Growth Incidence Curves in Reallocating Geometric Brownian Motion

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Objectives

- The goal of this project is to study the anonymous (**GIC**) and non-anonymous (**NAGIC**) growth incidence curves predicted by the GBM and RGBM models.
- GBM and RGBM as models for growth of wealth.
- GIC and NAGIC as measures of inequality over time.
- And compare these predictions to empirical evidence on growth incidence curves.
- Study their properties.

Growth Incidence Curves

- Distributional changes of wealth are commonly represented by the growth incidence curves (GIC and NAGIC).
- GIC shows the relative change in wealth in the same wealth quantile between the initial and final periods.
- $$G_f^a(p) = \frac{F_{t'}^{-1}(p) - F_t^{-1}(p)}{F_t^{-1}(p)}$$

  where $F_t^{-1}(p)$ is the wealth of p^{th} quantile at time t.

Non-Anonymous Growth Incidence Curves

- $$G_f^{na}(p) = \frac{\int_0^1 R_f(p,p')F_{t'}^{-1}(p')dp' - F_t^{-1}(p)}{F_t^{-1}(p)}$$

  shows the relative change in wealth between times t and t' of people at rank p at time t.

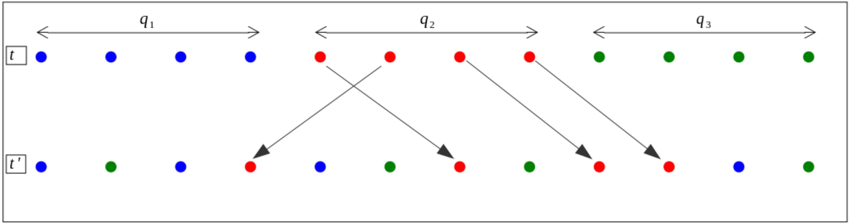


Figure 1: population sorted by wealth at two points in time (t and t' where $t < t'$). q : quantile.

Growth Incidence Curves

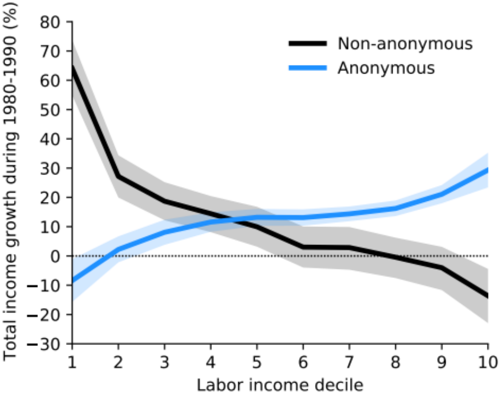


Figure 2: Growth incidence curves for the period 1980-1990 in the United States.

Berman and Bourguignon, 2021

Properties of GIC

- GICs in general are upward sloping when inequality is increasing.
- They ignore the identity of individuals within quantiles.
- The poorest (richest) in the initial period are compared to poorest (richest) in the final period.
- Thus the comparison is anonymous.

Properties of NAGIC

- NAGICs takes into account wealth mobility.
- NAGICs are more informative of the individual experience of wealth changes.
- Thus the comparison is non-anonymous.

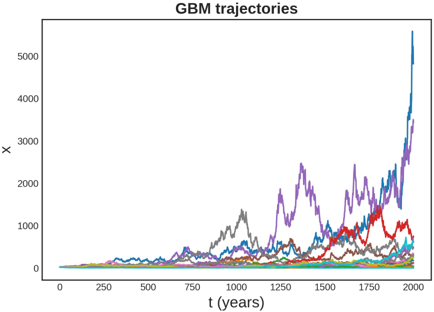
Geometric Brownian Motion

- GBM is a simple model for the evolution of individuals wealth.
- Let $x_i(t)$ be the **wealth** of i^{th} person at time t.

$$dx_i = x_i[\mu dt + \sigma dW_i(t)] \tag{1}$$

where
μ : drift term
σ : volatility parameter
dW : Wiener process or Brownian motion.

