"Recurrence Relation"

Tower of Hanoi:

$\boxed{T(n) = 2T(n-1) + 1}$ → Recursive Method } to find number of movement

e.g: $T(0) = 0$

$T(1) = 2T(1-1) + 1 = 2 * T(0) + 1$

$= 2 * 0 + 1 = 1$

$T(2) = 2T(2-1) + 1 = 2 * T(1) + 1$

$= 2 * 1 + 1$

$= 3$

Find the number of movement by iterative method:

$T(n) = 2T(n-1) + 1$

$= 2\{2T(n-1-1) + 1\} + 1$

$= 2\{2T(n-2) + 1\} + 1$

$= 4T(n-2) + 2 + 1$

$= 4\{2T(n-3) + 1\} + 2 + 1$

Plug & Chug Problem of ToH *

$\boxed{SP'22 \mid 1(a)}$

$$= 8T(n-3)+4+2+1$$

$$= 1+2+2^2+2^3 T(n-3)$$

$$= 2^0+2^1+2^2+2^3 T(n-3)$$

$$= 2^0+2^1+2^2+ \cdots + 2^{n-1} T[\cancel{n-}\,n-(n-1)]$$

$$= 2^0+2^1+2^2+ \cdots + 2^{n-1} . T(1)$$

$$= 2^0+2^1+2^2+ \cdots + 2^{n-1} * 1 \quad [\because T(1) = 1]$$

$$\boxed{T(n) = 2^n - 1}$$

$$= 2^3 T(n-3) + 2^2 + 2^1 + 2^0$$

$$= 2^n T(n-n) + 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2^1 + 2^0$$

$$= 2^n T_0 + 2^{n-1} + 2^{n-2} + \cdots + 2^0$$

$$= 2^0 + 2^1 + 2^2 + \cdots + 2^{n-2} + 2^{n-1}$$

$$= \frac{1-2^{n+1-1}}{1-2} = \frac{1-2^n}{-1} = (2^n + 1)$$

Ans

Recurrence To Sum :

$$T_n = 2T_{(n-1)} + 1$$

$$= \frac{T_{(n-1)}}{2^{-1}} + 1$$

$$\Rightarrow \frac{T_n}{2^n} = \frac{T_{(n-1)}}{2^{-1} \cdot 2^n} + \frac{1}{2^n}$$

$$\Rightarrow S_n = \frac{T_{(n-1)}}{2^{n-1}} + 1/2^n$$

$$\Rightarrow S_n = S_{n-1} + 2^{-n} \quad (n>0)$$

So, $\boxed{S_n = \sum_{k=1}^{n} 2^{-k}}$ *

* Considering, $S_n = \frac{T_n}{2^n}$

So, $S_{n-1} = \frac{T_{n-1}}{2^{n-1}}$

"Generating function"

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad [a_k = 1]$$

$$1 - x + x^2 - x^3 + x^4 - \cdots = \frac{1}{1+x} = \sum_{k=0}^{\infty} \underbrace{C(n+k-1,k)(-1)^k}_{a_k} x^k$$

$$1 + ax + a^2x^2 + a^3x^3 + \cdots = \frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$$

$$1 + x^2 + x^4 + x^6 + x^8 + \cdots = \frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k} \quad [a^k = 1]$$

$$1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n = \sum_{k=0}^{n} C(n,k)\, x^k \quad [a^k = C(n,k)]$$

$$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k$$

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)\, x^k$$

$$1 + C(n,1)x + C(n+1,2)x^2 + \cdots = \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)\, x^k$$

Fruit Salad Problem :

(i) The number of apple must be even,

$$A(x) = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

(ii) The number of bananas must be a multiple of 5,

$$B(x) = 1 + x^5 + x^{10} + x^{15} + \cdots = \frac{1}{1-x^5}$$

(iii) There can be at most 4 oranges,

$$O(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$$

(iv) There can be at most one pear,

$$P(x) = 1 + x$$

So, the convolution rule for among the fruit to make salad is,

$$A(x) \cdot B(x) \cdot O(x) \cdot P(x) = \frac{1}{1-x^2} * \frac{1}{1-x^5} * \frac{1-x^5}{1-x} * 1+x$$

$$= \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{k=0}^{\infty} (k+1) x^k \quad \text{Ans}$$

