lec 4

Ex: show that $\lim\limits_{z \to 0} \frac{z}{\bar{z}}$ does not exist

$y - y_0 = m(x - x_0)^k$

$$\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{(x,y)\to(0,0)} \frac{x+iy}{x-iy} = \lim_{(x,y)\to(0,0)} \frac{(x+iy)(x+iy)}{(x-iy)(x+iy)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2 + i\,2xy}{x^2 + y^2}$$

$y = mx$

$$L = \lim_{(x,y)\to(0,0)} \frac{x^2 - m^2x^2 + 2ixmx}{x^2 + m^2x^2} \begin{matrix} \div x^2 \\ \div x^2 \end{matrix} = \lim_{\substack{\cancel{(x,y)\to(0,0)} \\ x \to 0}} \frac{1 - m^2 + 2im}{1 + m^2}$$

$$= \frac{1 - m^2 + 2im}{1 + m^2}$$

Limit depends on m

$\therefore$ The limit does not exist

Continuty of a Function:

- the Function $F(z)$ is said to be Continous at z_0 if $F(z)$ satisfy the Following Conditions:

(1) $F(z_0)$ defined

(2) $\lim\limits_{z \to z_0} F(z)$ exists

(3) $F(z_0) = \lim\limits_{z \to z_0} F(z)$

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ex:

$$F(z) = \frac{1-z}{1+z}$$

$F(z)$ is continous For $\forall z$ except at $z=-1$

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$$F(z) = \frac{1-z}{e^z+2}$$

$$e^z = -2$$

$$z = \ln(-2) = \ln(2) + i(\pi + 2\pi k)$$

$F(z)$ is continous For $\forall z$ except at $z = \ln(2) + i(\pi + 2\pi k)$

→ Differentiability of a Function:

$$\frac{d\,f(z)}{dz} = f'(z) = \lim_{l \to 0} \frac{f(z+l) - f(z)}{l}$$

ex. $F(z) = z^2$

$F'(z) = 2z$ #

→ using the definition

$$f'(z) = \lim_{l \to 0} \frac{(z+l)^2 - z^2}{l} = \lim_{l \to 0} \frac{\cancel{z^2} + 2lz + l^2 - \cancel{z^2}}{l}$$

$$= \lim_{l \to 0} \frac{\cancel{l}(2z+l)}{\cancel{l}} = 2z \text{ \#}$$

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ex: $F(Z) = Z^4 + Z^3 + Z^2 + 1$

$F'(Z) = 4Z^3 + 3Z^2 + 2Z$

→ Cauch - Riemann conditions:

$$F(Z) = u + iv$$

$\boxed{1}$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ → $(u_x = v_y)$ $\boxed{2}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ → $(u_y = -v_x)$

∴ $F'(z)$ exist & $F(z)$ analytic Function

$$\boxed{F'(z) = u_x + i v_x}$$

$$= v_y - i u_y$$

→ if $F(z)$ is analytic Function then $F(z)$ is entire Function

ex: $F(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x[\underbrace{\cos y}_{u} + i\sin y]$

$u = e^x \cos y$ $v = e^x \sin y$

$u_x = e^x \cos y$ $v_y = e^x \cos y$

$u_y = -e^x \sin y$ $v_x = e^x \sin y$

∴ $u_x = v_y$ & $u_y = -v_x$ → Cauch - Riemann satisfied

∴ $F(z)$ is analytic Function

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$F'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y$

Find $F'(z)$ in terms of z & y

$z = x + iy$
$\bar{z} = x - iy$ $\Big\}$ $x = \frac{z+\bar{z}}{2}$

$F'(z) = e^x(\cos y + i \sin y)$

$= e^x e^{iy} = e^{x+iy} = e^z$ #

$\leftarrow$ or $\rightarrow$ substitude $y = \frac{z-\bar{z}}{2i}$

Cauch - Riemann Conditions in Polar Form :

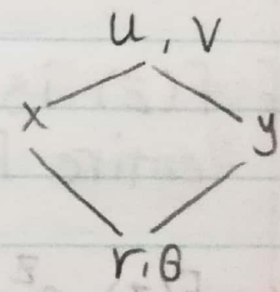
$F(z) = u + iv$

$x = r\cos\theta$, $y = r\sin\theta$

$\boxed{1}$ $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ $\boxed{2}$ $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$