Geometric Brownian Motion



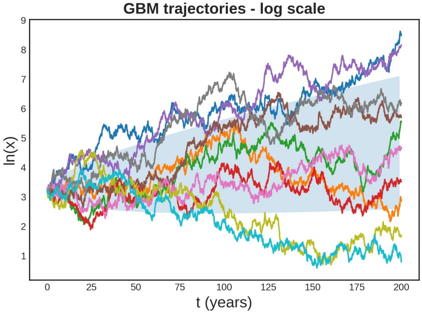


Figure 3: $\mu = 0.02 year^{-1}, \sigma = 0.15 year^{-1/2}$

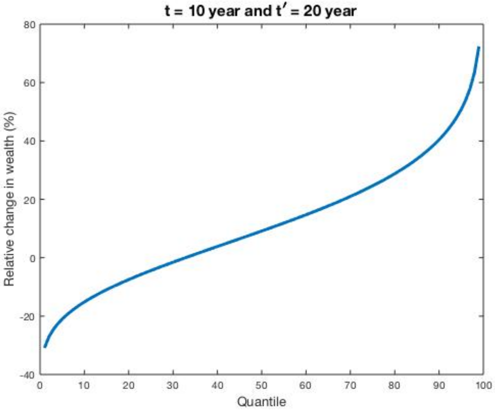
X follows log-normal distribution

$$ln(x(0)) + (\mu - 0.5\sigma^2)t \pm \sigma\sqrt{t}$$

Analytical GIC in GBM

- Analytical GIC is given by

$$G_t(p) = \frac{\exp[(\mu - 0.5\sigma^2)t' + \sigma\sqrt{t'}\phi^{-1}(p)]}{\exp[(\mu - 0.5\sigma^2)t + \sigma\sqrt{t}\phi^{-1}(p)]} - 1$$



GIC and NAGIC in GBM between two points in time

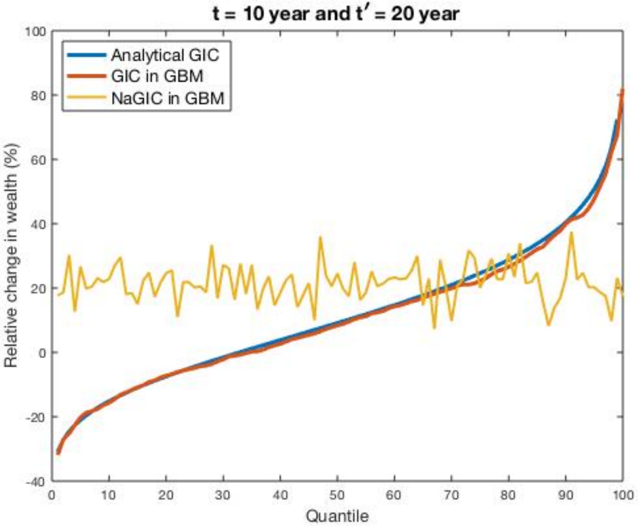


Figure 4: $\mu = 0.02 year^{-1}, \sigma = 0.15 year^{-1/2}, N = 10^4, X_0 = 25$

Wealth Lognormality

- Wealth X is lognormally distributed with $EX = X_0 e^{\mu t}$ and $Var(X) = X_0^2 e^{2\mu t}(e^{\sigma^2 t - 1})$.

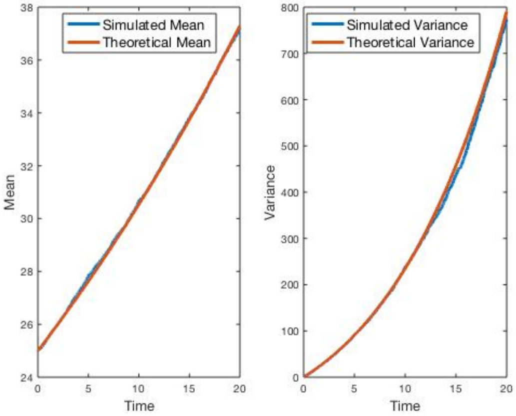


Figure 5: $\mu = 0.02 year^{-1}, \sigma = 0.15 year^{-1/2}$

Reallocating Geometric Brownian Motion

- RGBM : GBM + A reallocation mechanism
- Each individual pays a fixed proportion of its wealth, into a central pot (contributes to society)
- Let $x_i(t)$ be the **wealth** of i^{th} person at time t.

