The Calculus of Variations

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Basic Ideas

We will discuss certain method for solving the boundary value problem(BVP) for some partial differential equation(PDE)s which can be written as,

$$\mathcal{A}[u] = 0. \tag{1}$$

where,the nonlinear operator $\mathcal{A}[\cdot]$ is the derivative of an appropriate energy functional $\mathcal{I}[u]$ and u is unknown.
The Calculus of Variations identifies an important class of problems which can be solved using relatively simple techniques and this needs to adapt the notions of differential calculus. The stationarity of a functional $\mathcal{A}[u]$ is "simply" characterized by the equation,

$$\mathcal{A}[u] = \mathcal{I}'[u] = 0 \tag{2}$$

From this, we have the advantage of solving equation (1) (at least weakly) is equivalent to finding the critical numbers of $\mathcal{I}$.

First Variation, Euler-Lagrange Equation

Let $U \subset \mathbb{R}^n$ be a bounded, open set with ∂U (smooth boundary) and for a smooth function,$u : U \to \mathbb{R}^N$ (if $N = 1$, then u is scalar and if $N \geq 2$, then u is vector), let $u = (u^1, u^2, \cdots, u^N)$ and use,

$$Du = D_i u^k = \partial_{x_i} u^k$$

(where $k = 1, 2, \cdots, N; i = 1, 2, \cdots, n$) to denote the Jacobi Matrix of u; for each $x \in U, Du(x) \in \mathbb{M}^{N\times n}$.
Now we clear our previous vague ideas by assuming $\mathcal{I}[\cdot]$ to have the explicit form, for a given function $L : U \times \mathbb{R}^N \times \mathbb{M}^{N\times n} \to \mathbb{R}$,

$$\mathcal{I}[u] = \int_U L(Du(x), u(x), x)dx. \tag{3}$$

We call $L(\xi, s, x)$ the Lagrangian pf the functional $\mathcal{I}$.

We suppose that $L(\xi, s, x)$ is continuous in (ξ, s, x) and smooth in (ξ, s). We also assume $u \in C^1(\bar{U})$ minimizer of $\mathcal{I}[u]$ with it's own boundary data; that is,

$$\mathcal{I}[u] \leq \mathcal{I}[u + \tau v], \quad (\tau \in \mathbb{R} \ \ \& \ \ v \in C_0^\infty(U))$$

Taking derivative of $\mathcal{I}[u + \tau v]$ at $\tau = 0$, we see that u satisfies,

$$\int_U \sum_{i=1}^n \Big(L_{\xi_i^k}(Du(x), u(x), x) D_i v^k + L_{s^k}(Du(x), u(x), x) v^k \Big) \, dx = 0 \tag{4}$$

Left hand side of equation (4) is the first vartiation of $\mathcal{I}$ at u (in direction of v) and it can also be expressed by using $\langle \mathcal{I}'[u], v\rangle$. Since v has compact support,we can integrate by parts,we get,

$$\int_U \left[-\sum_{i=1}^n D_i(L_{\xi_i^k}(Du(x), u(x), x)) + L_{s^k}(Du(x), u(x), x) \right] v^k \ dx = 0 \tag{5}$$

Since this holds for $\forall\ v$ (test functions), we conclude that u solves the nonlinear PDE,

$$-\sum_{i=1}^n D_i(L_{\xi_i^k}(Du(x), u(x), x)) + L_{s^k}(Du(x), u(x), x) = 0 \tag{6}$$

in U.
This is the Euler-Lagrange equation associated with the energy functional $\mathcal{I}[\cdot]$ defined by (3).

In brief, any smooth minimizer of $\mathcal{I}[u]$ is a solution of the Euler-Lagrange PDE(6) and so we can try to solve PDEs similar to this kind by looking for minimizers of functional $\mathcal{I}[u]$. This is the method of Calculus of Variations (= Variational method) for PDE. The main issues of this method are if minimizers exist or not and are also smooth enough to be a solution of the PDE.

The method we have discussed so far, we are going to apply this on an example.

