Finite Element Method in 1D

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Abstract

Finite Element Method (FEM) is a powerful numerical approach,used to approximate the solutions of many intricate problems using an array of mathematical techniques. The name comes from the fact that the method subdivides a larger problem into simpler parts which are called Finite Elements. The equations that model these finite elements are solved and assembled back into the larger system of equations which model the entire problem. In this project, we will try to solve Linear One dimensional equations using this method. We have the following problem,

$$\frac{d^2v}{dx^2} + p(x)\frac{dv}{dx} + q(x)v = r(x), x \in (0, \pi)$$
$$v(0) = v(\pi) = 0$$

We assume that $r(x)$ is integrable function, $q(x)$ and $p(x)$ are absolutely integrable functions throughout the interval (a, b) where $a, b \in \mathbb{R}$.
Since we do not have any exact solution to our problem, to validate our code, we set,

$$G(w) \equiv \frac{d^2w}{dx^2} + p(x)\frac{dw}{dx} + q(x)w$$

We will study the following cases:
1) 2nd Order linear ODE with constant coefficients
2) 2nd Order linear ODE with variable coefficients
3) 2nd Order non-linear ODE with variable coefficients
4) 1st order linear PDE with variable coefficients

Constant coefficients:

We choose, $p = q = 1$. where, $w = \sin(x)$

Then, we have on the RHS,

$$\frac{d^2w}{dx^2} + \frac{dw}{dx} + w = -\sin(x) + \cos(x) + \sin(x) = \cos(x)$$

We introduce the integrating factor e^x then the above equation becomes,

$$e^x\frac{d^2v}{dx^2} + e^x\frac{dv}{dx} + e^x v = e^x\cos(x)$$

which can be written as,

$$\left(e^x v'\right)' + e^x v = e^x\cos(x) \tag{1}$$

with the boundary conditions,

$$v(0) = v(\pi) = 0.$$

First we discritize the domain (π) by dividing into subintervals i.e.

$$(0, \frac{\pi}{4}), (\frac{\pi}{4}, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{3\pi}{4}), (\frac{3\pi}{4}, \pi)$$

Obtain weak form, by integrating by parts the product of the above equation multiplied by a test function, $w \in C_c^\infty(0, \pi)$ with an additional condition that $w(0) = w(\pi) = 0$,

$$\int_0^\pi \Big((e^x v')' + e^x v\Big) w dx = \int_0^\pi w e^x \cos(x) dx$$

Then integrating by parts, we have,

$$v'e^x w\Big|_0^\pi - \int_0^\pi e^x v'w'dx + \int_0^\pi e^x vwdx = \int_0^\pi we^x\cos(x)dx$$

Then,we have,

$$-\int_0^\pi e^x v'w'dx + \int_0^\pi e^x vwdx = \int_0^\pi we^x\cos(x)dx$$

where the first term has vanished because $w(0) = w(\pi) = 0$.

Introduce a basis solution and make an anzats

$$v \approx \sum_{j=1}^N a_j w_j(x) \tag{2}$$

where, $w_j : (a, b) \to \mathbb{R}, j = 1, \cdots, N$ is basis,has the form,

$$w_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & \text{if } x_{j-1} < x < x_j. \\ \frac{x_{j+1} - x}{x_{j+1} - x_j}, & \text{if } x_j < x < x_{j+1} \\ 0, & \text{Otherwise} \end{cases}$$

Using the defined v from above and letting $w = w_k, k = 1, \cdots, N$,we obtain,

$$-\int_0^\pi e^x w_k'(x)\left(\sum_{j=1}^N a_j w_j'(x)\right)dx + \int_0^\pi e^x w_k(x)\left(\sum_{j=1}^N a_j w_j(x)\right)dx = \int_0^\pi w_k(x)e^x \cos(x)dx \tag{3}$$

which can be written as

$$MV = b$$

where

$$M = -\int_0^\pi e^x w_k'(x)w_j'(x)dx + \int_0^\pi e^x w_k(x)w_j(x)dx$$

$$b = \int_0^\pi w_k(x)e^x \cos(x)dx$$

and $V = \sum_{j=1}^N a_j$ is to be found.

