Sheet 1

1. Find (in the form $x+iy$): let: $Z_1 = 3+4i$ & $Z_2 = 5-2i$

* $(Z_1 - Z_2)^2 = (3-5+(4+2)i)^2 = (-2+6i)^2 = 4-36-24i = -32-24i$

* $\frac{Z_1}{Z_2} = \frac{3+4i}{5-2i} \times \frac{5+2i}{5+2i} = \frac{(3\times5-4\times2)+(3\times2+4\times5)i}{5\times5+2\times2} = \frac{7}{29}+\frac{26}{29}i$

* $\frac{1}{Z_1^2} = \frac{1}{(3+4i)^2} = \frac{1}{(3^2-4^2)+(2\times3\times4)i} = \frac{1}{-7+24i} \times \frac{-7-24i}{-7-24i} = \frac{-7-24i}{(-7)^2+(24)^2} = \frac{-7}{625}+\frac{-24}{625}i$

* $Re(Z_1^3) = 3\times3\times3+3\times 4i\times4i\times3 = -117$, $Re(Z_2^3) = 5\times5\times5+3\times5\times(-2i)\times(-2i) = 185$

* $Img\left(\frac{1}{Z_1^2}\right) = \frac{-24}{625}\cancel{i}$, $Img\left(\frac{1}{Z_2^2}\right) = \frac{20}{841}\cancel{i}$

2. Find:

a. $\left|\frac{1+4i}{4+i}\right| = \left|\frac{(1+4i)\times(4-i)}{4^2+1^2}\right| = \left|\frac{(1\times4-4i\times i)+(4\times4-1\times1)i}{17}\right| = \sqrt{\left(\frac{8}{17}\right)^2+\left(\frac{15}{17}\right)^2} = \underline{1}$

b. $\left|\frac{z-1}{z+1}\right| = \left|1-\frac{2}{z+1}\right| = \left|1-\frac{2}{(a+1)+bi}\right| = \left|\frac{a-1+bi}{(a+1)+bi} \times \frac{(a+1)-bi}{(a+1)-bi}\right| = \left|\frac{(a^2+b^2-1)\overset{-bai+bi}{-b(a-1)i}+\overset{bai+bi}{b(a+1)i}}{(a+1)^2+b^2}\right| = \sqrt{\left(\frac{a^2+b^2-1}{(a+1)^2+b^2}\right)^2+\left(\frac{2b}{(a+1)^2+b^2}\right)^2}$

3. Represent in polar form:

a. $-3-3i$

$r = \sqrt{(-3)^2+(-3)^2} = \underline{3\sqrt{2}}$

$\theta = \tan^{-1}\left(\frac{-3}{-3}\right) = 45^\circ \rightarrow$ third quarter $\theta = 180+45 = \underline{225^\circ}$

$Z = 3\sqrt{2}\left[\cos(225)+i\sin(225)\right]$

b. -5

$r = \sqrt{(5)^2+(0)^2} = \underline{5}$

$\theta = \tan^{-1}\left(\frac{0}{-5}\right) = \underline{0^\circ}$

$Z = 5\left[\underset{\downarrow 1}{\cos(0)} + i\underset{\downarrow 0}{\sin(0)}\right] = \underline{5}$

c. $-4i$

$r = \sqrt{(0)^2+(-4)^2} = \underline{4}$

$\theta = \tan^{-1}\left(\frac{-4}{0}\right) = \frac{\pi}{2} = \underline{90^\circ}$ $\rightarrow$ $-ve \rightarrow$ $90+180 = 270$

$Z = 4\left[\underset{\downarrow 0}{\cos\left(\overset{90}{\frac{\pi}{2}}\right)} + i\underset{\downarrow -1}{\sin\left(\overset{270}{\frac{3\pi}{2}}\right)}\right] = -4i$

d. $\frac{1}{4+3i} = \frac{4-3i}{4^2+3^2} = \frac{4}{25}-\frac{3}{25}i$

$r = \sqrt{\left(\frac{4}{25}\right)^2+\left(\frac{-3}{25}\right)^2} = \underline{\frac{1}{5}}$

$\theta = \tan^{-1}\left(\frac{-3}{4}\right) = \underline{-36.87^\circ} \rightarrow$ fourth quarter $\theta = -36.87^\circ$