Find Co-efficient of x^{15} for

$$\boxed{\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k}$$

$n=6$

$K=15$

(i) $$\frac{1}{(1-x)^6} = \sum_{K=0}^{\infty}\binom{K+5}{K}x^K$$

$$= \binom{15+5}{15} = \binom{20}{15} = 15,504$$

$^{20}C_{15}$

(ii) $$\frac{(1+x^2)^3}{(1-x)^6} = (1+x^2)^3\sum_{K=0}^{\infty}\binom{K+5}{K}x^k$$

$$= (1+3x^2+3x^4+x^6)\sum_{K=0}^{\infty}\binom{K+5}{K}x^5$$

$$= \binom{20}{5}+3\binom{18}{5}+3\binom{16}{5}+\binom{14}{5}$$

$$= 56314 \text{ Ans}$$

5, 5, 5, 5 ----.

$$\Rightarrow 5+5x+5x^2+\cdots = \sum \frac{5}{1-x}$$

5, 0, 5, 0, ---- ----.

$$5+5x^2+5x^4+\cdots = \frac{5}{1-x^2}$$

How many ways can n balls be put into 5 boxes if no ~~ball~~ box has exactly 2 balls?

Solⁿ:

$$1+x+x^3+x^4 \cdots = \frac{1}{1-x} - x^2$$

$$= \frac{1-x^2+x^3}{1-x}$$

$$\text{Generating function} = \left(\frac{1}{1-x} - x^2\right)^5$$

n balls 5 boxes → 1st box even balls
→ 2nd box odd balls

Solⁿ:

$$\frac{x}{(1-x^2)^2 (1-x)^3}$$

Tower of Hanoi ⇒ $T_n = 2T_{(n-1)} + 1$

Merge Sort ⇒ $T_n = 2T_{n/2} + n - 1$ [$n \geq 2$]

Fibonacci ⇒ $T_n = T_{(n-1)} + T_{(n-2)}$ [$n \geq 2$]

Pizza Cutting ⇒ $b_n = \frac{n(n+1)}{2}$ [$n \geq 0$]

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$\boxed{\text{SP'22/3a}}$ Cement & Sand Problem

Let, $C(n)$ = Amount of Cement in the first bag after n rounds

S = Amount of Sand (1 Kg)

Iteration 1:

Cement transferred to 2nd bag = $C(0)/4$

Total Cement in the 2nd bag = $C(0)/4 + S(0)$

Amount of mixture transfer back to 1st bag

$$= \{C(0)/4 + S(0)\}/4 = (C_0 + 4S)/16$$

Cement in the 1st bag after round 1 :

$$C(1) = C(0) - C(0)/4 + (C(0) + 4S)/16$$

Generalized Model :

$$C(n) = C(n-1) - C_{(n-1)}/4 + (C_{(n-1)} + 4S)/16$$

$$= (3C(n-1) + 4S)/16$$

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Closed-form Formula:

$$C(n) = C(0) * [(3+93)/16]^n$$

Limiting amount of Cement in the 1st bag,

$$\Rightarrow \lim \sum_{n=0}^{\infty} C(n) = C(0) * 3/16 = 3C(0)/16$$

3(a) or

(i) Chocolate and at least 3:

$$C(x) = x^3 + x^4 + x^5 + \cdots = \frac{x^3}{1-x}$$

(ii) Glazed and at most 2:

$$G(x) = 1 + x + x^2 = \frac{1-x^3}{1-x}$$

(iii) Coconut and exactly 2 or more:

$$C_o(x) = 1 + x^2$$

(iv) Plain and multiple of 4:

$$P(x) = 1 + x^4 + x^8 + x^{12} + \cdots = \frac{1}{1-x^4}$$

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Combining Case :