We call M and b as stiffness matrix and load vector respectively.

Integration by Hand

We consider the case when $N = 3$.

$$M_{j,j} = \int_{x_{j-1}}^{x_{j+1}} -e^x \left(w_j'(x)\right)^2 dx + \int_{x_{j-1}}^{x_{j+1}} e^x w_j(x)^2 dx$$

For diagonal entries,

$$\begin{aligned}
M(w_1, w_1) &= \int_0^{\frac{\pi}{4}} -e^x \left(\frac{1}{\frac{\pi}{4}}\right)^2 dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \left(\frac{-1}{\frac{\pi}{4}}\right)^2 dx + \int_0^{\frac{\pi}{4}} e^x (\frac{x-0}{\frac{\pi}{4}})^2 dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x (\frac{\frac{\pi}{2}-x}{\frac{\pi}{4}})^2 dx \\
&= -1.9345 - 4.2428 + 0.4771 + 0.707 = -4.9932 \\
M(w_2, w_2) &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -e^x \left(\frac{1}{\frac{\pi}{4}}\right)^2 dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} e^x \left(\frac{-1}{\frac{\pi}{4}}\right)^2 dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x (\frac{x-\frac{\pi}{4}}{\frac{\pi}{4}})^2 dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} e^x (\frac{\frac{3\pi}{4}-x}{\frac{\pi}{4}})^2 dx \\
&= -4.2428 - 9.306 + 1.0464 + 1.551 = -10.9514 \\
M(w_3, w_3) &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} -e^x \left(\frac{1}{\frac{\pi}{4}}\right)^2 dx - \int_{\frac{3\pi}{4}}^{\pi} e^x \left(\frac{-1}{\frac{\pi}{4}}\right)^2 dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} e^x (\frac{x-\frac{\pi}{2}}{\frac{\pi}{4}})^2 dx + \int_{\frac{3\pi}{4}}^{\pi} e^x (\frac{\pi-x}{\frac{\pi}{4}})^2 dx \\
&= -24.021
\end{aligned}$$

For super and sub-diagonal entries,

$$\begin{aligned}
M_{j,j+2} &= M_{j+2,j} = 0. \\
M_{j,j+1} &= -\int_{x_j}^{x_{j+1}} e^x w_j'(x) w_{j+1}'(x) dx + \int_{x_j}^{x_{j+1}} e^x w_j(x) w_{j+1}(x) dx \\
M(w_1, w_2) &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -e^x \left(\frac{1}{\frac{\pi}{4}}\right)\left(\frac{-1}{\frac{\pi}{4}}\right) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \big(\frac{\frac{\pi}{2} - x}{\frac{\pi}{4}}\big)\big(\frac{x - \frac{\pi}{4}}{\frac{\pi}{4}}\big) dx \\
&= 1.13878 = M(w_2, w_1) \\
M(w_2, w_3) &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} -e^x \left(\frac{1}{\frac{\pi}{4}}\right)\left(\frac{-1}{\frac{\pi}{4}}\right) dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} e^x \big(\frac{\frac{3\pi}{4} - x}{\frac{\pi}{4}}\big)\big(\frac{x - \frac{\pi}{2}}{\frac{\pi}{4}}\big) dx \\
&= 10.253 = M(w_3, w_2)
\end{aligned}$$

Now it's time for load vector formulation.

