Non-integrability criterion for homogeneous Hamiltonian systems via theory of H. Yoshida

Class Project for fulfilling the requirements of MATH 551: Differential and Integral Equations

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Contents

Abstract

The discussion of a dynamical system being non-integrable is very vast and complicated. In this project we listed some of the conditions required for the existence of algebraic first integral for Homogeneous Hamiltonian systems the sense of H.Yoshida[5, 6, 7, 8]. All the concepts were analyzed using examples.

1 Introduction

The N-Body problem has been considered always as an epitome of chaotic behavior in Applied mathematics. Solving N-Body Problem is an important to understand the motions of the solar system (Sun, Moon, planets), and visible stars, as well as understanding the dynamics of globular cluster star systems.This problem have been studied

for quite long time by researchers from different fields of science.
In 1887, Bruns showed the non-existence of first integrals for 3 body problem is algebraic with respect to positions, linear momenta and times. But his work contained some mistakes which was lately pointed and modified by Henri Poincar$\acute{e}$ in 1896. Poincar$\acute{e}$ concluded by saying that the problem had no uniform integral other than the energy and angular momentum [4].
In 1889, as a celebration of birthday of King Oscar II (of Norway and Sweden), a famous competition was held in mathematics. Henri Poincar$\acute{e}$ accepted the challenge and submitted a paper on restricted three body problem. He won the competition and shortly thereafter an error came upon in his proof. After working relentless hours, he removed the error but he could not actually solve the problem instead he uncovered the splitting of separatrices of a hyperbolic equilibrium.
Kovalevskaya had pioneered another theory in this field. By studying the property of singularities in the rigid body model she discovered a new integrable case. Ziglin [9] developed the singularity theory as a continuation of her approach to prove the non-integrability. By applying the Ziglin analysis, Yoshida [8] given a criterion for the homogeneous 'non-integrability.

2 ABC's for the N body problem

2.1 Equations of Motion

The N-Body problem is treated as the problem of predicting the motion of a set of N number objects that interact independently with each other over a long range of period-scale. It's dynamics is studied under the consideration of positive masses m_i and its positions $r_i \in \mathbb{R}^d$ which is generally expressed as,

$$m_i \ddot{r}_i = \sum_{j \neq i} G m_j m_i \frac{r_i - r_j}{r_{ij}^3}, i = 1, \cdots, N \tag{2.1}$$

where, $r_{ij} = \|r_i - r_j\|\; i.e$, the Euclidean distance between r_i and r_j and
For our convenient we choose $G = 1$[4].

2.2 Introducing As Hamiltonian System

We get the Hamiltonian form of the equations by replacing the velocities $\dot{r}$ by the linear momentum $y = \sum_{i=1}^{N} m_i \dot{r}_i$ and introducing the Hamiltonian, H. We denote the coordinates of position by, $x = (r_1, \cdots, r_n)$, then we get,

$$\dot{x} = \frac{\partial H}{\partial y}$$
$$\dot{y} = -\frac{\partial H}{\partial x} \quad \text{where, } H = \frac{|y|^2}{2} - \sum_{j \neq i} \frac{m_i m_j}{r_{ij}}$$

3 Non-integrability

3.1 Poincare's approach

We study the set of Hamiltonian equations that we got in section 2.2. Now we expand the function $H(x, y, \epsilon)$ with a small parameter,

$$H = H_0(x) + \epsilon H_1(x, y) + \cdots$$

where, H_0 is completely integrable and the co-ordinates (x, y) are called action angle co-ordinates. Now, the core concept of this approach is that an invariant torus of an integrable approximation that is the union of periodic solutions that has no dynamic meaning and is therefore highly likely to be broken by sufficiently enough perturbation which gives rise to "generic" periodic solutions which are the reason behind non-Integrability.

