$$\frac{d^2v}{dx^2}+p(x)\frac{dv}{dx}+q(x)v=r(x) \tag{1}$$

We choose, $p=\sin(x), q=\cos(x)$ and $r=\frac{d^2w}{dx^2}+p(x)\frac{dw}{dx}+q(x)w$. where, $w=\sin(x)$
Then, we have on the RHS,

$$\frac{d^2w}{dx^2}+p(x)\frac{dw}{dx}+q(x)w=-\sin(x)+\sin(x)\cos(x)+\cos(x)\sin(x)=-\sin(x)+2\cos(x)\sin(x)$$

(1) becomes,

$$\frac{d^2v}{dx^2}+\sin(x)\frac{dv}{dx}+\cos(x)v=-\sin(x)+2\cos(x)\sin(x) \tag{2}$$

We introduce the integrating factor $e^{\int\sin(x)}$ then (2) becomes,

$$e^{\int\sin(x)}\frac{d^2v}{dx^2}+e^{\int\sin(x)}\sin(x)\frac{dv}{dx}+e^{\int\sin(x)}\cos(x)v=e^{\int\sin(x)}\big(-\sin(x)+2\cos(x)\sin(x)\big)$$

which can be written as,

$$\big(\varphi(x)v'\big)'+\psi(x)v=g \tag{3}$$

where, $\varphi=e^{\int\sin(x)}, \psi=\cos(x)\varphi$ and $g=e^{\int\sin(x)}\big(-\sin(x)+2\cos(x)\sin(x)\big)$.
First we discritize the domain (π) by dividing into subintervals i.e.

$$(0,\frac{\pi}{4}),(\frac{\pi}{4},\frac{\pi}{2}),(\frac{\pi}{2},\frac{3\pi}{4}),(\frac{3\pi}{4},\pi)$$

Obtain weak form, by integrating by parts the product of the above equation multiplied by a test function, $w\in C_c^\infty(0,\pi)$ with an additional condition that $w(0)=w(\pi)=0$,

$$\int_0^\pi\Bigg((e^{\int\sin(x)}v')'+\cos(x)e^{\int\sin(x)}v\Bigg)wdx=\int_0^\pi e^{\int\sin(x)}\big(-\sin(x)+2\cos(x)\sin(x)\big)dx$$

Then integrating by parts, we have,

$$v'e^{\int\sin(x)}w\Big|_0^\pi-\int_0^\pi e^{\int\sin(x)}v'w'dx+\int_0^\pi\cos(x)e^{\int\sin(x)}vwdx=\int_0^\pi we^{\int\sin(x)}\big(-\sin(x)+2\cos(x)\sin(x)\big)dx$$

Then,we have,

$$-\int_0^\pi e^{\int\sin(x)}v'w'dx+\int_0^\pi\cos(x)e^{\int\sin(x)}vwdx=\int_0^\pi we^{\int\sin(x)}\big(-\sin(x)+2\cos(x)\sin(x)\big)dx$$

where the first term has vanished because $w(0)=w(\pi)=0$.
Introduce a basis solution and make an anzats

$$v\approx\sum_{j=1}^{3}a_jw_j(x) \tag{4}$$

where, $w_j:(a,b)\to\mathbb{R}, j=1,\cdots,3$ is basis,has the form,

$$w_j(x)=\begin{cases}\frac{x-x_{j-1}}{x_j-x_{j-1}}, & \text{if } x_{j-1}<x<x_j.\\ \frac{x_{j+1}-x}{x_{j+1}-x_j}, & \text{if } x_j<x<x_{j+1}\\ 0, & \text{Otherwise}\end{cases}$$

Inserting (4) and letting $w=w_k, k=1,\cdots,3$,we obtain,

$$-\int_0^\pi e^{\int\sin(x)}w_k'(x)\Bigg(\sum_{j=1}^{3}a_jw_j'(x)\Bigg)dx+\int_0^\pi\cos(x)e^{\int\sin(x)}w_k(x)\Bigg(\sum_{j=1}^{3}a_jw_j(x)\Bigg)dx=$$

$$\int_0^\pi w_k(x)e^{\int\sin(x)}\big(-\sin(x)+2\cos(x)\sin(x)\big)dx$$

which can be written as

$$MV=b$$

where

$$M = -\int_0^\pi e^{\int \sin(x)} w_k'(x) w_j'(x) dx + \int_0^\pi \cos(x) e^{\int \sin(x)} w_k(x) w_j(x) dx$$

$$b = \int_0^\pi w_k(x) e^{\int \sin(x)} \big(-\sin(x) + 2\cos(x)\sin(x) \big) dx$$

and $V = \sum_{j=1}^3 a_j$ is to be found.