$$b = \int_{x_{j-1}}^{x_{j+1}} e^x \cos(x) w_j dx$$

$$\begin{aligned}
b_1 &= \int_0^{\frac{\pi}{4}} e^x \cos(x) \frac{x - 0}{\frac{\pi}{4}} dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \cos(x) \frac{\frac{\pi}{2} - x}{\frac{\pi}{4}} dx \\
&= 1.08776
\end{aligned}$$

$$\begin{aligned}
b_2 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \cos(x) \frac{x - \frac{\pi}{4}}{\frac{\pi}{4}} dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} e^x \cos(x) \frac{\frac{3\pi}{4} - x}{\frac{\pi}{4}} dx \\
&= -.3881 \\
b_3 &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} e^x \cos(x) \frac{x - \frac{\pi}{2}}{\frac{\pi}{4}} dx + \int_{\frac{3\pi}{4}}^{\pi} e^x \cos(x) \frac{\pi - x}{\frac{\pi}{4}} dx \\
&= -5.9177
\end{aligned}$$

Solving

$$a_1 = -0.1216; a_2 = .4221; a_3 = .4265;$$

Summary of the results:

Changing the number of elements to 5,10,20 and 40, we can see that the order of accuracy of the method is 2.

N	Error	Order
5	0.6267	
10	0.1820	1.98
20	0.0592	2.01
40	0.0201	2.05

Variable coefficients:

We have the following problem,

$$\frac{d^2v}{dx^2} + p(x)\frac{dv}{dx} + q(x)v = r(x)$$

Since we do not have any exact solution to our problem, to validate our code, we set,

$$G(w) \equiv \frac{d^2w}{dx^2} + p(x)\frac{dw}{dx} + q(x)w$$

We choose, $p = \sin(x)$, $q = \cos(x)$ and $w = \sin(x)$

Then, we have on the RHS,

$$\frac{d^2w}{dx^2} + \sin(x)\frac{dw}{dx} + \cos(x)w = -\sin(x) + 2\sin(x)\cos(x)$$

We introduce the integrating factor $e^{\int p(x)dx}$ then the above equation becomes,

$$e^{\int p(x)dx}\frac{d^2v}{dx^2} + e^{\int p(x)dx}p(x)\frac{dv}{dx} + e^{\int p(x)dx}q(x)v = e^{\int p(x)dx}\big(-\sin(x) + 2\sin(x)\cos(x)\big)$$

which can be written as,

$$\big(e^{\int p(x)dx}v'\big)' + e^{\int p(x)dx}q(x)v = e^{\int p(x)dx}\Bigg(-\sin(x) + 2\sin(x)\cos(x)\Bigg) \tag{4}$$

with the boundary conditions,

$$v(0) = v(\pi) = 0.$$

We use the mid-point integration[1] to handle $\int p(x)dx$.

First we discritize the domain (π) by dividing into subintervals i.e.

$$(0, \frac{\pi}{4}), (\frac{\pi}{4}, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{3\pi}{4}), (\frac{3\pi}{4}, \pi)$$

Obtain weak form, by integrating by parts the product of the above equation multiplied by a test function, $w \in C_c^\infty(0, \pi)$ with an additional condition that $w(0) = w(\pi) = 0$,

$$\int_0^\pi \Big((e^{\int p(x)dx} v')' + e^{\int p(x)dx} q(x) v \Big) w dx = \int_0^\pi w e^{\int p(x)dx} \Big(-\sin(x) + 2\sin(x)\cos(x) \Big) dx$$

Then integrating by parts, we have,

$$v' e^{\int p(x)dx} w \Big|_0^\pi - \int_0^\pi e^{\int p(x)dx} v' w' dx + \int_0^\pi e^{\int p(x)dx} v w q(x) dx = \int_0^\pi w e^{\int p(x)dx} \Big(-\sin(x) + 2\sin(x)\cos(x) \Big) dx$$

Then,we have,

$$- \int_0^\pi e^{\int p(x)dx} v' w' dx + \int_0^\pi e^{\int p(x)dx} v w q(x) dx = \int_0^\pi w e^{\int p(x)dx} \Big(-\sin(x) + 2\sin(x)\cos(x) \Big) dx$$

where the first term has vanished because $w(0) = w(\pi) = 0$.