Example 1: (Dirichlet's principle)
Let,

$$L(\xi, s, x) = \frac{1}{2}|p|^2,$$

Then, $L_{\xi_i} = \xi_i (i = 1, \cdots, n), L_s = 0$; and so the Euler-Lagrange equation associated with the functional

$$\mathcal{I}[u] := \frac{1}{2}\int_U |Du|^2 dx$$

is

$$\triangle u = 0$$

in U.

Example 2: (Generalized Dirichlet's principle)
Let,

$$L(\xi, s, x) = \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j - sf(x),$$

where $a_{ij} = a_{ji}$ $(i, j = 1, 2, \cdots, n)$ and $f : U \to \mathbb{R}$ are given functions. Then the associated Euler-Lagrange equation is,

$$-\sum_{i,j=1}^{n} \left(a^{ij}u_{x_j}\right)_{x_i} = f,$$

a generalization of Poisson's equation.
In order to show this, first we consider, u is a minimizer of (given Lagrangian L)

$$\mathcal{I}_L(u) = \int_U L(Du, u, x)dx$$

over an admissible class of function (Let),

$$\mathcal{A} \equiv \left\{u \in C^2(U), u = g \text{ for } x \in \partial U\right\}.$$

Let $v \in C^\infty(U)$ such that v has compact support within U. We denote this space of functions by $C_c^\infty(U)$.
Define,

$$i(\epsilon) = \mathcal{I}[u + \epsilon v]$$

If u is a minimizer of $\mathcal{I}$,then $i'(0) = 0$.

$$\begin{aligned} i(\epsilon) &= \mathcal{I}(u + \epsilon v) \\ &= \int_U L(Du + \epsilon Dv, u + \epsilon v, x)dx. \end{aligned}$$

Therefore,

$$i'(\epsilon) = \int_U \sum_{i=1}^{n} L_{\xi_i}(Du + \epsilon Dv, u + \epsilon v, x)v_{x_i} + L_s(Du + \epsilon Dv, u + \epsilon v, x)v \ dx.$$

Now $i'(0) = 0$ implies

$$0 = i'(0) = \int_U \sum_{i=1}^{n} L_{\xi_i}(Du, u, x)v_{x_i} + L_s(Du, u, x)v \ dx.$$

Integrating by parts and using the fact that $v = 0$ for $x \in \partial U$, we conclude that

$$\int_U -\left[\sum_{i=1}^{n}(L_{\xi_i}(Du,u,x))_{x_i} + L_s(Du,u,x)\right] v dx = 0.$$

Since this is valid for $\forall v \in C_c^\infty(U)$, we can say that u is a solution of the Euler-Lagrange equation associated with L.

Now we will mood forward to Second Variation which is denoted as $\langle \mathcal{I}''[u]v, v\rangle$.

Second Variation, Legendre-Hadamard Conditions

If L, u are smooth enough (say, C^2 class) then, at the minimizer u, $\forall v \in C_0^\infty(U)$, we have,

$$\langle \mathcal{I}''[u]v, v\rangle := \frac{d^2}{dt^2} I[u+\tau v]\bigg|_{t=0} \geq 0,$$

which gives,

$$\langle \mathcal{I}''[u]v, v\rangle = \int_U \left(L_{\xi_i^k \xi_j^l}(Du,u,x) D_i v^k D_j v^l + 2L_{\xi_i^k s^l}(Du,u,x) v^l D_i v^k + L_{s^k s^l}(Du,u,x) v^k v^l \right) dx \geq 0, \quad (7)$$

which is true $\forall v \in C_0^\infty$ and this is known as second variation of $\mathcal{I}$ at u (in direction of v).
From (7), we get some useful information. We note that after a routine approximation argument that (7) is valid for any Lipschitz continuous function v vanishing on ∂U.
Let $\rho : \mathbb{R} \to \mathbb{R}$ be the periodic zig-zag function of period 1 defined by

$$\rho(t) = \begin{cases} t, & \text{if } 0 \leq t \leq \frac{1}{2}. \\ 1-t, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Given, $\varphi \in \mathbb{R}^n, \psi \in \mathbb{R}^N, \epsilon > 0$ and $\zeta \in C_0^\infty(U)$, we define,

$$v(x) = \epsilon \rho\left(\frac{x.\varphi}{\epsilon}\right)\zeta(x)\psi, \quad \forall x \in U \quad (8)$$