3.2 Yoshida's Theory of Non-integrability

Yoshida [5, 6, 7] derived a relationship between scale invariant autonomous system of ordinary differential equations

$$\dot{x} = f(\boldsymbol{x}), x \in \mathbb{C}^n$$

algebraic first integrals and Kowalevskaya exponents. The starting point in his analysis is the scale invariance of the system under,

$$t \to a^{-1}t \quad x_1 \to a^{w_1}x_1 \quad \cdots \quad x_n \to a^{w_n}x_n$$

where,the weights $(w_1, \cdots, w_n) = \boldsymbol{w} \in \mathbb{Q}$. Then he assumed a non-trivial solution for the system which has the form,

$$x(t) = \alpha t^{-\boldsymbol{w}} = \alpha t^{\boldsymbol{p}}$$

where, $\boldsymbol{w} = -\boldsymbol{p}$ and the co-efficients $\alpha \in \mathbb{C}^n$ are given by non vanishing solutions of the algebraic equation

$$\boldsymbol{p}\alpha = f(\alpha)$$

For a given $\boldsymbol{p}$, there may exist different sets of values of α (balances). Then we compute the Kovalevskaya matrix given by

$$K = Df(\alpha) - \text{diag}(\boldsymbol{p})$$

where the jacobian at $x = \alpha$ is $Df(\alpha)$ and the eigenvalues from K is called Kovalevskaya exponents denoted by ρ.
The idea of Yoshida to find out a system is integrable or not is a complicated one which consists of two stages.
First, he shows that under certain conditions, the weighted degree of a first integral is a Kovalevskaya exponent.
Then he proves that if one of the exponents is not rational, then the system is not algebraically integrable.

To make our discussion more interesting, we choose to discuss with an example that has been mentioned several times by Yoshida in his papers [5, 7].
Example: Consider the quartic homogeneous potential,

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}(q_1^4 + q_2^4) + \frac{\epsilon}{2}q_1^2 q_2^2 \tag{3.1}$$

Aim is to find all the values for ϵ under which the Hamiltonian system does not possess a second analytic first integral.

Yoshida's Analysis

In [5], he showed that, the system is integrable for

$$\begin{aligned}
&\epsilon = 0,\ I = \frac{1}{2}(p_1^2 - p_2^2) + \frac{1}{4}(q_1^4 - q_2^4) \\
&\epsilon = 1,\ I = p_1 q_2 - p_2 q_1 \\
&\epsilon = 3,\ I = p_1 p_2 + q_1 q_2 (q_1^2 + q_2^2)
\end{aligned}$$

But the existence of only these values of ϵ for which the system is integrable doesn't make impossible of the possibility of existence of other values for which the system might be integrable with an analytic first integral.
To back up, Yoshida's calculation, we can check the system is integrable for these values of ϵ using Singularity Analysis done by Yoshida [5, 6].
First we do balance of order. Since we have order 2 and 4 terms in the system, we get, a balance of order 2 for $q_1 = p_1 = 0$ or $q_2 = p_2 = 0$ and a balance of order 4 which leads us to two set of Kovalevskaya exponents which are,

$$\begin{aligned}
\mathcal{R}_1 &= \{-1, 4, \frac{3}{2} \pm \frac{1}{2}\sqrt{1+8\epsilon}\}, \\
\mathcal{R}_2 &= \{-1, 4, \frac{3a + 3 \pm \sqrt{-7\epsilon^2 + 18\epsilon + 25}}{2(\epsilon+1)}\}
\end{aligned}$$

both corresponding to the weight vector $\boldsymbol{w} = (1, 1, 2, 2)$, from $\boldsymbol{w} = -\boldsymbol{p}$, dominant exponents are calculated and $\boldsymbol{p} = (-1, -1, -2, -2)$. On a quick observation, we notice that the exponents are integer valued only if $\epsilon = 0, 1, 3$, that is in the integrable cases. For other values of ϵ the system does not possess any rational exponents so by extension-accordingly Yoshida's analysis, the system is non-integrable when $\epsilon \neq \{0, 1, 3\}$.

3.3 Homogeneous Potentials

We now study the case of two DOF (degree of freedom) Hamiltonian which has a kinetic part (diagonal) and homogeneous potentials [5].
So the Hamiltonian is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2)$$

where, $V(\epsilon q_1, \epsilon q_2) = \epsilon^k V(q_1, q_2)$ is of degree k. Let $\boldsymbol{r} = (r_1, r_2)$ be the solution of

$$r_i = \frac{\partial V}{\partial q_i}(\boldsymbol{r}), i = 1, 2.$$

We define the integrability coefficient λ,

$$\begin{aligned}\lambda &= \mathrm{Tr}V_H(r_1,r_2)-(k-1)\\ &= \frac{\partial^2 V}{\partial q_1^2}(\boldsymbol{r})+\frac{\partial^2 V}{\partial q_2^2}(\boldsymbol{r})-k+1\end{aligned}$$

where, V_H is the hessian matrix of $V(r_1,r_2)$. and if λ lies in the regions S_k defined below then the system is non-integrable.