$Z = \frac{1}{5}\left[\cos(-36.87)+i\sin(-36.87)\right]$

Q: Show that $Z = X + iy$ is a pure imaginary if and only if $-Z = \bar{Z}$

Solution: let $Z = X + iy$, then: $\bar{Z} = X - iy$

$$R.H.S = X - iy$$

$$L.H.S = -Z = -(X + iy) = -X - iy$$

$$L.H.S = R.H.S \longrightarrow -X - iy = X - iy$$

$$-X = X$$

$$X = 0$$

$$\therefore Re(Z) = 0 \rightsquigarrow \therefore Z = Im(Z) \cdot i = iy$$

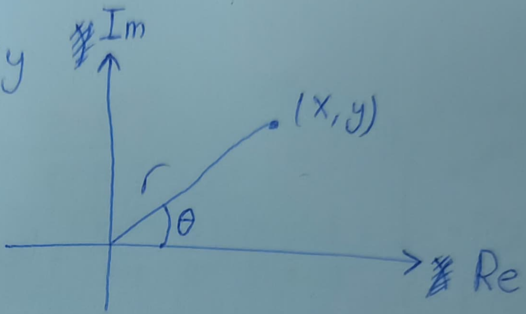
$\therefore Z = X + iy$ is pure imaginary if real part equals zero.

☆ Section - 2 ☆ mathematics ☆

Recap: complex number $z = x + iy$

$$Re(z) = x \quad , \quad Im(z) = y$$

$$z = r\angle\theta = r[\cos(\theta) + i\sin(\theta)]$$



$x = r\cos\theta$

$y = r\sin\theta$

$r = |z| = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \arg(z)$

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $|z_1 + z_2| \leq |z_1| + |z_2|$

***powers of z:**

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

***roots of z:**

$$\sqrt[m]{z} = \sqrt[m]{r}\left[\cos\left(\frac{\theta + 2\pi k}{m}\right) + i\sin\left(\frac{\theta + 2\pi k}{m}\right)\right]$$

$k = 0, 1, 2, \ldots, m-1$

✯ section-3 ✯ mathematics ✯

Recap

complex numbers $z = x+iy = re^{i\theta}$

- $e^{i\theta} = \cos\theta + i\sin\theta$
- $F(z) = u + iv$

$F'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

or

$F'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

- $F(z)$ is analytic function if satisfy Cauchy Reimann equations

$U_x = V_y$, $U_y = -V_x$

$U_r = \frac{1}{r}V_r$, $U_\theta = \frac{-1}{r}V_\theta$

- $F(z)$ is harmonic if

$U_{xx} + U_{yy} = 0$

$V_{xx} + V_{yy} = 0$

find $f'(z)$

$*\ f(z) = z^2$

Sol. $z = x+iy$

$f(x,y) = f(z) = (x+iy)^2 = x^2-y^2+2ixy$

$u(x,y) = x^2-y^2$, $v(x,y) = 2xy$

$f'(z) = U_x + iV_x = 2x + i2y$

$*\ f(z) = \frac{z+1}{z-2}$

Sol. $f(x,y) = \frac{x+1+iy}{x-2+iy} \cdot \frac{(x-2-iy)}{(x-2-iy)}$

$f(x,y) = \frac{(x+1)(x-2)+y^2}{(x-2)^2+y^2} + i\left(\frac{-3y}{(x-2)^2+y^2}\right)$

$U(x,y) = \frac{(x+1)(x-2)+y^2}{(x-2)^2+y^2} \longrightarrow U_x$

$V(x,y) = \frac{-3y}{(x-2)^2+y^2} \longrightarrow V_x$

$U = x$, $V = y$

Sol.

$U_x = 1$ $\quad V_x = 0$

$U_y = 0$ $\quad V_y = 1$

$\because U_x = V_y$, $U_y = -V_x$

$\therefore$ analytic function

$U = e^x\cos(y)$, $V = e^x\sin(y)$

Sol.

$U_x = e^x\cos y$ $\quad V_x = e^x\sin y$

$U_y = -e^x\sin y$ $\quad V_y = e^x\cos y$

$\because U_x = V_y$, $U_y = -V_x$

$\therefore$ analytic function

$U = x^3-3xy^2$, $V = e^x\sin(y)$

Sol.