We observe that $D_i v^k(x) = \rho'(\frac{x.\varphi}{\epsilon})\varphi_i \psi^k \zeta + O(\epsilon)$ as $\epsilon \to 0^+$. Substituting (8) into (7) and let $\epsilon \to 0^+$ to obtain,

$$\int_U \left(\sum_{i,j=1}^{n} \sum_{k,l=1}^{N} L_{\xi_i^k \xi_j^l}(Du,u,x)\varphi_i\varphi_j\psi^k\psi^l \right) \zeta^2 dx \geq 0.$$

SInce this holds $\forall \zeta \in C_0^\infty(U)$, we deduce that,

$$\sum_{i,j=1}^{n} \sum_{k,l=1}^{N} L_{\xi_i^k \xi_j^l}(Du,u,x)\varphi_i\varphi_j\psi^k\psi^l \geq 0, \ \forall x \in U, \varphi \in \mathbb{R}^n, \psi \in \mathbb{R}^N \quad (9)$$

and this is known as the (weak) Legendre-Hadamard condition for L at minimum point u.

Now we will identify some conditions on L which ensure that the functional $\mathcal{I}[\cdot]$ does have a minimizer (at least within an appropriate Sobolev space).

Existence of Minimizers

We start by giving some defintions in Non-linear Functional Analysis and then we will discuss existence under two assumptions on L: coercivity and convexity.
Definitions: Let X be Banach Space and X^* be its dual space.

1. *A sequence u_ν in X is called "weakly converge" to an element $u \in X$ if*

$$\langle f, u_\nu\rangle \to \langle f, u\rangle \ \forall f \in X^*.$$

2. *A set $\mathcal{C} \subset X$ is called "(sequentially) weak closed" provided $u \in \mathcal{C}$ whenever $\{u_\nu\} \subset \mathcal{C}, u_\nu \rightharpoonup u$.*

3. *A function $\mathcal{I} : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is called "(sequentially) weakly lower semicontinuous (WLSC) on X provided*
$$\mathcal{I}[u] \leq \liminf_{\nu \to \infty} \mathcal{I}[u_\nu] \text{ whenever } u_\nu \rightharpoonup u \text{ in } X$$

4. *A function $\mathcal{I} : X \to \bar{\mathbb{R}} = \mathbb{R}$ is said to be "coercive" on an unbounded set $\mathcal{C} \subseteq X$provided $I[u] \to \infty$ as $||u|| \to \infty$ in $\mathcal{C}$.*

We will now undertstand the multiple integral functionals of the type,

$$\mathcal{I}[u] = \int_U L(Du(x), u(x), x)dx,$$

on Sobolev space $W^{1,p}(U)$.

We define
$$D_v = \left\{u \in W^11, p(U)|u - v \in W_0^{1,p}(U)\right\}$$
as the Dirichlet class. We note that if $\partial U \in C^1$ then by using γ_0(tracer operator), we say that D_v is weakly closed in $W^{1,p}(U)$.
By Poincarè's inequality,

$$||Du||_{L^p(U)} \leq ||u||_{W^{1,p}(U)} \leq C(||Du||_{L^p(U)} + ||v||_{W^{1,p}(U)}) \ \forall u \in D_v \tag{10}$$

Coercivity

We assume $L(\xi, s, x)$ is continuous on (ξ, s) and we then suppose $\exists \alpha > 0, \beta > 0$ such that

$$L(\xi, s, x) \geq \alpha|\xi|^p - \beta \quad \forall x \in U, s \in \mathbb{R}^N, \xi \in \mathbb{M}^{N \times n}, \tag{11}$$

where $\alpha > 0$ is a constant and $\beta \in L^1(U)$ is a function of x. Then $I : W^{1,p}(U) \to \mathbb{R}$ is well-defined and

$$\mathcal{I}[w] \geq \alpha||Dw||^p_{L^p(U)} - ||\beta||_{L^1(U)} \quad \forall w \in W^{1,p}(U).$$

By (10), we have for some constrants $\delta > 0$ and $\gamma \in \mathbb{R}$,

$$\mathcal{I}[w] \geq \delta||w||^p_{W^{1,p(U)}} - \gamma, \quad \forall w \in D_v \tag{12}$$

We get the coercivity of $\mathcal{I}$ on the D_v.