$$G(x) = C(x) * G(x) * C_o(x) * P(x)$$

$$= \frac{x^3}{1-x} * \frac{1-x^3}{1-x} * (1+x^2) * \frac{1}{1-x^4}$$

$$= \frac{x^3}{1-x} * \frac{1-x^3}{1-x} * \frac{1+x^2}{1-x^4}$$

$$= \frac{x^3 * (1-x^3)}{(1-x)^2} * \frac{1+x^2}{\{(1-x)(1+x)\}^2}$$

$$= \frac{x^3(1-x^3)(1+x^2)}{(1-x)^4\,(1+x)^2} \quad \text{Ans}$$

$\boxed{3(b)}$ Lilies, Roses & Tulips Problem :

(i) $L(x) = 1+x+x^2+x^3 = \frac{1-x^4}{1-x}$

(ii) $R(x) = 1+x+x^2+x^3+\ldots = \frac{1}{1-x}$

(iii) $T(x) = 1+x^4+x^8+x^{12}+\ldots = \frac{1}{1-x^4}$

$$G(x) = \frac{1-x^4}{1-x} * \frac{1}{1-x} * \frac{1}{1-x^4}$$

For n numbers of flowers,

$$G(n) = \frac{1-n^9}{(1-n)^6} \quad \text{Ans}$$

Password Problem:

$$F = \{a, b, \ldots, z, A, B, \ldots, Z\}$$

$$S = \{a, b, \ldots, z, A, B, \ldots, Z, 0, 1, \ldots, 9\}$$

In this term, the set of all Password is:

$$\underbrace{(F \times S^5)}_{\text{Length } 6} \cup \underbrace{(F \times S^6)}_{\text{Length } 7} \cup \underbrace{(F \times S^7)}_{\text{Length-}8}$$

$$= |F \times S^5| + |F \times S^6| + |F \times S^7| \rightarrow [\text{Sum Rule}]$$

$$= |F| \cdot |S|^5 + |F| \cdot |S|^6 + |F| \cdot |S|^7 \rightarrow [\text{Product Rule}]$$

$$= 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7$$

$$= 1.8 * 10^{14} \text{ different Password} \quad \text{Ans}$$

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$\boxed{SP'22 | 2(C)}$ License Plate Problem :

(a) Let's express L in terms of set A and D:

(i) Standard Plate : 3 Letters followed by 3 digits :

$A \times A \times A \times D \times D \times D$

(ii) Vanity Plate : 5 letters

$A \times A \times A \times A \times A$

(iii) Big shot Plate : 2 chars (letter or number)

$(A \cup D) \times (A \cup D)$

Now, we express L as the union of this sets :

$$L = (A \times A \times A \times D \times D \times D) \cup (A \times A \times A \times A \times A) \cup (A \cup D) \times (A \cup D)$$

(b) $|L|$, the numbers of different license Plate :

(i) Standard Plate : $A \times A \times A \times D \times D \times D$

$$= |A^3 \times D^3| = 26^3 \times 10^3$$

(ii) Vanity Plate : $|A \times A \times A \times A \times A| = |A|^5 = 26^5$

(iii) Big shot : $|(A \cup D) \times (A \cup D)|$

$$= |A \cup D|^n = (|A| \cup |D|)^n$$

$$= (26 + 10)^n = 36^n$$

Now, sum these result,

$$|L| = 26^3 \times 10^3 + 26^5 + 36^n \text{ Ans}$$

SP'22 | 1(b)

Given, $f(1) = 1$

$f(n) = 2f(n-1) + n$ for $n \geq 2$

Now,

$$f(n) = 2f(n-1) + n$$

$$\Rightarrow f(n) - 2f(n-1) = n \quad \text{---(i)}$$

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Applying, $n=2$, $n=3$, to $n=K$ in eq^n (i)

$$T(2) - 2T(1) = 2$$
$$T(3) - 2T(2) = 3$$
$$\text{--- --- ---}$$
$$T(K) - 2T(K-1) = K$$