Introduce a basis solution and make an anzats

$$v \approx \sum_{j=1}^{3} a_j w_j(x) \tag{5}$$

where, $w_j : (a, b) \to \mathbb{R}, j = 1, \cdots, N$ is basis,has the form,

$$w_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & \text{if } x_{j-1} < x < x_j. \\ \frac{x_{j+1} - x}{x_{j+1} - x_j}, & \text{if } x_j < x < x_{j+1} \\ 0, & \text{Otherwise} \end{cases}$$

[1]**Mid point Integration:** Suppose any function $f(x)$ is continuous on an interval $[0, 1]$. If the interval is divived into n sub-intervals, with length $\triangle x = \frac{1}{n}$ and then,

$$\begin{aligned} M_{\frac{1}{2}} &= \int_0^{x^{\frac{1}{2}}} f(x')dx' \approx \frac{\triangle x}{2} f(0) \\ M_{\frac{3}{2}} &= \int_0^{x^{\frac{3}{2}}} f(x')dx' \approx \frac{\triangle x}{2} f(0) + h f(1) \\ &\vdots \\ M_{j+\frac{1}{2}} &= \triangle x \big[\frac{f(0)}{2} + f(1) + \cdots + f(j) \big] \end{aligned}$$

Inserting (5) and letting $w = w_k, k = 1, \cdots, N$,we obtain,

$$\int_0^\pi e^{\int p(x)dx} w_k'(x)\Big(\sum_{j=1}^N a_j w_j'(x)\Big)dx - \int_0^\pi e^{\int p(x)dx} w_k(x)\Big(\sum_{j=1}^N a_j w_j(x)\Big)q(x)dx$$
$$= -\int_0^\pi w_k(x) e^{\int p(x)dx}\Big(-\sin(x) + 2\sin(x)\cos(x)\Big)dx \tag{6}$$

which can be written as

$$MV = b$$

where

$$M = \int_0^\pi e^{\int p(x)dx} w_k'(x) w_j'(x)dx - \int_0^\pi e^{\int p(x)dx} w_k(x) w_j(x) q(x)dx$$
$$b = -\int_0^\pi w_k(x) e^{\int p(x)dx}\Big(-\sin(x) + 2\sin(x)\cos(x)\Big)dx$$

and $V = \sum_{j=1}^N a_j$ is to be found.
We introduce .$\int_0^\pi \varphi(x) w_k'(x) w_j'(x)dx$ and $\int_0^\pi \varphi(x) w_k(x) w_j(x)dx$ as S, T respectively.Now we use mid-point quadrature to approximate S, T. And we let $\varphi_{j-\frac{1}{2}}$ and $\varphi_{j+\frac{1}{2}}$ be the mid-point value at $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ respectively.

$$\begin{aligned}
S_{j,j} &= \int_{x_{j-1}}^{x_{j+1}} \varphi(x)\big(w_j'(x)\big)^2 dx \\
&= \int_{x_{j-1}}^{x_j} \varphi(x)\big(w_j'(x)\big)^2 dx + \int_{x_j}^{x_{j+1}} \varphi(x)\big(w_j'(x)\big)^2 dx \\
&\approx \varphi_{j-\frac{1}{2}} \frac{h_j}{h_j^2} + \varphi_{j+\frac{1}{2}} \frac{\big(-1\big)^2 h_{j+1}}{h_{j+1}^2} \\
&= \frac{\varphi_{j-\frac{1}{2}}}{h_j} + \frac{\varphi_{j+\frac{1}{2}}}{h_{j+1}}, j = 1, \cdots, N
\end{aligned}$$

$$\begin{aligned}
S_{j,j+1} &= \int_{x_j}^{x_{j+1}} \varphi(x) w_j'(x) w_{j+1}'(x)dx \\
&\approx \varphi_{j+\frac{1}{2}} \frac{1}{h_{j+1}} h_{j+1} \frac{\big(-1\big)}{h_{j+1}} \\
&= -\frac{\varphi_{j+\frac{1}{2}}}{h_{j+1}}, j = 1, \cdots, N
\end{aligned}$$

$$\begin{aligned}
S_{j,j-1} &= \int_{x_{j-1}}^{x_j} \varphi(x) w_j'(x) w_{j-1}'(x)dx \\
&\approx -\frac{\varphi_{j-\frac{1}{2}}}{h_j}, j = 1, \cdots, N
\end{aligned}$$