$U_x = 3x^2-3y^2$ $\quad V_x = e^x\sin(y)$

$U_y = -6xy$ $\quad V_y = e^x\cos(y)$

$\because U_x \neq V_y$, $U_y \neq -V_x$

$\therefore$ not analytic function

$f(z) = z^3 + z$

Sol. $f(x,y) = (x+iy)^3 + x + iy$

$f(x,y) = x^3 + 3ix^2y - 3xy^2 - iy^3 + x + iy$

$f(x,y) = \underbrace{x^3-3xy^2+x}_{u(x,y)} + i\underbrace{(3x^2y-y^3+y)}_{V(x,y)}$

$U_x = 3x^2-3y^2$ $\quad V_x = 6xy$

$U_y = -6xy$ $\quad V_y = 3x^2-3y^2$

$\because U_x = V_y$, $U_y = -V_x$

$\therefore f(z)$ is analytic function

$f(z) = |z|^2$

Sol. $f(x,y) = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$

$U = x^2+y^2$, $V = 0$

$U_x = 2x$, $V_x = 0$

$U_y = 2y$, $V_y = 0$

$\because U_x \neq V_y$, $U_y \neq -V_x$

$\therefore f(z)$ is not analytic function

$f(z) = e^x\cos y$

Sol. $f(x,y) = e^x\cos y$

$U = e^x\cos y$ $\quad V = 0$

$U_x = e^x\cos y$ $\quad V_x = 0$

$U_y = -e^x\sin y$ $\quad V_y = 0$

$f(x)$ is not analytic

$f(z) = Im(z)$

Sol. $f(x,y) = Im(x+iy) = y$

$U = y$, $V = 0$

$U_x = 0$ $\quad V_x = 0$

$U_y = 1$ $\quad V_y = 0$

$\because U_x = V_y$ but $U_y \neq -V_x$

$\therefore f(z)$ is not analytic

$f(z) = \bar{z}$

Sol. $f(x,y) = \overline{(x+iy)} = x - iy$

$U = x$, $V = -y$

$U_x = 1$ $\quad V_x = 0$

$U_y = 0$ $\quad V_y = -1$

$\because U_x \neq V_y$, $U_y = -V_x$

$\therefore f(z)$ is not analytic

$f(z) = \arg(z)$

$f(r,\theta) = \arg(re^{i\theta}) = \theta$

$U = \theta$, $V = 0 \longrightarrow U_r = 0, V_r = 0$; $U_\theta = 1, V_\theta = 0$

$f(z)$ is not analytic

$U = x$ is harmonic & find harmonic conjugate

Sol.

$\left.\begin{array}{l} U_x = 1 \\ U_{xx} = 0 \\ U_y = 0 \\ U_{yy} = 0 \end{array}\right\}$ harmonic

$U_x = V_y = 1$

$V_y = 1$

$V = \int dy = y + g(x)$

$U_y = -V_x$

$0 = -(0+g'(x))$

$g'(x) = 0$

$g(x) = C$

$V = y + C$

$V = xy$

$\left.\begin{array}{l} V_x = y \\ V_{xx} = 0 \\ V_y = x \\ V_{yy} = 0 \end{array}\right\}$ harmonic

~~$U_x = V_y = x$~~

~~$U = \int x\,dx$~~

$U_y = -V_x$

$U_y = -y$

$U = \int -y\,dy = -\frac{y^2}{2} + g(x)$

$U_x = V_y$

$g'(x) = x$

$g(x) = \frac{x^2}{2} + C$

$U = \frac{-y^2}{2} + \frac{x^2}{2} + C = \frac{1}{2}(x^2-y^2+C)$

$f(z) = \frac{1}{2}(x^2-y^2+C) + ixy$

Sheet : $U = \frac{x}{x^2+y^2}$ $\quad$ *harmonic? Find harmonic conj.

☆ section-4 ☆ complex numbers ☆ mathematics ☆

* $\sin(1-4i) = \cos(1)\sin(-4i) + \sin(1)\cos(-4i)$

$= -i\cos(1)\sinh(4) + \sin(1)\cosh(4)$

$= \sin(1)\cosh(4) - i\cos(1)\sinh(4)$

$= 22.98 - 14.74i$

*Recap:

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$\cosh(\theta) = \cos(i\theta) = \frac{1}{2}(e^{\theta} + e^{-\theta})$

$\sinh(\theta) = \frac{1}{i}\sin(i\theta) = -i\sin(i\theta) = \frac{1}{2}(e^{\theta} - e^{-\theta})$

$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

$\ln(z) = \ln(|z|) + i(\arg(z) + 2\pi k)$

$k = 0, 1, \ldots$

* $\sin(z) = 3i$. find z.