$$dx_i = x_i[\mu dt + \sigma dW_i(t)] - \tau x_i dt + \tau \langle x \rangle_N dt \tag{2}$$

where:
μ : drift term
σ : volatility parameter
dW : Wiener process or Brownian motion
τ : reallocation rate

Reallocating Geometric Brownian Motion

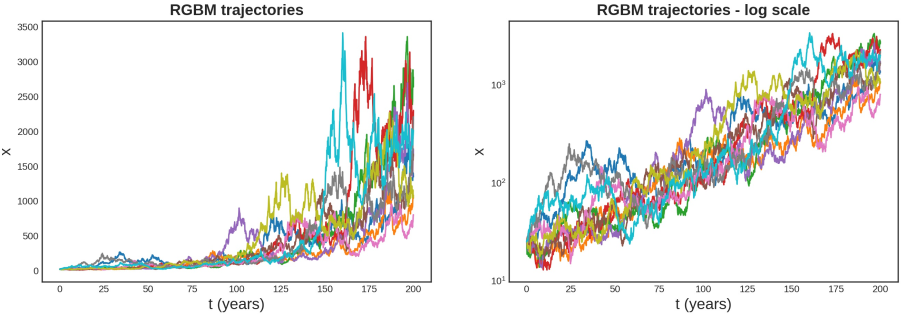


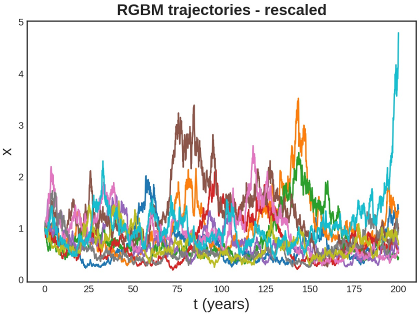
Figure 6: $\mu = 0.01 year^{-1}, \sigma = 0.15 year^{-1/2}, \tau = 0.04 year^{-1}$

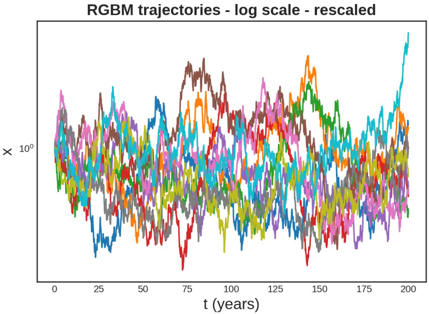
Rescaled Wealth

- Individual wealth divided by the population average

$$y_i(t) \equiv \frac{x_i(t)}{\langle x(t) \rangle_N} \tag{3}$$

$$dy_i = \sigma y_i dW_i(t) - \tau(y_i - 1)dt \tag{4}$$





RGBM stationary distribution

- For $\tau > 0$ and large N approximation
- Solving the stationary Fokker-Planck equation
- A stationary distribution exists
- Inverse Gamma Distribution with a Pareto tail

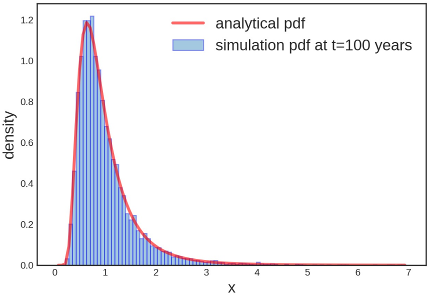
$$P(y) \ = \ \frac{(\zeta - 1)^{\zeta}}{\Gamma(\zeta)} \, e^{-\dfrac{\zeta - 1}{y}} \, y^{-(1+\zeta)} \tag{5}$$