Sum both sides :

$$T(2) - 2T(1) + T(3) - 2T(2) + T(K) - 2T(K-1) = 2+3+ \text{---} + K$$

$$\Rightarrow T(K) - T(1) = 2+3+ \text{---} + K$$

$$= K(K+1)/2 \quad \left[\because 2+3+\text{---}+K = \frac{K(K+1)}{2}\right]$$

$$\Rightarrow T(K) = T(1) + K(K+1)/2$$

$$= 1 + K(K+1)/2 \quad [T(1) = 1]$$

So, $T(n) = 1 + n(n+1)/2$ Ans

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SP'22 | 2(a)

Block / Book Staking Problem:

(i) For 1 book, $\Rightarrow L/2$ → Maximum over hang edge

(ii) For, 2 books $\Rightarrow \left(L/2 + L/4\right)$

3 books $\Rightarrow \frac{L}{2} + \frac{L}{4} + \frac{L}{6}$

4 books $\Rightarrow \frac{L}{2} + \frac{L}{4} + \frac{L}{6} + \frac{L}{8}$

So, for n-books maximum over hang edge from table,

$$\frac{L}{2} + \frac{L}{4} + \frac{L}{6} + \frac{L}{8} + \cdots$$

$$= L/2 \left(1 + 1/2 + 1/3 + 1/4 + \cdots\right)$$

$$= L/2 \sum_{n=1}^{N} (1/n) \quad \text{Ans}$$

$(0,0,-1,-4,-9,-16,-25 \cdots)$

$$G(x) = -1x^2 - 4x^3 - 9x^4 - 16x^5 - 25x^6 - \cdots$$

$$-xG(x) = \quad +1x^3 + 4x^4 + 9x^5 + 16x^6$$

$$G(x) - xG(x) = -x^2 - 3x^3 - 5x^4 - 7x^5 - 9x^6 \cdots$$

$$G(x)(1-x) = -x^2(1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots) \quad \text{--- (i)}$$

$$N(x) = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots$$

$$-xN(x) = -x - 3x^2 - 5x^3 - 7x^4 + \cdots$$

$$N(x) - xN(x) = 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots$$

$$\Rightarrow N(x)(1-x) = 1 + 2x(1 + x + x^2 + x^3 + \cdots)$$

$$= 1 + 2x * \frac{1}{1-x}$$

$$\Rightarrow N(x) = \frac{1 + \frac{2x}{1-x}}{1-x} = \frac{1-x+2x}{1-x} * \frac{1}{1-x}$$

$$= \frac{(1+x)}{(1-x)^2}$$

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N(x) (i) ~~সমা~~ বাকিগুলো,

$$G(x)\cdot(1-x) = -x^2 \cdot \frac{1+x}{(1-x)^2} \Rightarrow G(x) = \frac{-x^2-x^3}{(1-x)^2} \times \frac{1}{(1-x)}$$

$\boxed{\text{Aut'22 | 2(a) or}}$

(i) Given,

Client debit initially $= m$

Service fee $= f$

interest ~~r~~ate $= p$

(i) On the first day client debit $= (m+f)*p$

On, 2nd day $= [\{\underbrace{(m+f)*p}_{\text{1st day}}\}+f]*p$

(ii) For 'k' days Client debit,

$(m+f)p^k + fp^{k-1} + fp^{k-2} + fp^{k-3} + \dots + fp^1 + fp^0$

$= mp^k + fp^k + fp^{k-1} + fp^{k-2} + \dots + fp^1 + fp^0$

$= mp^k + f(p^k + p^{k-1} + p^{k-2} + \dots + p^1 + 1)$

$= mp^k + f(1 + p + p^2 + \dots + p^{k-1} + p^k)$

$$= mp^k + f * \frac{1-p^{k+1}}{1-p}$$

$$= mp^k + \frac{f(1-p^{k+1})}{1-p} \quad \text{Ans}$$

Given, 1, 1, 0, 1, 1, 1 — —

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots$$

$$= 1x^0 + 1x^1 + 0.x^2 + 1x^3 + \cdots$$

$$= 1 + x + 0 + x^3 + \cdots$$

$$= 1 + x + x^3 + \text{——}$$

$$= (1 + x + x^2 + x^3 + \text{——}) - x^2$$

$$= \frac{1}{1-x} - x^2 \quad \text{Ans}$$

Counting Problem:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \left[\begin{array}{l} n = \text{total} \\ k = \text{Choosen items} \end{array}\right]$$

