Fourier Transform Derivation and Basic Properties

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1 Derivation of Fourier Integral from Fourier Series

We know the General Fourier Series of a periodic functions $f(x)$ in the interval $(-L, L)$ is given by,

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L}dt + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}dt \tag{1}$$

Now, Lets work with the summation part of the 1 and try to convert it in integral form only,
Here, we take the integral of the periodic function $f(t)$ common and combine the two sum under one summation notation,

$$\frac{1}{L}\int_{-L}^{L} f(t)\left[\sum_{n=1}^{\infty}\{\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}\ \}\right] dt \tag{2}$$

Then the summation becomes,

$$\frac{1}{2L}\int_{-L}^{L} f(t)\left[2\sum_{n=1}^{\infty}\cos\frac{n\pi(x-t)}{L}\right] dt \tag{3}$$

Now, We use this transformed value of the summation 3 in 1,

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt\ + \frac{1}{2L}\int_{-L}^{L} f(t)\left[2\sum_{n=1}^{\infty}\cos\frac{n\pi(x-t)}{L}\right] dt \tag{4}$$

Now, merging both the integral within the same range $(-L, L)$ as one,

$$\begin{aligned} f(x) &= \frac{1}{2L}\int_{-L}^{L} f(t)\left[1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi(x-t)}{L}\right] dt \\ &= \frac{\pi}{2\pi L}\int_{-L}^{L} f(t)\left[1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi(x-t)}{L}\right] dt \\ &= \frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\frac{\pi}{L} + 2\frac{\pi}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi(x-t)}{L}\right] dt \end{aligned} \tag{5}$$

We know, $\cos 0^\circ = 1$. So we can write, $\frac{\pi}{L}$ as $\frac{\pi}{L}\cos 0.\frac{\pi(x-t)}{L}$. So we rewrite 5 as,

$$f(x) = \frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\frac{\pi}{L}\cos 0.\frac{\pi(x-t)}{L} + 2\frac{\pi}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi(x-t)}{L}\right] dt \tag{6}$$

We also know that, $\cos(-\Theta) = \cos\Theta$. Now we can again, rewrite the summation inside the integral of 6 as,

$$f(x) = \frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\frac{\pi}{L}\cos 0.\frac{\pi(x-t)}{L} + \frac{\pi}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi(x-t)}{L} + \frac{\pi}{L}\sum_{n=1}^{\infty}\cos\frac{-n\pi(x-t)}{L}\right]dt \tag{7}$$

Now, As $\sum_{n=1}^{\infty}\cos\frac{-n\pi(x-t)}{L} = \sum_{n=-\infty}^{-1}\cos\frac{n\pi(x-t)}{L}$ so 7 becomes,

$$f(x) = \frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\frac{\pi}{L}\cos 0.\frac{\pi(x-t)}{L} + \frac{\pi}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi(x-t)}{L} + \frac{\pi}{L}\sum_{n=-\infty}^{-1}\cos\frac{n\pi(x-t)}{L}\right]dt \tag{8}$$

Now we merge the summation together in 8 to get,

$$f(x) = \frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\cos\frac{n\pi(x-t)}{L}\right]dt \tag{9}$$

Rearranging we get,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-L}^{L} f(t)\sum_{n=-\infty}^{\infty}\left[\frac{1}{\frac{L}{\pi}}\cos\frac{r\pi(x-t)}{L}\right]dt \\ &= \frac{1}{2\pi}\int_{-L}^{L} f(t)[lim_{n\to\infty}\sum_{r=-n}^{n}\left[\frac{1}{\frac{L}{\pi}}\cos\frac{r\pi(x-t)}{L}\right]dt \end{aligned} \tag{10}$$

Now when, $L \to \infty$. $\frac{L}{\pi} \to \infty$ and we have,

$$lim_{L\to\infty}\sum_{r=-\infty}^{\infty}\frac{1}{\frac{L}{\pi}}\{\cos\frac{r(x-t)}{\frac{L}{\pi}}\} = lim_{\Delta u\to\infty}\sum_{r=-\infty}^{\infty}\cos r\Delta u(x-t)\Delta u$$

Where , $\Delta u = \frac{1}{\frac{L}{\pi}}$.