Like before for T we let $\psi_{j+\frac{1}{2}}$ and $\psi_{j-\frac{1}{2}}$ be the mid-point value at $[x_j, x_{j+1}]$ and $[x_{j-1}, x_j]$ respectively.

$$\begin{aligned}
T_{j,j} &= \int_{x_{j-1}}^{x_{j+1}} \psi(x) w_j(x)^2 dx \\
&= \int_{x_{j-1}}^{x_j} \psi(x) w_j(x)^2 dx + \int_{x_j}^{x_{j+1}} \psi(x) w_j(x)^2 dx \\
&= \int_{x_{j-1}}^{x_j} \psi(x) \big(\frac{x - x_{j-1}}{h_j}\big)^2 dx + \int_{x_j}^{x_{j+1}} \psi(x) \big(\frac{x_{j+1} - x}{h_j}\big)^2 dx \\
&\approx \psi_{j-\frac{1}{2}} \frac{h_j^3}{3h_j^2} + \psi_{j+\frac{1}{2}} \frac{h_{j+1}^3}{3h_{j+1}^2} \\
&= \frac{1}{3}\big[\psi_{j-\frac{1}{2}} h_j + \psi_{j+\frac{1}{2}} h_{j+1}\big] \\
T_{j,j+1} &= \int_{x_j}^{x_{j+1}} \psi(x) w_j(x) w_{j+1}(x) dx \\
&= \int_{x_j}^{x_{j+1}} \psi(x) \big(\frac{x - x_{j-1}}{h_j}\big)\big(\frac{x_{j+1} - x}{h_j}\big) dx \\
&\approx \psi_{j+\frac{1}{2}} \frac{h_{j+1}^3}{6h_{j+1}^2} \\
&= \frac{1}{6} \psi_{j+\frac{1}{2}} h_{j+1} \\
T_{j,j-1} &= \frac{1}{6} \psi_{j-\frac{1}{2}} h_j
\end{aligned}$$

Combining all these matrices lead us to the actual matrix,

$$M = S_{j,k} + T_{j,k},\ j = 1, \cdots, N, k = 1, \cdots, N$$

which is a tridiagonal matrix and the entries of the matrix are,

$$\begin{aligned}
M_{j,j} &= \frac{\varphi_{j-\frac{1}{2}}}{h_j} + \frac{\varphi_{j+\frac{1}{2}}}{h_{j+1}} - \frac{1}{3}\left[\psi_{j-\frac{1}{2}} h_j + \psi_{j+\frac{1}{2}} h_{j+1}\right], j = 1, \cdots, N, \\
M_{j,j+1} &= -\frac{\varphi_{j+\frac{1}{2}}}{h_{j+1}} - \psi_{j+\frac{1}{2}} h_{j+1}\left(\frac{1}{6}\right), j = 1, \cdots, N, \\
M_{j,j-1} &= -\frac{\varphi_{j-\frac{1}{2}}}{h_j} - \psi_{j-\frac{1}{2}} h_j\left(\frac{1}{6}\right), j = 1, \cdots, N
\end{aligned}$$