sol.1

$\frac{1}{2i}(e^{iz} - e^{-iz}) = 3i$

$e^{iz} - e^{-iz} = -6$

$e^{2iz} - 1 = -6e^{iz}$

$e^{2iz} + 6e^{iz} - 1 = 0$

$e^{iz} = -3 + \sqrt{10}$ | $e^{iz} = -3 - \sqrt{10}$ X

$iz = \ln(-3 + \sqrt{10})$

$z = -i\ln(-3 + \sqrt{10})$

sol.2

$\sin(z) = 3i$

$\sin(x)\cosh(y) + i\cos(x)\sinh(y) = 3i$

$\sin(x)\cosh(y) = 0$ | $\cos(x)\sinh(y) = 3$

$\sin(x) = 0$ | $\cos(n\pi)\sinh(y) = 3$

$\boxed{x = n\pi}$ | $(-1)^n \sinh(y) = 3$

n is even $\rightarrow \sinh(y) = 3$

$\boxed{y = \sinh^{-1}(3)}$

$z = n\pi + i\sinh^{-1}(3)$

Sheet.

* $e^{3z} = 3$

sol

$\ln(e^{3z}) = \ln(3)$

$3z = \ln(3)$

$z = \frac{1}{3}\ln(3)$

$\because \ln(z) = \ln(|z|) + i(\theta + 2\pi k)$

$z = \frac{1}{3}[(\ln(3) + i(0 + 2\pi k))]$, $k = 0, 1, \ldots$

* $\ln(z) = \frac{-\pi}{2} i$

$\cancel{\ln(z) = \ln(|z|) + i(\theta + 2\pi k)}$

$\cancel{z = \ln(1 - \frac{\pi}{2} i) + i(}$

$e^{\ln(z)} = e^{\frac{-\pi}{2} i}$

$z = e^{-\frac{\pi}{2} i + 2\pi k}$, $k = 0, 1, \ldots$

$z = e^{2\pi k}(\cos(\frac{\pi}{2}) - i\sin(\frac{\pi}{2}))$

* $(1+i)^{1-i}$

$\because e^{\ln(z)} = z$

$e^{\ln[(1+i)^{1-i}]} = z$

$e^{(1-i)\ln(1+i)} = z$

$e^{(1-i)} e^{\ln(1+i)} = z$

$(1+i)e^{(1-i)} = z$

$(1+i)\, e \cdot e^{-i} = z$

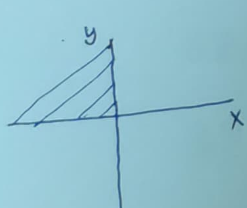
$(1+i)e(\cos(1) - i\sin(1)) = z$

$z = e[(\cos(1) - i\sin(1)) + (\sin(1) + i\cos(1))]$

$z = e(\cos(1) + \sin(1)) + ie(\cos(1) - \sin(1))$

* Conformal mapping:

• $\arg(z)$: $\frac{\pi}{2} \le \arg(z) \le \pi$, $w = z^2$



(z-plane)

(w-plane)

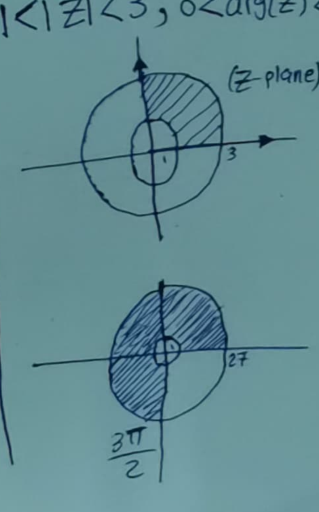
• $w = e^z$

$u + iv = e^{x+iy} = e^x(\cos(y) + i\sin(y))$

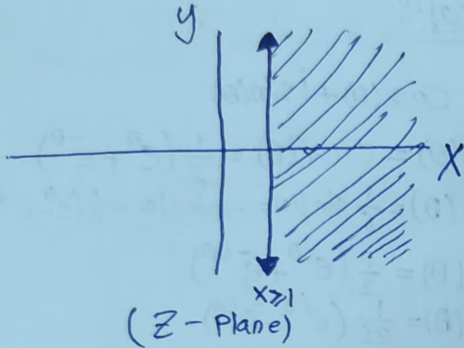
$u = e^x\cos(y)$, $v = e^x\sin(y)$

$w = z^3$

• $1 < |z| < 3$, $0 < \arg(z) < \frac{\pi}{2}$



$* \quad x \geq 1 \ , \ w = \frac{1}{z}$



$$w = \frac{1}{z}$$

$$\rho e^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \longrightarrow \rho = \frac{1}{r} \ , \ \phi = -\theta$$