1. $k \geq 3$:
   $$S_k = \{\lambda<0, 1<\lambda<k-1,\cdots,\frac{k}{2}j(j-1)+j<\lambda<\frac{k}{2}j(j+1)-j,\cdots\};$$
2. $S_1 = \mathbb{R}-\{0,1,3,6,10,\cdots,\frac{j(j+1)}{2},\cdots\};$
3. $S_{-1} = \mathbb{R}-\{1,0,-2,-5,-9,\cdots,-\frac{j(j+1)}{2}+1,\cdots\};$
4. $k \leq -3$:
   $$\begin{aligned}S_k = \{&\lambda>1, 0>\lambda>|k|+2, -|k|-1>\lambda>-3|k|+3,\\ &\cdots,\frac{|k|}{2}j(j-1)-j+1>\lambda>\frac{|k|}{2}j(j+1)-j,\cdots\};\end{aligned}$$

When $k=0,\pm 2$ such regions are not defined by this approach. To have a better understanding, we provide the following example from *Almeida et al.* [10].
Example: We consider the reduced Hamiltonian Strömer problem in cylindrical coordinates (ρ,ϕ,z) which is (with $P_\phi=0$)

$$H=\frac{(p_\rho^2+p_z^2)}{2}+\frac{1}{2}a^2\rho^2(\rho^2+z^2)^{-3}$$

which has a homogeneous potential with degree $k=-4$ given by

$$V=\frac{1}{2}a^2\rho^2(\rho^2+z^2)^{-3}$$

Now like before we Let $\boldsymbol{r}=(r_1,r_2)$ be the solution of

$$r_i=\frac{\partial V}{\partial q_i}(\boldsymbol{r}), i=1,2.$$

Clearly,

$$\begin{aligned}r_1 &= \frac{\partial V}{\partial \rho}(\boldsymbol{r})\\ r_2 &= \frac{\partial V}{\partial z}(\boldsymbol{r})\end{aligned}$$

from the system, we get,

$$\begin{aligned}r_1 &= a^2 r_1(r_1^2+r_2^2)^{-3}-3a^2r_1^3(r_1^2+r_2^2)^{-4}\\ r_2 &= -3a^2r_1^2r_2(r_1+r_2^2)^{-4}\end{aligned}$$

simplifying and solving for r_1, r_2, we get,

$$r_1=(-2a^2)^{\frac{1}{6}},\quad r_2=0$$

Then we calculate the Hessian matrix, is

$$V_H = \begin{bmatrix} -5 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

and hence,

$$\lambda = (-5 + \frac{3}{2}) + 5 = \frac{3}{2}$$

Hence λ falls in the following region (according to previous discussion),

$$S_4 = \{\lambda > 1, 0 > \lambda > -2, -5 > \lambda > -9, \cdots\}$$

So we conclude that the system is non-integrable.

3.4 The 1D 3- Body Problem

The Hamiltonian is given by,

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + g\big[|q_1 - q_2|^k + |q_2 - q_3|^k + |q_3 - q_1|^k\big] \tag{3.2}$$

with g as a constant and we use the following canonical transformation,

$$(Q, P) \to (q, p) : Q = Uq, P = Up$$

where U is the orthogonal matrix given by,

$$U = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Following [8], we ignore the momentum p_3 and we get,

$$V(q_1, q_2) = g\big[(\sqrt{3}q_1 + q_2)^k + (-\sqrt{3}q_1 + q_2)^k + (2q_2)^k\big]$$

Again letting $\boldsymbol{r} = (r_1, r_2)$ be the solution of

$$r_i = \frac{\partial V}{\partial x_i}(\mathbf{r}),$$

We get,

$$(I).\ r_1 = 0,\ r_2^{2-k} = 2kg(1 + 2^{k-1}) \tag{3.3}$$

$$(II).\ (\text{if } k = 2g > 0)\ \ r_1^{2-k} = 2kg(\sqrt{3})^k, r_2 = 0 \tag{3.4}$$

Then we can get the Integrability constant,

$$\begin{aligned} \lambda = k(k-1)\big[3(\sqrt{3}r_1 + r_2)^{k-2} + (-\sqrt{3}r_1 + r_2)^{k-2} + (\sqrt{3}r_1 + r_2)^{k-2} \\ +3(\sqrt{3}r_1 + r_2)^{k-2} + 8r_2^{k-2}\big] - (k-1) \end{aligned} \tag{3.5}$$

