CS 419 Homework 1

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The trapezoidal method is defined as:

$$x_{n+1} = x_n + (h/2)(f(t_n, x_n) + f(t_{n+1}, x_{n+1})) \tag{1}$$

The theoretical general formula for methods of integration is defined as:

$$\sum_{k=0}^{r} \alpha_k x_{j+k} = h \sum_{k+0}^{1} \beta_k f(t_{j+k}, x_{j+k}) \tag{2}$$

In the case of r=1, this equation becomes:

$$\alpha_0 x_j + \alpha_1 x_{j+1} = h[(\beta_0 f(t_j, x_j)) + (\beta_1 f(t_{j+1}, x_{j+1}))] \tag{3}$$

Which can be solved as:

$$\alpha_1 x_{j+1} = -\alpha_0 x_j + h[(\beta_0 f(t_j, x_j)) + (\beta_1 f(t_{j+1}, x_{j+1}))] \tag{4}$$

We can now solve for alpha and beta:

$$\alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{5}$$

and

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \tag{6}$$

The stability of the trapezoidal rule with consideration of the amplification factor can be defined as:

$$Q(h\lambda) = (1 + (h/2)\lambda)/(1 - (h/2)\lambda) \tag{7}$$

Where stability is defined as:

$$lim_{h->0} |Q(h\lambda)| <= 1 \tag{8}$$

And we get our Q function from

$$w_{n+1} = Q(h\lambda) w_n = (Q(h\lambda))^{n+1} y_0 \tag{9}$$

This means that for the trapezoidal rule to be stable, h must always be greater than 0. If this is not the case, then the defenition of unstable would be true, which is:

$$lim_{h->0}|Q(h\lambda)| > 1 \tag{10}$$

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Convergence in the case of integration approximation can be defined as:

stability + consistency = convergence

So we need to estimate stability and consistency to prove convergence. We have already solved for stability above. For consistency, we'll need the local truncation error. The gerenal form of local truncation error is:

$$\tau_{n+1}(h) = (y_{n+1} - z_{n+1})/h \tag{11}$$

And consistency is

$$\lim_{h->0} \tau_{n+1} = 0 \tag{12}$$

So if we take the expanded taylor series of the trapezoidal rule:

$$(1 + (h/2)\lambda)/(1 - (h/2)\lambda) = 1 + h\lambda + (1/2)(h\lambda)^2 + (1/4)(h\lambda)^3 + O(h^4) \tag{13}$$

And plug that into the truncation error

$$\tau_{n+1}(h) = ((e^{h\lambda} - Q(h\lambda))y_n)/h = (1/h)(-(1/6)(h\lambda)^3 + O(h^4))y_n \tag{14}$$

$$-(1/6)h^2\lambda^3 + O(h^3))y_n = \tau_{n+1}(h) = O(h^2) \tag{15}$$

The local truncation error is on the order of two. So the trapezoidal rule is a second order method.

If we take the same equation for the Forward Euler method, where

$$Q(h\lambda) = 1 + h\lambda \tag{16}$$

Then our truncation turns out to be

$$((1/2!)h\lambda^2 + O(h^2))y_n = \tau_{n+1}(h) = O(h) \tag{17}$$

And we can see that this is a first order convergence. Backward Euler comes out to be the same, but with the first term negative:

$$(-(1/2!)h\lambda^2 + O(h^2))y_n = \tau_{n+1}(h) = O(h) \tag{18}$$