$$|w| = \frac{1}{|z|} \ , \ \arg(w) = -\arg(z)$$

$$\arg(w) = -\arg(z)$$

or

$$w = \frac{1}{z}$$

$$\rho\left(\cos(\phi) + i\sin(\phi)\right) = \frac{1}{r}\left(\cos(\theta) - i\sin(\theta)\right)$$

$$\rho\cos(\phi) = \frac{1}{r}\cos(\theta) \ , \ \rho\sin(\phi) = \frac{-1}{r}\sin(\theta)$$

$$x \geq 1$$

$$r\cos(\theta) \geq 1$$

$$\frac{1}{\rho}\cos(\phi) \geq 1$$

$$\boxed{\rho \leq \cos(\phi)}$$

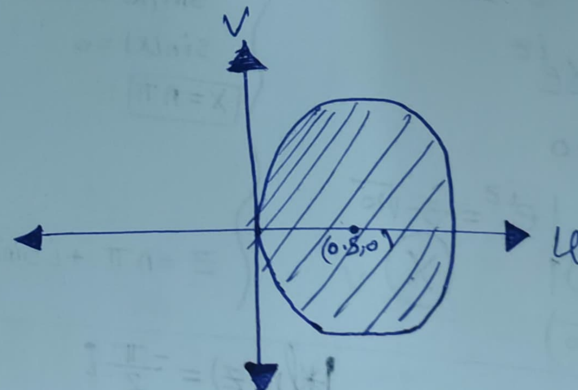
$$\cancel{\cos(\phi)}$$

$$\sqrt{u^2+v^2} \leq \frac{u}{\sqrt{u^2+v^2}}$$

$$u^2+v^2 \leq u$$

$$u^2 - u + v^2 \leq 0$$

$$\left(u-\frac{1}{2}\right)^2 + v^2 \leq \frac{1}{4}$$



☆ section-5 ☆ mathematics ☆

* 17.4: Find the image and sketch for $w = e^{z}$

$$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1}$$

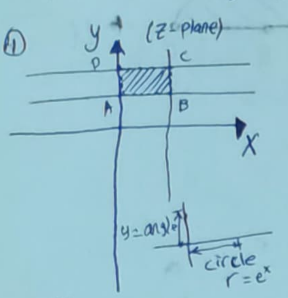
4. $0<x<1,\ \frac{1}{2}<y<1$

$$W(u,v) = f(z(x,y))$$
$$w(R,\phi) = f(z(r,\theta))$$
$$w = e^{z} = e^{x+iy}$$
$$w = u+iv = e^{x}\left(\cos(y)+i\sin(y)\right)$$

$u = e^{x}\cos(y)$ | $|w| = R = e^{x}$

$v = e^{x}\sin(y)$ | $\arg(w) = \phi = y$

①



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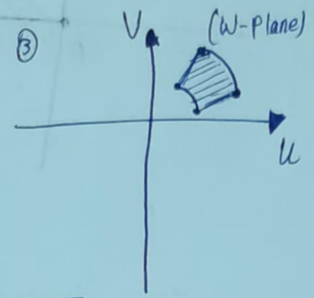
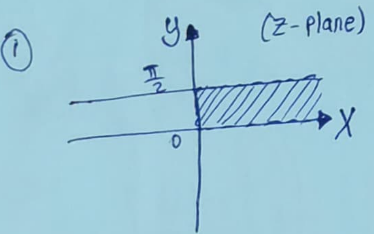


②

	Z	W		
A	(0, 0.5)	(cos(0.5), sin(0.5))		(0.89, 0.48)
B	(1, 0.5)	(e cos(0.5),	e sin(0.5))	(2.4, 1.3)
C	(1, 1)	(e cos(1),	e sin(1))	(1.5, 2.3)
D	(0, 1)	(cos(1),	sin(1))	(0.5, 0.8)

6. $0<x<\infty,\ 0<y<\frac{\pi}{2}$

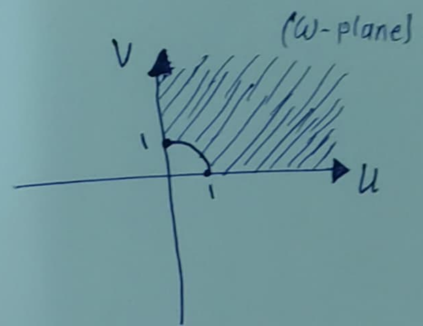
①



②

z	w	P
(0,0)	(cos(0), sin(0))	(1,0)
(0, π/2)	(cos(π/2), sin(π/2))	(1,1)