Convexity

From our second variation analysis, we get the inequality,

$$\sum_{i,j=1}^{n} \sum_{k,l=1}^{N} L_{\xi_i^k \xi_j^l}(Du, u, x)\varphi_i\varphi_j\psi^k\psi^l \geq 0, \ \forall x \in U, \varphi \in \mathbb{R}^n, \psi \in \mathbb{R}^N$$

holding as a necessary condition, whenever u is a smooth minimizer. This inequality strongly suggests that it is reasonable to assume that L is convex in its argument.

Theorem on Existence of Minimizers

We can establish that $\mathcal{I}[\cdot]$ has a minimizer among a certain class of admissible functions $\mathcal{A}$.

Theorem 1:

Assume that L satisfies the coercivity and convexity condition. We also the admissible set $\mathcal{A}$ is nonempty. Then $\exists$ at least one function $u \in \mathcal{A}$ solving

$$\mathcal{I}[u] = \min_{w \in \mathcal{A}} \mathcal{I}[w].$$

Proof:

1. We know that the functional $\mathcal{I}[\cdot]$ is bounded below by the coercivity condition. Therefore, $\mathcal{I}$ has an infimum. We set,
$$m := \inf_{w \in \mathcal{A}} \mathcal{I}[w],$$
if $m = +\infty$, we don't have to prove anything. So we assume $-\infty < m < +\infty$.

2. We select a minimizing sequence $\{u_k\}_{k=1}^{\infty} \subset W^{1,q}(U)$[1] which satisfies,
$$\mathcal{I}[u_k] \to m, \text{as } k \to \infty. \tag{13}$$

3. We take $\beta = 0$ in (11),
$$\mathcal{I}[w] \geq \alpha |\xi|^q$$
So,
$$\mathcal{I}[w] \geq \alpha \int_U |dw|^q dx, \tag{14}$$
From (13) and (14), we conclude,
$$\sup_k ||Du_k||_{L^q(U)} < \infty. \tag{15}$$

4. We fix any function $w \in \mathcal{A}$. Since $u_k = g = w$ on ∂U, $u_k - w = 0$ for $x \in \partial U$. Therefore, we can use Poincarè's inequality. Therefore, we have,
$$\begin{aligned} ||u_k||_{L^q(U)} &\leq ||u_k - w||_{L^q(U)} + ||w||_{L^q(U)} \\ &\leq C||Du_k - Dw||_{L^q(U)} + C \leq C \end{aligned}$$
Therefore,
$$\sup_k ||u_k||_{L^q(U)} < +\infty$$
This estimate together with (15) implies $\{u_k\}_{k=1}^{\infty}$ is bounded in $W^{1,q}(U)$.

5. Applying the weak compactness theorem for $W^{1,q}(U)$, $\exists$ a subsequence $\left\{u_{k_j}\right\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and a function $u \in W^{1,q}(U)$ such that
$$u_{k_j} \rightharpoonup u \text{ weakly in } W^{1,q}(U)$$

6. Now we show that $u \in \mathcal{A}$ and which is the closedness condition in our proof. To see this, we note that for $w \in \mathcal{A}, u_k - w \in W_0^{1,q}(U)$.
Now, $W_0^{1,q}(U)$ is a linear subspace of $W^{1,q}(U)$. We know that every linear subspace is convex. $W_0^{1,q}(U)$ is closed (by definition). We know from Mazur's theorem that *A convex, closed subset of a Banach space X is weakly closed.* Hence, $W_0^{1,q}(U)$ is weakly closed and $u - w \in W_0^{1,q}(U)$.Consequently the trace of u on ∂U is g.The Dirichlet boundary condition is satisfied in the trace sense, so $u \in \mathcal{A}$.

7. From *Weak Lower Semicontinuity Theorem (Evans Chapter 8, Theorem 1)* , we can say that under the given conditions,
$$\mathcal{I}[u] \leq \liminf{}_{j \to \infty} \mathcal{I}[u_{k_j}] = m.$$
But, since $u \in \mathcal{A}, \mathcal{I}[u] \geq m$. Therefore, $\mathcal{I}[u] = m$.