From (I), we get, $r_1 = 0,\ r_2^{k-2} = \frac{1}{2k(1+2^{k-1})}$ substituting these to (3.5), we get,

$$\lambda = \frac{3(k-1)}{1+2^{k-1}}$$

and for $k \leq -3$ which lies in the region of,

$$-6|k| + 4 < \lambda < -3|k| - 2$$

that means it falls in the non-integrable region. In some similar manner,we get,

$$\lambda = \frac{k-1}{3}$$

when $k = 2g \geq 6$, it falls under the non-integrable region i.e.

$$1 < \lambda < k - 1$$

.

4 Conclusion

The notion of Integrability is confined within the existence of first integral. Proving a system is non-integrable is a hard nut to crack issue. Yoshida's method for the analysis of the non-Integrability under domains of Hamiltonian system has the advantage of solving perturbed problems and in extension, to general situations as well. It is a good procedure to learn about the non-integrable regions. We also note that it has some limitations as well such as, not all the integrable cases are fully determined by this method.

5 Some Volunteer work

5.1 Linear System

Consider a linear system with constant coefficients

$$\dot{x} = Ax,\ Ax \in M_n(\mathbb{C}), x \in \mathbb{C}^n \tag{5.1}$$

The system(5.1) will be noted as integrable if it has a sufficient number set of first integrals such that the solutions of the system can be expressed by these integrals. To clear the idea about first integral, we consider, a function which depends on x only i.e. $\phi(x)$ and it will be called a first integral of the system if it is constant along any solution curve of system i.e.$\langle \frac{d\phi}{dx}, Ax\rangle = 0$. Otherwise, we call it non-integrable if the system does not have any nontrivial first integral.
Citing [1], we get the following theorems. Consider the given system with eigenvalues λ. Then,$\Lambda = \{\lambda_1, \cdots, \lambda_n\}$ are $\mathbb{Z}$-independent[1] $\Leftrightarrow$ there is no polynomial first integral. Consider the given system with eigenvalues λ and and assume it has no rational first integral. Then the Jordan form of the matrix A is diagonal matrix and all the diagonal entries are $\mathbb{Z}$-independent.

[1] A finite subset $\{b_1, \cdots, b_n\}$ of a complex vector space E is $\mathbb{Z}$-independent if $b_1c_1 + \cdots + b_nc_n = 0$ with $c_i \in \mathbb{Z}^+$ implies $c_1 = \cdots = c_n = 0$.

5.2 Nonlinear Systems

Following [2],We assume that $f(x) = Ax$ is analytic and $f(0) = 0$. Let B denote the Jacobi matrix of the vector field $f(x)$ at $x = 0$ we write the nonlinear system as,

$$\dot{x} = Bx + \tilde{f}(x) \tag{5.2}$$

near some neighborhood of the origin $x = 0$. Consider the eigenvalues $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$ of A. If (5.2) has a rational integral, then $\exists$ a nonzero integral vector $k = (k_1, \cdots, k_n) \in \mathbb{Z}_n$ such that $\langle \Lambda, k\rangle = 0$. To understand the theorem, we use the following example provided on [2].

Example: Consider the following system which models three-wave interaction with quadratic nonlinearities

$$\begin{cases} \dot{x} &= ax + by + z - 2y^2 \\ \dot{y} &= -bx + ay + 2xy \\ \dot{z} &= -2z - 2xz \end{cases} \quad \text{where } \ a, b \text{ are constants.}$$

Equilibrium point: $(x, y, z) = (0, 0, 0)$.
Then we have the jacobian A is formed by

$$A = \begin{pmatrix} a & b & 1 \\ -b & a & 0 \\ 0 & 0 & -2 \end{pmatrix} \text{whose eigenvalues are } \Lambda = (-2,\ a + bi,\ a - bi)$$

Then by Theorem 2.3, the given system has no rational integral if,

$$-2k_1 + (k_2 + k_3)a + (k_2 - k_3)bi \neq 0$$

for any $k_1, k_2, k_3 \in \mathbb{Z}$ and $|k_1| + |k_2| + |k_3| \neq 0$. We also note that if $b \neq 0, a \neq \mathbb{Q}$ then the deduced inequality holds.

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