Now, writing $r\Delta u = u$ and $\Delta u = du$ after differentiating we get the limit sum as an integral form by definition of the integral as the limit of the sum as,

$$\int_{-\infty}^{\infty}\cos u(x-t)du$$

Now, we replace it in our equation 10 and get

$$f(x) = \frac{1}{2\pi}\int_{-L}^{L} f(t)dt\int_{-\infty}^{\infty}\cos u(x-t)du \tag{11}$$

This double integral is known as **Fourier Integral** and holds if x is a point of continuity of $f(x)$. Moreover, $f(t)$ is called the Fourier transform of $f(x)$.

2 Finite Fourier Sine and Cosine transform

The **finite Fourier Sine transform** of $F(x)$, $0 < x < l$ is defined as,

$$f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{L} dx \tag{12}$$

where n is an integer.

The function $F(x)$ is then called the **inverse finite Fourier Sine transform** of $f_s(n)$ and is given by,

$$F(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{L} \tag{13}$$

The **finite Fourier Cosine transform** of $F(x)$, $0 < x < l$ is defined as,

$$f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{L} dx \tag{14}$$

where n is an integer.

The function $F(x)$ is then called the **inverse finite Fourier Cosine transform** of $f_c(n)$ and is given by,

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{L} \ dx \tag{15}$$

3 Infinite Fourier Sine and Cosine transform

3.1 Infinite Fourier Sine transform

The **infinite Fourier Sine Transform** of a function $F(x)$ of x such that $0 < x < \infty$ is denoted by $f_s(n)$ and defined by,

$$f_s(n) = \int_0^\infty F(x) \sin nx \; dx \tag{16}$$

The function $F(x)$ is called the **inverse Fourier Sine transform** of $f_s(n)$ and is given by,

$$F(x) = \frac{2}{\pi} \int_0^\infty f_s(n) \sin nx \; dx \tag{17}$$

3.2 Infinite Fourier Cosine transform

The **infinite Fourier Cosine Transform** of a function $F(x)$ of x such that $0 < x < \infty$ is denoted by $f_c(n)$ and defined by,

$$f_s(n) = \int_0^\infty F(x) \cos nx \; dx \tag{18}$$

The function $F(x)$ is called the **inverse Fourier Cosine transform** of $f_c(n)$ and is given by,

$$F(x) = \frac{2}{\pi} \int_0^\infty f_c(n) \cos nx \; dx \tag{19}$$

4 Complex form of Fourier Integral and Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(t)dt \; e^{iu(x-t)} \right] du \tag{20}$$

is called the complex form of **Fourier Integral**.

5 Alternative form of Fourier transform

If,

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux} dx \tag{21}$$

then,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u)\, e^{iux} du \tag{22}$$

The function $F(u)$ is called Fourier transform of $f(x)$ and sometimes written as,

$$F(u) = F[f(x)] \tag{23}$$

6 Properties of Fourier Transform

Suppose $f \in L^1$, then

6.1 Time Shifting

(a) $F[f(x-a)] = e^{-iua}\hat{f}(u)$ and
(b) $F[e^{iau}f(x)] = \hat{f}(u-a)$

Proof. **(a)**If time is shifted by a in the Fourier transform where $a \in \mathbb{R}$,

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)\ e^{-iux}\ dx \tag{24}$$

Then by substituting $x-a$ by t,

$$\begin{aligned} x - a &= t \\ \implies x &= t + a \\ \therefore dx &= dt \end{aligned}$$

Substituting these values in equation 24 we get,

$$\begin{aligned} F[f(x-a)] &= \int_{-\infty}^{\infty} f(t)\ e^{-iu(t+a)}\ dt \\ &= e^{-iua}\int_{-\infty}^{\infty} f(t)\ e^{-iut}\ dt \\ &= e^{-iua}\hat{f}(u) \end{aligned} \tag{25}$$