e.g: 5 books from 100 →

$$\binom{100}{5} = \frac{100!}{5!\,(100-5)!} = \cdots$$

* n donut from k flavours of donut →

$$\binom{n+(k-1)}{n}$$

Manipulation of Sum:

$$\# \sum_{k=50}^{100} k^2 \Bigg/ \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2$$

Solve this?

$$\Rightarrow \sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$$

$$\left[\because \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\right] \qquad = \frac{100 \times 101 \times 201}{6} - \frac{49 \times 50 \times 99}{6}$$

$$= 297925 \text{ Ans}$$

Annuity Value:

$$V = m\left(\frac{1-x^n}{1-x}\right)$$

$$= m\,\frac{1+P-(1/P+1)^{n-1}}{P} \quad \left[x = 1/_{1+P}\right]$$

$$V = m\frac{1+P}{P} > V = \frac{m(1+P-(1/P+1)^{n-1})}{P}$$

Merge Sort Problem ToH: $\boxed{SP22 \mid 2ⓐ}$

$T_1 = 0$

$T_2 = 2T_1 + 2 - 1 = 1$

$T_4 = 2T_2 + 4 - 1 = 5$

$T_8 = 2T_4 + 8 - 1 = 17$

$T_{16} = 2T_8 + 16 - 1 = 49$

There is no obvious pattern. So let's try the Plug and Chug method instead.

Generating Function for Fibonacci Series:

$$f(0) = 0$$

$$f(1) = 1$$

$$f(n) = f(n-1) + f(n-2) \text{ for } n \geq 2$$

e.g.: $F(x) = x + 2x^2 + 3x^3 + 5x^4 + \cdots$

Akra-Bazzi (Divide & Conquer):

General form of akra-bazzi $= \boxed{\sum_{i=1}^{K} a_i T(b_i n) + g(n)}$

e.g.: solve, $T(n) = 2T(n/4) + 3T(n/6) + \theta(n \log n)$ with akra-bazzi.

Soln: Here, $a_1 = 2$, $a_2 = 3$ | $g(n) = n \log n$

$b_1 = 1/4$, $b_2 = 1/6$ | $g(x) = x \log x$

Now, $\boxed{\sum_{i=1}^{K} a_i b_i^P = 1}$

$\Rightarrow a_1 \cdot b_1^P + a_2 b_2^P = 1$

$\Rightarrow 2 * \left(\frac{1}{4}\right)^P + 3 * \left(1/6\right)^P = 1$

For, $P=1$

$$2 * \frac{1}{4} + 3/6 = 1 \quad [\text{satisfied}]$$

$$\boxed{T(n) = \theta\left(n^P\left(1+\int_1^n \frac{g(x)}{x^{P+1}}\,dx\right)\right)}$$

$$= \theta\left(n^1 \cdot \left(1+\int_1^n \frac{x\log x}{x^{1+1}}\,dx\right)\right)$$

$$= \theta\left(n * \left(1+\int_1^n \frac{x\log x}{x^2}\,dx\right)\right)$$

$$= \theta\left(n\left(1+\frac{1-n^2-\log n}{n^2}\right)\right)$$

$$= \theta\left(n\left(\frac{\cancel{n^2}+1-\cancel{n^2}-\log n}{n^2}\right)\right)$$