③



* 17.3: $W = \dfrac{az+b}{cz+d}$

$0, 1, 2 \longrightarrow 1, \frac{1}{2}, \frac{1}{3}$

$z_1\ z_2\ z_3 \qquad w_1\ w_2\ w_3$

$$\frac{w-1}{w-\frac{1}{3}}\cdot\frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{2}-1} = \frac{z-0}{z-2}\cdot\frac{1-2}{1-0}$$

$$\frac{1}{3}\left(\frac{w-1}{w-\frac{1}{3}}\right) = \frac{z}{z-2}$$

$$3z\left(w-\tfrac{1}{3}\right) = (z-2)(w-1)$$

$$3zw-\cancel{z} = zw-\cancel{z}-2w+2$$

$$3zw-zw+2w = 2$$

$$w(3z-z+2) = 2 \longrightarrow w = \frac{2}{2z+2} = \boxed{\frac{1}{z+1}}$$

$a=0,\ b=1,\ c=1,\ d=1$

$0, -i, i \longrightarrow -1, 0, \infty$

$$\frac{w+1}{w-\infty}\cdot\frac{0-\infty}{0+1} = \frac{z-0}{z-i}\cdot\frac{-i-i}{-i-0}$$

$$\frac{w+1}{1}\cdot\underbrace{\boxed{\frac{0-\infty}{w-\infty}}}_{=1} = \frac{2z}{z-i}$$

$$w+1 = \frac{2z}{z-i}$$

$$w = \frac{2z-(z-i)}{z-i} = \frac{z+i}{z-i}$$

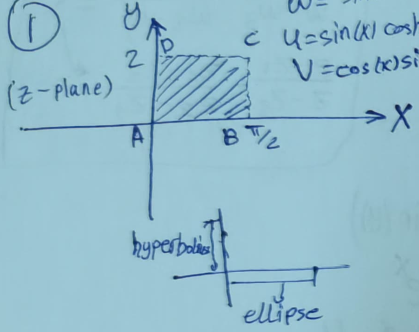
$a=1,\ b=i,\ c=1,\ z=-i$

* 17.4:

• **11.** $w = \sin(z)$, $0 < x < \frac{\pi}{2}$, $0 < y < 2$

$w = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

①

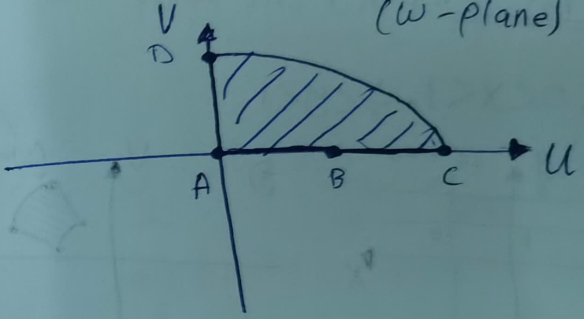


$u = \sin(x)\cosh(y)$
$V = \cos(x)\sinh(y)$

②

	Z	W
A	(0,0)	(0,0)
B	$(\frac{\pi}{2},0)$	(1,0)
C	$(\frac{\pi}{2},2)$	(3.7, 0)
D	(0,2)	(0, 3.6)

③



☆section-6 ☆mathematics ☆

☆Recap:

$$\int_{z_0}^{z_1} f(z)\,dz$$

*cauchy's Integral Formula

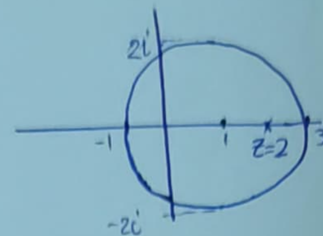
$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i\, F(z_0)$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

*Q1: $\oint_C \frac{z+2}{z-2}dz$, $c: |z-1|=2$

Sol:

$$I = 2\pi i\left(z+2\Big|_{z=2}\right) = 8\pi i$$

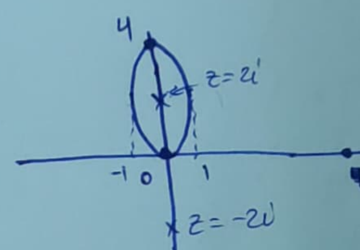


*Q2: $\oint_C \frac{1}{z^2+4}$, $C: 4x^2+(y-2)^2=4$

Sol: $z^2+4=0 \rightarrow z=\pm 2i$

$$I = \int \frac{1}{(z+2i)(z-2i)}dz = \int \frac{\left(\frac{1}{z+2i}\right)}{z-2i}dz$$