(b) Using the definiton it follows that,

$$\begin{aligned} F[e^{iax}f(x)] &= \int_{-\infty}^{\infty} e^{iax}f(x)\ e^{-iux}\ du \\ &= \int_{-\infty}^{\infty} f(x)e^{-i(u-a)x}\ du \\ &= \hat{f}(u-a) \end{aligned} \tag{26}$$

$\square$

6.2 Time Scaling

If $a > 0$, then we have the scaling formula,

$F[f(ax)](u) = \frac{1}{a}\hat{f}(\frac{u}{a})$

Proof. If $F[f(x)] = f(u)$ then the Fourier tansform of $f(ax)$ can be determined by substituting $x' = ax$ in the Fourier integral,

$$\begin{aligned} F[f(ax)] &= \int_{-\infty}^{\infty} f(ax)e^{-iux}\ dx \\ &= \int_{-\infty}^{\infty} f(X')e^{-iu\frac{x'}{a}}\ \frac{1}{a}dx' \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(x')e^{-i\frac{u}{a}x'}\ dx' \\ &= \frac{1}{a}\hat{f}(\frac{u}{a}) \end{aligned} \tag{27}$$

□

6.3 Transform of derivatives

If f is continuous and piecewise smooth and $f' \in L^1$, then,

$$F[f'(x)](u) = iu\hat{f}(u)$$

Proof. If the transformation of nth derivative $f^n(x)$ exists, then $f^n(x)$ must be integrable over $(-\infty, \infty)$. That means $f^n(x) \to 0$, as $t \to \pm\infty$. With this assumption, the Fourier transform of derivatives of $f(x)$ can be expressed in terms of $f(x)$. This can be shown as follows,

$$\begin{aligned} F[f'(x)] &= \int_{-\infty}^{\infty} f'(x)\, e^{-iux}\, dx \\ &= \int_{-\infty}^{\infty} \frac{d\,f(x)}{dx}\, e^{-iux} \end{aligned} \tag{28}$$

integrating by parts,

$$= f(x)\, e^{-iux}\Big|_{-\infty}^{\infty} + iu \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx$$

But since $f \in L^1$, the integrated term is equal to zero in both limits for the first part of the right hand side.
Thus,

$$\begin{aligned} F[f'(x)] &= iu \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx \\ &= iuF[f(x)] \\ &= iu\hat{f}(u) \end{aligned} \tag{29}$$

It follows that:

$$F[f"(x)] = iuF[f(x)] = (iu)^2 F[f(x)] = (iu)^2 \hat{f}(u).$$

Therefore,

$$F[f^n(x)] = (iu)^n F[f(x)] = (iu)^n \hat{f}(u).$$ □

6.4 Transform of Integral

If $xf(x)$ is integrable, then
$F[xf(x)] = i\hat{f}'(u)$

Proof. Since, $\frac{d}{du}\, e^{-iux} = -ix\, e^{-iux}$. We can write, $i\frac{d}{du}\, e^{-iux} = x\, e^{-iux}$.

$$\begin{aligned} F[xf(x)] &= \int_{-\infty}^{\infty} x\; f(x)\; e^{-iux}\; dx \\ &= \int_{-\infty}^{\infty} f(x)\; i\frac{d}{du}\; e^{-iux}\; dx \\ &= i\frac{d}{du}\int_{-\infty}^{\infty} f(x)\; e^{-iux}\; dx \\ &= i\; \hat{f}(u) \end{aligned} \tag{30}$$

□

6.5 Other Properties

* Linearity
If $Ff(x) = \hat{f}(u)$ and $Fg(x) = \hat{g}(u)$,then

$$\begin{aligned} F[af(x) + bg(x)] &= aF[f(x)] + bf[g(x)] \\ &= a\hat{f}(u) + b\hat{g}(u) \end{aligned} \tag{31}$$

*Symmetry

If $F[f(x)] = \hat{f}(u)$, then $F[\hat{f}(x)] = 2\pi f(-u)$