DIFFERENTIAL EQUATIONS

WITH BOUNDING-VALUE PROBLEMS

THIRD EDITION

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45. A one-parameter family of solutions for

$$y' = y^2 - 1 \quad \text{is} \quad y = \frac{1 + ce^{2x}}{1 - ce^{2x}}$$

By inspection,* determine a singular solution of the differen

46. On page 6 we saw that $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$ of $dy/dx = -x/y$ on the interval $(-2, 2)$. Explain why

$$y = \begin{cases} \sqrt{4 - x^2}, & -2 < x < 0 \\ -\sqrt{4 - x^2}, & 0 \le x < 2 \end{cases}$$

is not a solution of the differential equation on the interval

In Problems 47 and 48 find values of m so that $y = e^{mx}$ is a solu differential equation.

47. $y'' - 5y' + 6y = 0$

48. $y'' + 10y' + 25y =$

In Problems 49 and 50 find values of m so that $y = x^m$ is a solu differential equation.

49. $x^2y'' - y = 0$

50. $x^2y'' + 6xy' + 4y$

51. Show that $y_1 = x^2$ and $y_2 = x^3$ are both solutions of

$$x^2y'' - 4xy' + 6y = 0.$$

Are the constant multiples c_1y_1 and c_2y_2, with c_1 and c_2 ar solutions? Is the sum $y_1 + y_2$ a solution?

52. Show that $y_1 = 2x + 2$ and $y_2 = -x^2/2$ are both solutions

$$y = xy' + (y')^2/2.$$

Are the constant multiples c_1y_1 and c_2y_2, with c_1 and c_2 ar solutions? Is the sum $y_1 + y_2$ a solution?

53. By inspection determine, if possible, a real solution of the ential equation.

(a) $\left|\frac{dy}{dx}\right| + |y| = 0$ (b) $\left|\frac{dy}{dx}\right| + |y| + 1 = 0$ (c)

1.2 SOME MATHEMATICAL MODELS

In science, engineering, economics, and even psychology, we often scribe or model the behavior of some system or phenomenon in m terms. This description starts with

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* Translated, this means try a good guess and see if it works.

(ii) identifying the variables that are responsible for changing the system, and

(iii) a set of reasonable assumptions about the system.

These assumptions also include any empirical laws that are applicable to the system. The mathematical construct of all these assumptions, or the mathematical model of the system, is in many instances a differential equation or a system of differential equations. We expect a reasonable mathematical model of a system to have a solution that is consistent with the known behavior of the system.

A mathematical model of a physical system often involves the variable time. The solution of the model then gives the state of the system; in other words, for appropriate values of time t, the values of the dependent variable (or variables) describe the system in the past, present, and future.

Falling Body

The mathematical description of a body falling vertically under the influence of gravity leads to a simple second-order differential equation. The solution of this equation gives the position of the body relative to the ground.

EXAMPLE 1

It is well known that free-falling objects close to the surface of the earth accelerate at a constant rate g. Acceleration is the derivative of the velocity, and this, in turn, is the derivative of the distance s. Suppose a rock is tossed upward from the roof of a building as illustrated in Figure 1.4. If we assume that the upward direction is positive, then the mathematical statement

$$\frac{d^2s}{dt^2} = -g$$

is the differential equation that governs the vertical distance that the body travels. The minus sign is used because the weight of the body is a force directed opposite to the positive direction.

If we suppose further that the height of the building is s_0 and the initial velocity of the rock is v_0, then we must find a solution of the differential equation

$$\frac{d^2s}{dt^2} = -g, \qquad 0 < t < t_1,$$

that also satisfies the side conditions $s(0) = s_0$ and $s'(0) = v_0$. Here $t = 0$ is the initial time the rock leaves the roof of the building, and t_1 is the elapsed time when the rock hits the ground. Since the rock is thrown upward in the positive direction, it is naturally assumed that $v_0 > 0$.

Note that this formulation of the problem ignores other forces such as air resistance acting on the body. ■

Spring-Mass System

When Newton's second law of motion is combined with Hooke's law, we can derive a differential equation governing the motion of a mass attached to a spring.

