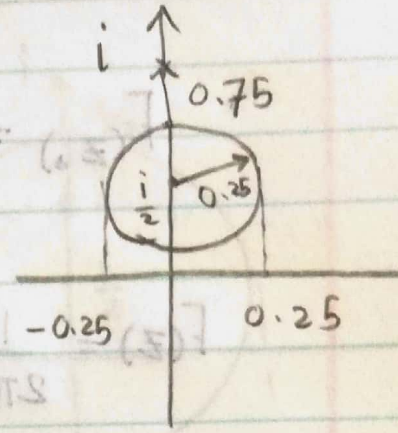
$$\int_C \frac{z^3}{2(z - \frac{i}{2})}\, dz$$

$F(z) = \frac{1}{2} z^3 \rightarrow F(z_0) = \frac{1}{2}\left(\frac{i}{2}\right)^3 = \frac{-i}{16}$

$$\int_C \frac{z^3}{2(z - \frac{i}{2})}\, dz = 2\pi i \ast \frac{-i}{16} = \frac{\pi}{8}$$

Ex: Page 106:

$$\oint_c \frac{z^3}{z^2+1}\,dz \qquad c:\ |2z-i| = 0.25$$



$$2z - i = 0$$
$$z = \frac{i}{2}$$

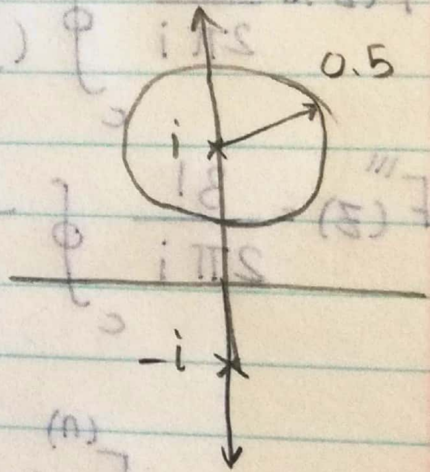
$$\oint_c \frac{z^3}{(z+i)(z-i)}\,dz$$

$$= 0$$

$\because \frac{z^3}{z^2+1}$ is analytic everywhere on c and inside c

Then the integration = 0

(b) $c:\ |z-i| = 0.5$



$$f(z) = \frac{z^3}{(z+i)} \rightarrow \frac{i^3}{2i} = \frac{-1}{2}$$

$$\int_c \frac{z^3/(z+i)}{(z-i)}\,dz = 2\pi i f(i)$$

$$= \cancel{2}\pi i * \frac{-1}{\cancel{2}} = -\pi i$$

→ Extended Cauchy's integral Formula:

$$F_{(z_0)} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)} dz$$

$$F(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)} d\xi$$

$$F'(z) = \frac{1}{2\pi i} \oint \frac{f(\xi)}{(\xi - z)^2} d\xi$$

$$F''(z) = \frac{2!}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)^3} d\xi$$

$$F'''(z) = \frac{3!}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)^4} d\xi$$

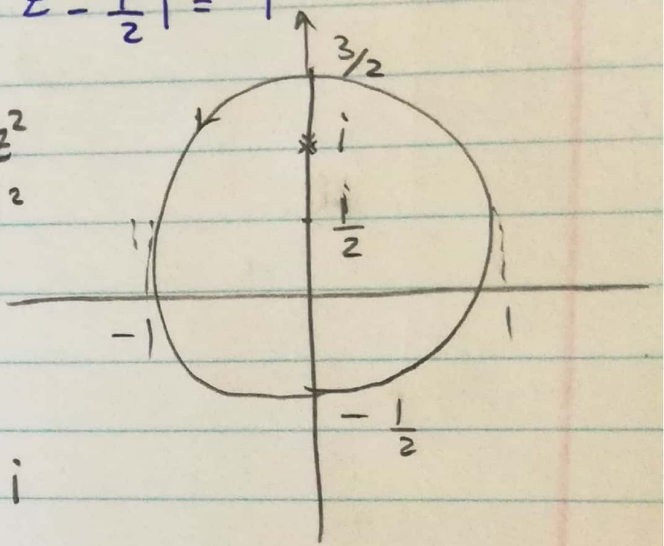
$$F^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$$

$$\boxed{F^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz}$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Ex: Page 107

$$\oint_C \frac{z^3}{(z-i)^2}\, dz \qquad c \quad \left| z - \frac{i}{2} \right| = 1$$



$z_0 = i \qquad f(z) = z^3 \rightarrow f'(z) = 3z^2$

$f'(i) = 3i^2$

$$\oint_C \frac{z^3}{(z-i)^2}\, dz = \frac{2\pi i}{1!}\, f'(i)$$

$$= 2\pi i * -3 = -6\pi i$$