→ Some Complex transformations :-

$$F(z) = u + iv$$

(1) Exponential transformation

$$F(z) = e^{z} = e^{x+iy} = e^{x} e^{iy}$$
$$= e^{x}(\cos y + i \sin y)$$

$|e^{z}| = e^{x}$, $\arg(e^{z}) = y$

$$u + iv = e^{x} \cos y + i e^{x} \sin y$$

$u = e^{x} \cos y$, $v = e^{x} \sin y$

(2) Trigonometric Functions

$$F(z) = \cos z$$
$$= \cos(x+iy)$$
$$= \cos(x)\cos(iy) - \sin(x)\sin(iy)$$

$$u + iv = \cos(x)\cosh(y) - i \sin(x)\sinh(y)$$

$u = \cos(x)\cosh(y)$, $v = -\sin(x)\sinh(y)$

$$|\cos z|^2 = \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y)$$
$$= \cos^2(x)\cosh^2(y) + (1 - \cos^2 x)\sinh^2(y)$$
$$= \cos^2(x)\underbrace{[\cosh^2 y - \sinh^2 y]}_{1} + \sinh^2 y$$

$= \cos^2 x + \sinh^2 y$

$f(z) = \sin z$

$= \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$u+iv = \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y \quad , \quad v = \cos x \sinh y$

$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$= \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y$

$= \sin^2 x \left[\cosh^2 y - \sinh^2 y \right] + \sinh^2 y$

$= \sin^2 x + \sinh^2 y$

(3) Hyperbolic Functions

$f(z) = \cosh z = \dfrac{e^z + e^{-z}}{2}$

$e^z = e^{x+iy}$

$e^{-z} = e^{-x-iy}$

نفكها زي ال exponential ونجمعهم

$f(z) = \sinh z = \dfrac{e^z - e^{-z}}{2}$

$f(z) = \tanh z = \dfrac{\sinh z}{\cosh z}$

$f(z) = \operatorname{sech} z = \dfrac{1}{\cosh z}$

$f(z) = \operatorname{csch} = \dfrac{1}{\sinh z}$

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(4) Logarithmic Function

$$\ln(z)$$

$$F(z) = \log(z) = \log(re^{i\theta})$$

$$= \ln r + \ln(e^{i\theta})$$

$u + iV$ $$= \ln r + i(\theta + 2\pi k)$$

$u = \ln r = \ln|z|$, $V = \theta + 2\pi k$

(5) Inverse Trigonometric Functions

$$\omega = \sin^{-1}(z)$$

$$\sin\omega = z = \frac{e^{i\omega} - e^{-i\omega}}{2i}$$

$e^{i\omega} = \cos\omega + i\sin\omega$

$e^{-i\omega} = \cos\omega - i\sin\omega$

$$e^{i\omega} - e^{-i\omega} = 2iz$$

$$e^{2i\omega} - 1 = 2iz\,e^{i\omega}$$

$$e^{2i\omega} - 2iz\,e^{i\omega} - 1 = 0$$

$$(e^{i\omega})^2 - 2iz(e^{i\omega}) - 1 = 0$$

$$e^{i\omega} = \frac{2iz \pm \sqrt{4i^2z^2 + 4}}{2}$$

$$e^{i\omega} = iz \pm \sqrt{1 - z^2}$$

$$\ln e^{i\omega} = \ln(iz \pm \sqrt{1-z^2})$$

$i\omega = \ln(iz \pm \sqrt{1-z^2})$ $\therefore\ \omega = \frac{1}{i}\ln(iz \pm \sqrt{1-z^2}) * \frac{-i}{-i}$

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(6) $(z_1)^{z_2} = e^{\ln (z_1)^{z_2}} = e^{z_2 \ln (z_1)}$

$= e^{z_2 \ln(r_1 e^{i\theta_1})} = e^{z_2[\ln r_1 + i(\theta_1 + 2\pi k)]}$

Page 97:

Solve $e^z = -2$ [take ln for both sides]

$z = \ln(-2)$ $\quad r=2 \quad \theta = \pi$

$z = \ln(2e^{i\pi}) = \ln 2 + i(\pi + 2\pi k)$

or

$e^{x+iy} = -2$

$e^x e^{iy} = -2$

$e^x(\cos y + i\sin y) = -2$

$e^x \cos y + i e^x \sin y = -2 \quad \therefore e^x \cos y = -2 \;\; \& \;\; e^x \sin y = 0$

• Solve the equation : Page 93 :

$$\sin z = 3$$

$$\sin(x+iy) = 3$$

$$\sin x \cosh y + i \cos x \sinh y = 3$$

$\sin x \cosh y = 3$

$y=0$

$\sin x = 3 \quad \because -1 \leq \sin x \leq 1$

unslovable equation

$\cos x \sinh y = 0$

$\cos x = 0$

$x = \left(\frac{2n+1}{2}\right)\pi$

$n = 0, \pm 1, \pm 2, \ldots$

$\sinh y = 0$

$y = 0$

refused

$\sin\left(\frac{2n+1}{2}\right)\pi \cosh y = 3$

$(-1)^n \cosh y = 3$

$n \to$ even

$\cosh y = 3$

$y = \cosh^{-1}(3)$

$$z = x+iy = \left(\frac{2n+1}{2}\right)\pi + i\cosh^{-1}(3)$$

$\to$ For n even

• Find the Principle value of, in the form of $u+iv$

$$(1+i)^{(1-i)}$$

$$e^{\ln(1+i)^{(1-i)}} = e^{(1-i)\ln(1+i)} = e^{(1-i)\ln(\sqrt{2}\, e^{i\frac{\pi}{4}})} = e^{(1-i)\left[\ln(\sqrt{2}) + i\frac{\pi + 2\pi k}{4}\right]}$$

Principle $\to k=0$

$$(1+i)^{(1-i)} = e^{(1-i)\left[\ln(\sqrt{2}) + i\frac{\pi}{4}\right]}$$

$u+iv = 2.80788 + i(1.31787)$

→ limit of a Function   Page 55

$\forall \varepsilon > 0 \ \exists \ \delta(\varepsilon) > 0 : |f(z) - L| < \varepsilon$ For $|z - z_0| < \delta$

i.e $\quad \lim\limits_{z \to z_0} f(z) = L$

Ex: Page 56

$$\lim_{z \to 2-i} \frac{z^2 + z + 1}{z^2 - z + 1} = \frac{(2-i)^2 + (2-i) + 1}{(2-i)^2 - (2-i) + 1}$$

$$= \frac{27}{13} + \frac{8}{13} i$$

Ex: Page 57

$$\lim_{z \to e^{\pi \frac{i}{3}}} \frac{z - e^{\pi \frac{i}{3}}}{z^4 + z^2 + 1} = \frac{0}{0}$$

use l'hpital

$$\lim_{z \to e^{\pi \frac{i}{3}}} \frac{1}{4z^3 + 2z}$$

$$= \frac{1}{4\left(e^{\frac{\pi i}{3}}\right)^3 + 2e^{\frac{\pi i}{3}}} = \frac{-1}{4} - \frac{\sqrt{3}}{12} i$$