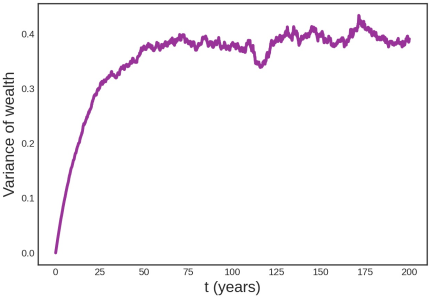
where :

$$\zeta \ = \ 1 + 2\tau/\sigma^2 \tag{6}$$

y is rescaled wealth
ζ is the Pareto tail index

RGBM stationary distribution



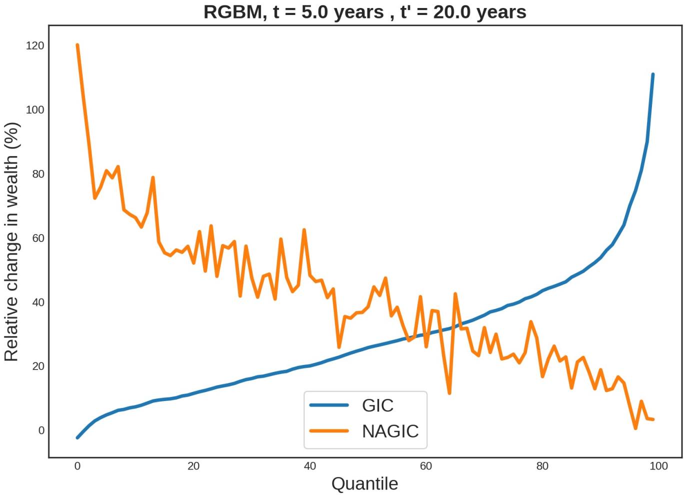


click on image for watching the video showing the evolution of pdf

Predicted GIC and NAGIC in RGBM

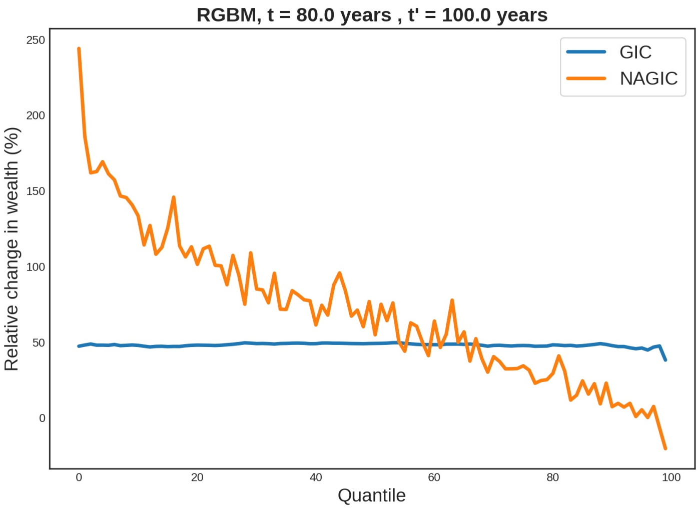
- GICs are expected to be upward sloping initially.
- GICs are expected to become flat after reaching stationary distribution.
- NAGICs are expected to be downward sloping for positive τ.

GIC and NAGIC in RGBM between two points in time



$$\mu = 0.02 year^{-1}, \sigma = 0.15 year^{-1/2}, N = 10^4, X_0 = 25, \tau = 0.04 year^{-1}$$

GIC and NAGIC in RGBM between two points in time



$$\mu = 0.02 year^{-1}, \sigma = 0.15 year^{-1/2}, N = 10^4, X_0 = 25, \tau = 0.04 year^{-1}$$

Conclusion

- For GBM analytical GICs agree with the empirical GICs.
- In GBM, GICs are upward sloping and NAGICs are flat.
- RGBM with positive τ reaches a stationary distribution for large N.
- In RGBM, GICs are flat for the stationary phase and NAGICs are downward sloping.
- RGBM with negative τ requires more investigation.