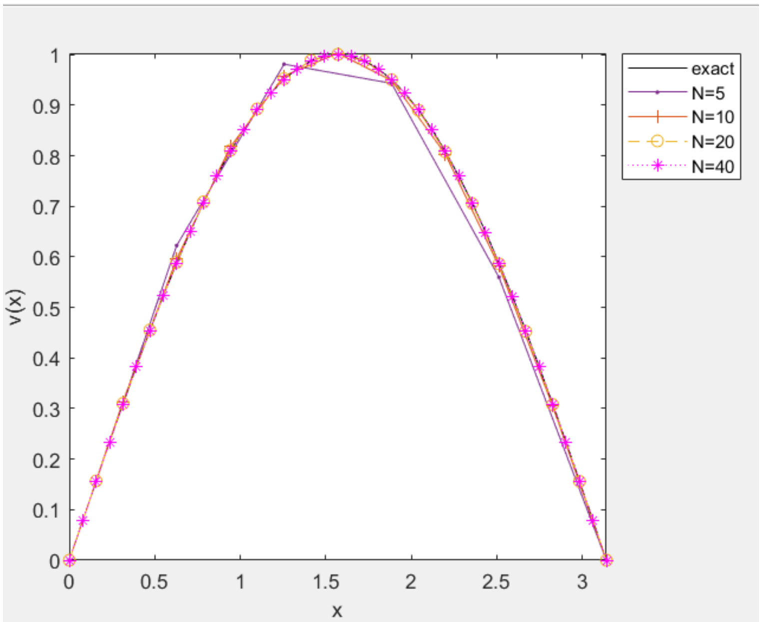

Figure 0.1: Integrating by hand

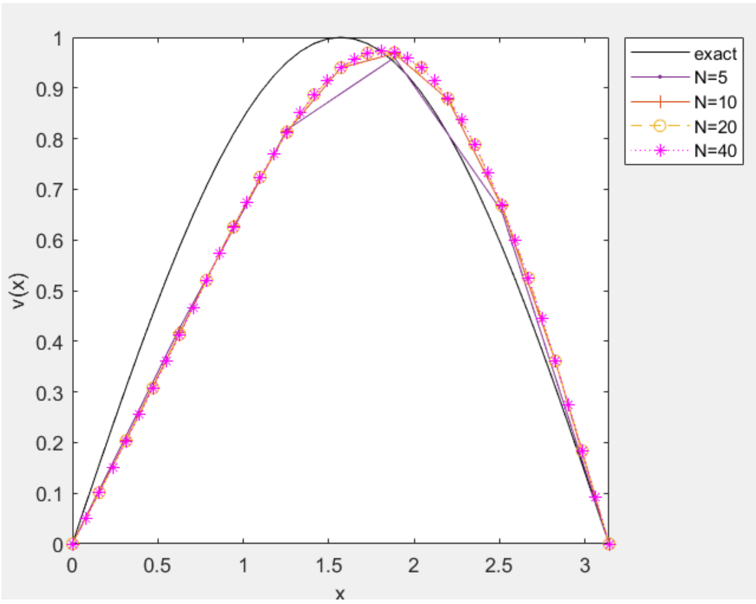

Figure 0.2: By coding

```
function y1 = p(x)
y1 = sin(x);
return
```

Trapezoidal rule for integrating factor

$$y = \int_{x_0}^{x_n} f_n(x)dx = (x_1 - x_0)\frac{f(x_0)+f(x_1)}{2} + (x_2 - x_1)\frac{f(x_1)+f(x_2)}{2} + \cdots + (x_n - x_{n-1})\frac{f(x_{n-1})+f(x_n)}{2}$$

```
function y = int_(x,y1)
s = 0;
n = length(x);
for k = 1:n-1
    s= s+ (x(k+1)-x(k))*(y1(k+1)-y1(k))/2;
end
y = s;
  return
```

```
  function y= phi(x)
        y1 = p(x);
        y=-exp(int_(x,y1));
        return
```

```
function y = g_ph(x1,x2)

  xm = (x1+x2)*0.5;
  y = phi(xm);

  return
```