$$I = 2\pi i\left(\frac{1}{z+2i}\Big|_{z=2i}\right) = \frac{\pi}{2}$$

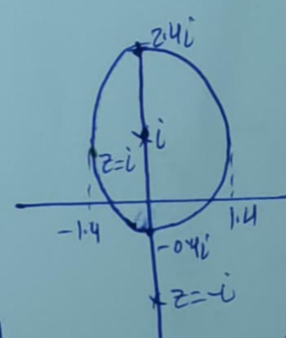


*Q3: $\oint_C \frac{\ln(z+1)}{z^2+1}dz$; $C: |z-i|=1.4$

Sol:

$$I = \int \frac{\ln(z+1)}{(z+i)(z-i)}dz = \int \frac{\left(\frac{\ln(z+1)}{z+i}\right)}{z-i}dz$$

$$I = 2\pi i\left(\frac{\ln(z+1)}{z+i}\Big|_{z=i}\right) = 2\pi i\left(\frac{\ln(i+1)}{2i}\right) = \pi\left(\ln(\sqrt{2}) + i\frac{\pi}{4}\right)$$

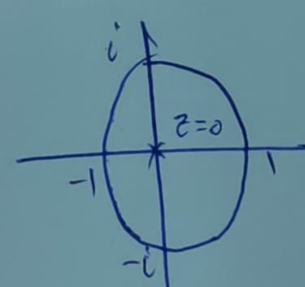


*Q4: $\oint_C \frac{\sin(z)}{z^4}dz$, c: unit circle

Sol:

$$I = \int \frac{\sin(z)}{z^4}dz \quad , \quad n+1=4, \quad n=3$$

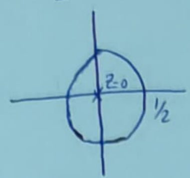
$$I = \frac{2\pi i}{3!}\left(\sin(z)\right)^{(III)}\Big|_{z=0} = \frac{-\pi i}{3}$$



☆section-8☆ mathematics☆

16.3:

21. $\oint_C \frac{\cos(\pi z)}{z^5}\,dz\ ;\ C: |z| = \frac{1}{2}$



Sol

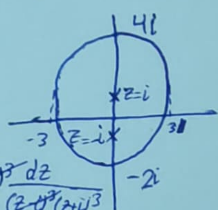
$f(z) = \frac{\cos(\pi z)}{z^5}$

- $\underset{z=0}{\text{Res}}\, f(z) : \frac{d^4}{dz^4}\,(z-0)^5\,\frac{\cos\pi z}{z^5} = \pi^4\cos(\pi z)$

$\frac{1}{4!}\lim_{z\to 0} \pi^4\cos(\pi z) = \frac{\pi^4}{4!}$

$\oint_C \frac{\cos(\pi z)}{z^5} = 2\pi i\left(\frac{\pi^4}{4!}\right)$

20. $\oint \frac{dz}{(z^2+1)^3}\ ;\ C: |z-i| = 3$



Sol

$I = \oint \frac{dz}{(z-i)^3(z+i)^3}$

$\underset{z=i}{\text{Res}}\, f(z) = \frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\,\cancel{(z-i)^3}\,\frac{dz}{\cancel{(z-i)^3}(z+i)^3}$

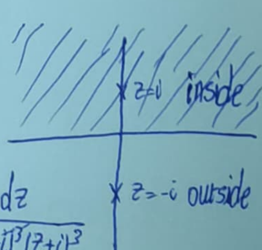
$= \frac{1}{2!}\lim_{z\to i} 12(z+i)^{-5} = -\frac{3}{16}i$

$\underset{z=-i}{\text{Res}}\, f(z) = \frac{1}{2!}\lim_{z\to -i}\frac{d^2}{dz^2}\,\cancel{(z+i)^3}\,\frac{dz}{(z-i)^3\cancel{(z+i)^3}}$

$= \frac{1}{2!}\lim_{z\to i} 12(z-i)^{-5} = \frac{3}{16}i$

$I = 2\pi i\left(\frac{-3}{16}i + \frac{3}{16}i\right) = \text{zero}$

10. $\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^3}$



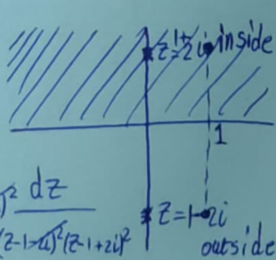
Sol let $x = z \to dx = dz$

$\underset{z=i}{\text{Res}}\, f(z) = \frac{1}{2!}\lim_{z\to i}\,\cancel{(z-i)^3}\,\frac{dz}{\cancel{(z-i)^3}(z+i)^3}$