$= u_x \cos\theta + u_y \sin\theta$



$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$

$= v_x * (-r\sin\theta) + v_y (r\cos\theta) = r[-v_x \sin\theta + v_y \cos\theta]$

$\because u_x = v_y$ & $u_y = -v_x$

$\frac{\partial v}{\partial \theta} = r(u_y \sin\theta + u_x \cos\theta) = r\frac{\partial u}{\partial r}$

$\boxed{\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$ #

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Harmonic Function:

$$F(z) = u + iv$$

$\boxed{1}$ $\nabla^2 u = \underset{\frac{\partial^2 u}{\partial x^2}}{u_{xx}} + \underset{\frac{\partial^2 u}{\partial y^2}}{u_{yy}} = 0 \Rightarrow u$ is harmonic

$\boxed{2}$ $\nabla^2 v = \underbrace{v_{xx} + v_{yy} = 0}_{\text{La Place equ.}} \Rightarrow v$ is harmonic

$\therefore F(z) \downarrow$ harmonic

F(z) analytic $\Rightarrow$ F(z) harmonic

Proof:

$$\nabla^2 u = u_{xx} + u_{yy} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial}{\partial y} * -\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

$u_x = v_y$

$u_y = -v_x$

$\therefore u$ is harmonic

$$\nabla^2 v = v_{xx} + v_{yy} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

$\therefore v$ is harmonic

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$F(z) = \bar{z} = x - iy = u + iv$

$u = x$ $\quad$ $v = -y$

$u_x = 1$ $\quad$ $v_y = -1$

$\therefore u_x \neq v_y \Rightarrow f(z) = \bar{z}$ is not analytic Function

$u_{xx} = 0$ $\quad$ $v_x = 0$, $v_{xx} = 0$

$u_y = 0$, $u_{yy} = 0$ $\quad$ $v_{yy} = 0$

$\nabla^2 u = u_{xx} + u_{yy} = 0$ $\quad$ $\therefore u$ is harmonic —(1)

$\nabla^2 v = v_{xx} + v_{yy} = 0$ $\quad$ $\therefore v$ is harmonic —(2)

From (1), (2) $\longrightarrow$ $F(z)$ is harmonic

Although $F(z)$ is harmonic it's not analytic

Theorem:

IF $f(z) = u + iv$ is harmonic Function Then u and v are Conjugate harmonic for each other

ex: Page 65:

$u = \sin x \cosh y$

Show that u is harmonic , Find it's Conjugate harmonic

$u_x = \cos x \cosh y$ $\quad$ $u_{xx} = -\sin x \cosh y$

$u_y = \sin x \sinh y$ $\quad$ $u_{yy} = \sin x \cosh y$

$\nabla^2 u = u_{xx} + u_{yy} = -\sin x \cosh y + \sin x \cosh y = 0$

$\therefore u$ is harmonic Function

$u_x = v_y$ or $u_y = -v_x = -\frac{\partial v}{\partial x}$

$u_x = \frac{\partial v}{\partial y}$

$v = \int -u_y \, \partial x$

$v = \int u_x \, \partial y$

$= \int (\cos x \cosh y) \, \partial y = \cos x \sinh y + h(x)$

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$v_x = -\sin x \sinh y + \frac{dh}{dx} = -u_y = -\sin x \sinh y$

$\frac{dh}{dx} = 0 \qquad h(x) = c$

$v = \cos x \sinh y + c$

$F(z) = u + iv = \sin x \cosh y + i(\cos x \sinh y + c)$

$= \sin x \cosh y + i\cos x \sinh y + ic$

$= \sin x \cos iy + \cos x \sin iy + ic$

$= \sin(x + iy) + ic = \sin(z) + ic$

$F'(z) = u_x + i v_x$

$= \cos x \cosh y + i(-\sin x \sinh y)$

$= \cos x \cos iy - \sin x \sin iy$

$= \cos(x + iy) = \cos(z)$