which takes the form,

$$M = \begin{bmatrix}
\frac{\varphi_{\frac{1}{2}}}{h_1} - \frac{h_1}{3}\psi_{\frac{1}{2}} & -\frac{\varphi_{\frac{1}{2}}}{h_1} - \psi_{\frac{1}{2}}\frac{h_1}{6} & & \\
-\frac{\varphi_{\frac{1}{2}}}{h_1} - \psi_{\frac{1}{2}}\frac{h_1}{6} & \frac{\varphi_{\frac{1}{2}}}{h_1} - \frac{h_1}{3}\psi_{\frac{1}{2}} + \frac{\varphi_{\frac{1}{2}}}{h_2} - \frac{h_2}{3}\psi_{\frac{1}{2}} & -\frac{\varphi_{\frac{1}{2}}}{h_2} - \psi_{\frac{1}{2}}\frac{h_2}{6} & \\
& -\frac{\varphi_{\frac{1}{2}}}{h_2} - \psi_{\frac{1}{2}}\frac{h_2}{6} & \frac{\varphi_{\frac{1}{2}}}{h_2} - \frac{h_2}{3}\psi_{\frac{1}{2}} + \frac{\varphi_{\frac{1}{2}}}{h_3} - \frac{h_3}{3}\psi_{\frac{1}{2}} & \frac{\varphi_{\frac{1}{2}}}{h_3} - \psi_{\frac{1}{2}}\frac{h_3}{6} \\
& & \ddots & \\
& & & -\frac{\varphi_{\frac{1}{2}}}{h_{n-1}} - \psi_{\frac{1}{2}}\frac{h_{n-1}}{3}
\end{bmatrix}$$

From this, we observe that, M can be written as sum of n matrices.

$$M = \begin{bmatrix} \frac{\varphi_{\frac{1}{2}}}{h_1} - \frac{h_1}{3}\psi_{\frac{1}{2}} & -\frac{\varphi_{\frac{1}{2}}}{h_1} - \frac{h_1}{6}\psi_{\frac{1}{2}} & & \\ -\frac{\varphi_{\frac{1}{2}}}{h_1} - \frac{h_1}{6}\psi_{\frac{1}{2}} & \frac{\varphi_{\frac{1}{2}}}{h_1} - \frac{h_1}{3}\psi_{\frac{1}{2}} & & \\ & & & \\ & & & \end{bmatrix} + \begin{bmatrix} & & & \\ & \frac{\varphi_{\frac{1}{2}}}{h_2} - \frac{h_2}{3}\psi_{\frac{1}{2}} & -\frac{\varphi_{\frac{1}{2}}}{h_2} - \frac{h_2}{6}\psi_{\frac{1}{2}} & \\ & -\frac{\varphi_{\frac{1}{2}}}{h_2} - \frac{h_2}{6}\psi_{\frac{1}{2}} & \frac{\varphi_{\frac{1}{2}}}{h_2} - \frac{h_2}{3}\psi_{\frac{1}{2}} & \\ & & & \end{bmatrix} + \cdots$$

Now it's time for load vector formulation.For b we let $g_{j+\frac{1}{2}}$ and $g_{j-\frac{1}{2}}$ be the mid-point value at $[x_j, x_{j+1}]$ and $[x_{j-1}, x_j]$.

$$\begin{aligned} b &= -\int_{x_{j-1}}^{x_{j+1}} g(x)w_j dx = -\int_{x_{j-1}}^{x_j} g(x)w_j dx - \int_{x_j}^{x_{j+1}} g(x)w_j dx \\ &= -\int_{x_{j-1}}^{x_j} g(x)\frac{x - x_j}{h_j} dx - \int_{x_j}^{x_{j+1}} g(x)\frac{x_{j+1} - x}{h_{j+1}} dx \\ &\approx -g_{j-\frac{1}{2}}\frac{h_j}{2} - g_{j+\frac{1}{2}}\frac{h_{j+1}}{2} \end{aligned}$$

where, $j = 1, \cdots, N$.