□

[1] Let $W^{1,q}(U) \equiv \{u : u \in L^q(U), Du \in L^q(U)\}$

Weak Solutions

We want to show that any minimizer $u \in \mathcal{A}$ of $\mathcal{I}[\cdot]$ solves the Euler-Lagrange equation(in some suitable sense). We can't do it using our previous discussion because we don't know u is smooth, only $u \in W^{1,q}(U)$. That's why we give some growth conditions on L and assume it's derivatives hold so that the weak solutions to the boundary value problem

$$-\sum_{i=1}^{n} D_i(L_{\xi_i}(Du(x), u(x), x)) + L_s(Du(x), u(x), x) = 0 \text{ in } U, \tag{16}$$

$$u = g \text{ on } \partial U. \tag{17}$$

Multiplying (16) by a test function $v \in C_c^\infty(U)$ and integrate by parts, we get,

$$\int_U \big[-\sum_{i=1}^{n} L_{\xi_i}(Du(x), u(x), x)D_i v + L_s(Du(x), u(x), x)v\big]dx = 0 \tag{18}$$

And the growth conditions are as follows:
We assume $L(\xi, s, x)$ is C^1 in (ξ, s) and

$$|L(\xi, s, x)| \leq C(|\xi|^p + |s|^p + 1), \tag{19}$$

$$|D_s L(\xi, s, x)| \leq C(|\xi|^{p-1} + |s|^{p-1} + 1), \tag{20}$$

$$|D_\xi L(\xi, s, x)| \leq C(|\xi|^{p-1} + |s|^{p-1} + 1). \tag{21}$$

for some constant C and under these growth conditions we can verify that the equation (18) is valid for a minimizer u and any $v \in W_0^{1,p}(U)$.

Theorem 2:

Assume L satisfies the conditions(19),(20) and (21) and u satisfies

$$\mathcal{I}[u] = \min_{w \in \mathcal{A}} \mathcal{I}[w]$$

Then u is a weak solution of (16).

proof:
Let $X = W^{1,p}(U), u, v \in X$ and $h(t) = \mathcal{I}[u+\tau v]$.By (19), h is finite valued and we show h is differentiable at $t = 0$ and (18) holds true.
Letting $\tau \neq 0$ and then we have,

$$\frac{h(t) - h(0)}{t} = \int_U \frac{L(Du + \tau Dv, u + \tau v, x) - L(Du, u, x)}{\tau} dx = \int_U L^\tau(x)dx, \tag{22}$$

Where,for almost every $x \in U$,

$$\begin{aligned}
L^\tau(x) &= \frac{1}{\tau}\big[L(Du + \tau Dv, u + \tau v, x) - L(Du, u, x)\big] \\
&= \frac{1}{\tau}\int_0^\tau \frac{d}{ds} L(Du + sDv, u + sv, x)ds \\
&= \int_0^\tau \big[L_{\xi_i}(Du + sDv, u + sv, x)D_i v + L_s(Du + sDv, u + sv, x)v\big]ds.
\end{aligned}$$

Hence,

$$\lim_{\tau \to 0} L^\tau(x) = L_{\xi_i}(Du, u, x)D_i v + L_s(Du, u, x)v \quad \text{for almost every } x \in U$$

Using (20), (21) and Young's Inequality, we get that $\forall\ 0 < |\tau| \leq 1$,

$$|L^\tau(x)| \leq C\big(|Du|^p + |Dv|^p + |u|^p + |v|^p + 1\big) \in L^1(U), \text{ for each } \tau \neq 0.$$

Thus, by Lebesgue dominated convergence theorem,

$$h'(0) = \lim_{\tau \to 0} \int_U L^\tau(x)dx = \int_U \big(L_{\xi_i}(Du, u, x)D_i v + L_s(Du, u, x)v\big)dx,$$

which proves $h'(0)$ exists and since $h(\cdot)$ has a minimum for $\tau = 0$, we know $h'(0) = 0$ and thus u is a weak solution.