$$= \theta\left(n * \frac{1-\log n}{n^2}\right)$$

$$= \theta\left(\frac{1-\log n}{n}\right) \text{ Ans}$$

$$\int_1^n \frac{x\log x}{x^2}\,dx$$

$$= x^2\left(x.\frac{1}{x}+\log x.1\right)$$

$$\frac{-x\log x * 2x}{x^4}$$

$$= \frac{x^2\log x + \cancel{2x^2\log x}}{x^4}$$

$$= \frac{\cancel{x^2}\log x\ \cancel{\cdots}\ (1-2)}{\cancel{x^4}\ x^2}$$

$$= \left[\frac{-\log x}{x^2}\right]$$

$$= \frac{x^2+x^2\log x-2x^2\log x}{x^4}$$

$$= 1+\log x - 2\log x/x^2$$

$$= \left[\frac{1-\log x}{x^2}\right]_1^n$$

$$\frac{1-\log n}{n^2} - \frac{1}{1}$$

$$= \frac{1-\log n-n^2}{n^2}$$

SP'23 | 1(a)

$f(0) = 1$

$f(1) = 1$

$f(n) = f(n-1) + f(n-2)$

Let, $f(x)$ be the generating function of fibonacci numbers,

$$f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$

Now,

$$f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$

$$-x\, f(x) = \quad - f_0 x - f_1 x^2 - f_2 x^3 + \cdots$$

$$-x^2 f(x) = \quad - f_0 x^2 - f_1 x^3 + \cdots$$

$$f(x)\left(1-x-x^2\right) = f_0 + (f_1 - f_0)x + (f_2 - f_1 - f_0)x^2 + \cdots$$

$$\Rightarrow f(x)\left(1-x-x^2\right) = 1 + x + (f_2 - 2)x^2$$

$$\Rightarrow f(x) = \frac{1+x+(f_2-2)^2}{1-x-x^2}$$

Ans

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$\boxed{SP'23 \mid 1(b)}$

$$T(n) = 2T(n/4) + T(n/3) + n$$

$$T(n) = a_i T(b_i n) + g(n)$$

Here, $a_1 = 2$, $b_1 = 1/4$

$a_2 = 1$, $b_2 = 1/3$

$g(n) = n \Rightarrow g(x) = x$

$$\sum_{i=1}^{k} a_i b_i^P = 1$$

$\Rightarrow a_1 b_1^P + a_2 b_2^P = 1$

$\Rightarrow 2 * (1/4)^P + (1/3)^P = 1$

For, $P = 1 \Rightarrow \frac{1}{2} + \frac{1}{3} \neq 1$

$P = 2 \Rightarrow \frac{1}{8} + \frac{1}{9} \neq 1$

Considering $P = 1$

$$T(n) = \theta\left(n\left(1 + \int_1^n \frac{g(x)}{x^{P+1}} dx\right)\right)$$

$$= \theta\left(n\left(1 + \int_1^n \frac{x}{P^2} dx\right)\right)$$

Find the GF,

$a_r = 2^r + 3^r \; , \; r \geq 0$

<u>Soln:</u> $G(x) = \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} (2^r + 3^r) x^r$

$= \sum_{r=0}^{\infty} 2^r x^r + \sum_{r=0}^{\infty} 3^r x^r$

$= (2^0 x^0 + 2^1 x^1 + 2^2 x^2 + \cdots) + (3^0 x^0 + 3^1 x^1 + 3^2 x^2 + \cdots)$

$= (1 + 2x + (2x)^2 + (2x)^3 + \cdots) + (1 + 3x + (3x)^2 + (3x)^3 + \cdots)$

$= \frac{1}{1-2x} + \frac{1}{1-3x}$

$= \frac{2-5x}{1-5x-6x^2}$ Ans

$\sum_{i=0}^{n} \sum_{j=0}^{m} (2m)^i = \sum_{i=0}^{n} (2m)^i \sum_{j=0}^{m} 1$

$= \sum_{i=0}^{n} 2^i m^i * (m+1)$

$= (m+1) * \sum_{i=0}^{n} 2^i * \sum_{i=0}^{n} m^i$

$$= (m+1) * \frac{2^{n+1} - 1}{2-1} * \frac{m^{n+1} - 1}{m-1} \quad \left[\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a-1} \right]$$

$$= \frac{(m+1)(2^{n+1} - 1)(m^{n+1} - 1)}{m-1}$$

Ans