$= -\frac{3}{16}i$

$I = 2\pi i\left(\frac{-3}{16}i\right) = \frac{3}{8}\pi$

12. $\int_{-\infty}^{\infty}\frac{dx}{(x^2-2x+5)^2}$



Sol let $x = z \to dx = dz$

$\underset{z=1+2i}{\text{Res}}\, f(z) = \frac{1}{1!}\lim_{z\to 1+2i}\,\cancel{(z-1-2i)^2}\,\frac{dz}{\cancel{(z-1-2i)^2}(z-1+2i)^2}$

$= \frac{-2}{(4i)^3} = \frac{-i}{32}$

$I = 2\pi i\left(\frac{-i}{32}\right) = \frac{\pi}{16}$

* Recap.

Res of $f(z)$ at m^{th} order pole at z_0

$\underset{z=z_0}{\text{Res}}\, f(z) = \frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)\right\}$

- $\oint_C f(z)\,dz = 2\pi i\sum \text{Res}\, f(z)$
- $\underset{z=z_0}{\text{Res}}\,\frac{P(z)}{q(z)} = \frac{P(z)}{q'(z)}$

special cases:

- $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\sum_{\text{upper half plane}} \text{Res}\, f(z)$
- $\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx = \pi i\sum \text{Res}\, f(z)$

if $f(z)$ is even

- $\int_{-\infty}^{\infty} f(x)\cos(mx)\,dx = -2\pi\sum \text{Im}\left\{\text{Res}\left(f(z)e^{imz}\right)\right\}$
- $\int_{-\infty}^{\infty} f(x)\sin(mx)\,dx = 2\pi\sum \text{Re}\left\{\text{Res}\left(f(z)e^{imz}\right)\right\}$

17. $\int_{-\infty}^{\infty}\frac{\sin(3x)}{x^4+1}\,dx$

Sol let $x = z \to dx = dz$

Res

☆ section-9 ☆ mathematics ☆ integration using series ☆

15.4

5. $\frac{1}{2+z^2} = \frac{1}{2} \frac{1}{1-\left(\frac{-z^4}{2}\right)}$

let $\omega = \frac{-z^4}{2} \rightsquigarrow f(z) = \frac{1}{2} \cdot \frac{1}{1-\omega}$

$$f(z) = \frac{1}{2}\left(1+\omega+\omega^2+\omega^3+\cdots\right)$$

$$f(z) = \frac{1}{2}\left(1-\frac{z^4}{2}+\frac{z^8}{4}-\frac{z^{12}}{8}+\cdots\right) \Leftarrow$$

for $\left|\frac{-z^4}{2}\right| < 1$

$\left|\frac{z^4}{2}\right| < 1 \rightarrow \boxed{|z^4| < 2}$

9. $\int_0^z \exp\left(\frac{-t^2}{2}\right) dt$

$$e^{\frac{-z^2}{2}} = 1-\frac{z^2}{2}+\frac{z^4}{8}-\frac{z^6}{8\cdot 6}+\cdots$$

$$I = \int_0^z 1-\frac{t^2}{2}+\frac{t^4}{8}-\frac{t^6}{}+\cdots$$

$$I = \left[t-\frac{t^3}{6}+\frac{t^5}{40}\cdots\right|_0^z = z-\frac{z^3}{6}+\frac{z^5}{40}\cdots$$

7. $\cos^2\left(\frac{1}{2}z\right)$

$\because \cos^2(\theta) = \frac{1}{2}\left(1+\cos(2\theta)\right)$

$$f(z) = \frac{1}{2}\left(1+\cos(z)\right)$$

$$f(z) = \frac{1}{2}\left(2-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)$$

* Recap:

- Taylor series:

$$f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n, \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

- Maclarin series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$$

$$\sin(z) = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots$$

$$\cos(z) = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots$$

$$\ln(1+z) = z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\cdots \quad |z|<1$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+z^3+\cdots, \quad |z|<1$$

$$\int_C f(z)\,dz = 2\pi i\left(\text{coeff. of } \frac{1}{z-z_0} \text{ in Taylor series}\right)$$

↳ has a singularity point at $z=z_0$

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{1}{2!}(z-z_0)^2 f''(z_0) + \cdots$$