EXAMPLE 2

To find the vertical displacement $x(t)$ of a mass attached to a spring, we use two different empirical laws: Newton's second law of motion and Hooke's law. The former law states that the net force acting on the system in motion is $F = ma$, where m is the mass and a is acceleration. Hooke's law states that the restoring force of a stretched spring is proportional to the elongation $s + x$; that is, the restoring force is $k(s + x)$, where $k > 0$ is a constant. As shown in Figure 1.5(b), s is the elongation of the spring after the mass has been attached and the system hangs at rest in the equilibrium position. When the system is in motion, the variable x represents a directed distance of the mass beyond the equilibrium position. In Chapter 5 we shall prove that when the system is in motion, the *net force* acting on the mass is simply $F = -kx$. Thus, in the absence of damping and other external forces that may be impressed on the system, the differential equation of the vertical motion through the center of gravity of the mass can be obtained by equating:

$$m\frac{d^2x}{dt^2} = -kx.$$

Here the minus sign means that the restoring force of the spring acts opposite to the direction of motion, that is, toward the equilibrium position. In practice, this second-order differential equation is often written as

$$\frac{d^2x}{dt^2} + \omega^2x = 0,$$

where $\omega^2 = k/m$.

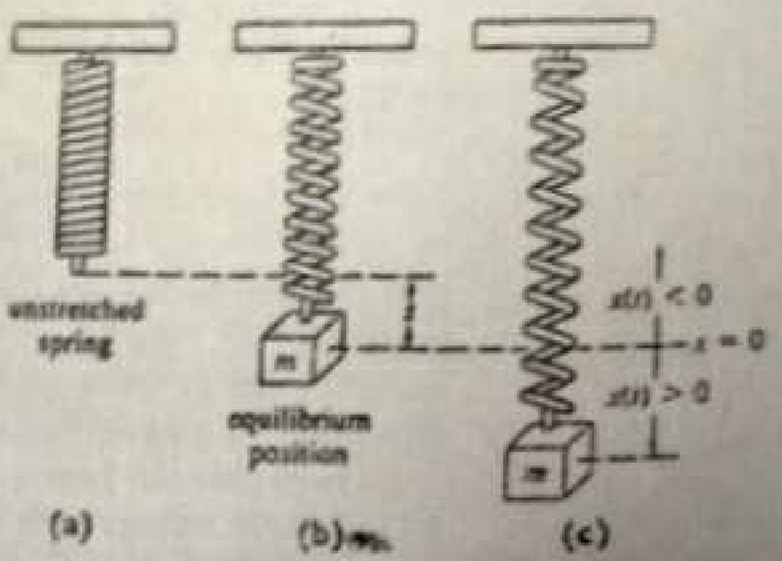

Figure 1.5

Units

A word is in order regarding the system of units that is used in describing dynamic problems such as those illustrated in the last two examples. Three commonly used systems of units are summarized in the following table. In each system the basic unit of time is the second.

Quantity	Engineering System*	SI System†	cgs
Force	pound (lb)	newton (N)	dyne
Mass	slug	kilogram (kg)	gram (g)
Distance	foot (ft)	meter (m)	centimeter (cm)
Acceleration of gravity g (approximate)	$32\ \text{ft/s}^2$	$9.8\ \text{m/s}^2$	$980\ \text{cm/s}^2$

The gravitational *force* exerted by the earth on a body of mass m is called its *weight* W. In the absence of air resistance, the only force acting on a freely falling body is its weight. Hence, from Newton's second law of motion, it follows that mass m and weight W are related by

$$W = mg.$$

For example, in the engineering system a mass of 1/4 slug corresponds to an 8-lb weight. Since $m = W/g$, a 64-lb weight corresponds to a mass of $64/32 = 2$ slugs. In the cgs system a weight of 2450 dynes has a mass of $2450/980 = 2.5$ grams. In the SI system a weight of 50 newtons has a mass of $50/9.8 = 5.1$ kilograms. We note that

$$1 \text{ newton} = 10^5 \text{ dynes} = 0.2247 \text{ pound}.$$

In the next example we derive the differential equation that describes the motion of a *simple pendulum.*

Simple Pendulum

Any object that swings back and forth is called a **physical pendulum.** The simple pendulum is a special case of the physical pendulum and consists of a rod to which a mass is attached at one end. In describing the motion of a simple pendulum, we make the simplifying assumptions that the mass of the rod is negligible and that no external damping forces act on the system (such as air resistance).

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* Also known as the English gravitational system or British engineering system.