This vector takes the form,

$$b = \begin{bmatrix} -g_{\frac{1}{2}}\frac{h_1}{2} \\ -g_{\frac{1}{2}}\frac{(h_1+h_2)}{2} \\ -g_{\frac{1}{2}}\frac{(h_2+h_3)}{2} \\ \vdots \\ -g_{\frac{1}{2}}\frac{(h_{n-1}+h_{n-1})}{2} \\ -g_{\frac{1}{2}}\frac{(h_n)}{2} \end{bmatrix}$$

Splitting b into a sum over the elements yields,

$$b = \begin{bmatrix} -g_{\frac{1}{2}}\frac{h_1}{2} \\ -g_{\frac{1}{2}}\frac{h_1}{2} \\ \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \\ -g_{\frac{1}{2}}\frac{h_2}{2} \\ -g_{\frac{1}{2}}\frac{h_2}{2} \\ \\ \\ \end{bmatrix} + \cdots + \begin{bmatrix} \\ \\ \\ -g_{\frac{1}{2}}\frac{h_n}{2} \\ -g_{\frac{1}{2}}\frac{h_n}{2} \end{bmatrix}$$

Summary of the results:

Changing the number of elements to 5,10,20 and 40, we can see that the order of accuracy of the method is 2.

N	Error	Order
5	0.0367	
10	0.0093	1.97
20	0.0022	2.08
40	5.206e-04	2.08

Time derivative:

We have the following problem,

$$\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} + p(x)\frac{\partial v}{\partial x} + q(x)v = r(x,t), x \in (0,\pi), t > 0$$
$$v(0,t) = v(\pi,t) = 0, \forall t$$
$$v(x,0) = g(x)$$

Since we do not have any exact solution to our problem, to validate our code, we set,

$$\begin{aligned} G(w) &\equiv \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + p(x)\frac{\partial w}{\partial x} + q(x)w \\ &= \sin(x) - t\sin(x) + t\sin(x)\cos(x) + t\cos(x)\sin(x) \\ &= \sin(x) - t\sin(x) + 2t\sin(x)\cos(x) \end{aligned}$$

where, $W = t\sin(x), p(x) = \sin(x), q(x) = \cos(x)$.Then,the IBVP becomes,

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} + p(x)\frac{\partial v}{\partial x} + q(x)v = \sin(x) - t\sin(x) + 2t\sin(x)\cos(x), t > 0 \\ v(0,t) = v(\pi,t) = 0, \forall t \\ v(x,0) = 0 \end{aligned}$$

Using a backward Euler difference on temporal resolution $\triangle t$, we solve at times $\triangle t, 2\triangle t, \cdots, k\triangle t, \cdots, T$, identifying variables at time level k via superscripts,the above equation, becomes,

$$\frac{v^k}{\triangle t} + \frac{\partial^2 v^k}{\partial x^2} + p(x)\frac{\partial v^k}{\partial x} + q(x)v^k = \sin(x) - k\triangle t\sin(x) + 2k\triangle t\sin(x)\cos(x) + \frac{v^{k-1}}{\triangle t}$$

We introduce the integrating factor $e^{\int p(x)dx}$ then the above equation becomes,

$$e^{\int p(x)dx}\frac{v}{\triangle t} + e^{\int p(x)dx}\frac{d^2 v}{dx^2} + e^{\int p(x)dx}p(x)\frac{dv}{dx} + e^{\int p(x)dx}q(x)v =$$
$$e^{\int p(x)dx}\left(\sin(x) - k\triangle t\sin(x) + 2k\triangle t\sin(x)\cos(x) + \frac{v^*}{\triangle t}\right)$$

v^* is the known solution at previous time step.

which can be written as,

$$\left(e^{\int p(x)dx}\frac{dv}{dx}\right)' + e^{\int p(x)dx}\left(\frac{1}{\triangle t} + q(x)\right)v =$$
$$e^{\int p(x)dx}\left(\sin(x) - k\triangle t\sin(x) + 2k\triangle t\sin(x)\cos(x) + \frac{v^*}{\triangle t}\right)$$

with the following conditions,

$$v(0) = v(\pi) = 0$$
$$v(x,0) = 0$$

Now we have the same case as before (ODE with continuous variable).

Summary of the results:

Changing the number of elements to 5,10 and number of time step to 1000 and 2000, we can see that the order of accuracy of the method is 2.86.

N,K	Error	Order
5,1000	0.0178	
10,2000	0.0024	2.86