Compressed Sensing

Fahy Gao, Marvin Meng

March 24, 2021

1 Motivation

We recently learned about bandlimited functions and the Nyquist-Shannon-Kotelnikov Sampling Theorem that can sample from continuous signals. The sampling theorem states that signals can be perfectly recovered from finite samples if the number of samples $N \geq 2K_c + 1$ (Nyquist rate). In other words, it requires the number of the measurement to be at least equal to the signal length in the discrete case.
However, in real life the Nyquist rate can be too high to acquire, requiring large storage space, heavy power consumption and large amounts of sensors. Consequently, people have been trying to find practical solutions to compress the signals, especially for high-dimensional data. This is where the Compressed Sensing comes from, a novel theory that helps to reconstruct the complete signals from a few of incoherent measurements. In this paper, we will be discussing the concept of Compressed Sensing from the signal's conditions—sparsity and incoherent nature, and following experiments to show these two properties.

2 Results

2.1 Compressed Sensing as a Linear Algebra problem

Typically, we want to solve Ax = b for an unknown x. $A \in \mathbb{R}^{m*n}, x \in \mathbb{R}^n, b \in \mathbb{R}^n$. In order to exactly solve for x, we require $m > n$ and A to be full-rank. Instead, if we know that the x is S-sparse, i.e. $x \in \mathbb{R}^n, S < n,$ S components of x $\neq$ 0, we then only need S equations instead of n. This is compressed sensing[5].

2.1.1 Compressed Sensing Lemma

Suppose there is a matrix $A \in R^{mxn}$ such that every set of 2S columns of A are linearly independent. Then an S-Sparse vector x $\in C^n$ can be reconstructed uniquely from $Ax \in C^m$. This implies that we can reconstruct an S-sparse vector with 2S measurements.

2.1.2 Proof by Contradiction

Suppose unique reconstruction fails; then there would be two S-Sparse vectors x, x' $\in C^n$ such that Ax' = Ax. Then A(x - x') = 0. Because x - x' is 2S-sparse, this means that 2S of the columns are linearly dependent, which contradicts our assumption[4].

2.1.3 Finding sparse solutions

Sparse solutions can be found by by minimizing the l_0-norm:

$$x = \underset{x:Ax=b}{argmin} ||x||_0 \tag{1}$$

However, this is intractable, so instead we use **basis pursuit**, i.e. minimize l_1 norm instead.

$$x = \underset{x:Ax=b}{argmin} ||x||_1 \tag{2}$$

This way we replace the non-convex l_0 with the convex l_1.

2.2 Relation to Fourier Series

We have stated that the Fourier series of a signal x(t) is given by:

$$x(t) = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}[k]\psi_k = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}[k] exp\left(\frac{i2\pi k}{T}\right)$$

$$\psi_k = \begin{bmatrix} 1 & exp\left(\frac{i2\pi k}{N}\right) & exp\left(\frac{i2\pi k 2}{N}\right) & ... & exp\left(\frac{i2\pi k(N-1)}{N}\right) \end{bmatrix}$$

This can be expressed as a linear algebra equation: $Ax = b$

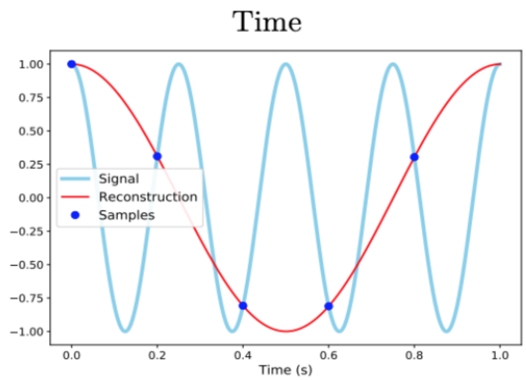
$$A = \frac{1}{T} \begin{bmatrix} \psi_{-k_c} \\ \psi_{-k_c+1} \\ ... \\ \psi_{k_c} \end{bmatrix} \quad x = \begin{bmatrix} \hat{x}[-k_c] \\ \hat{x}[-k_c+1] \\ ... \\ \hat{x}[k_c] \end{bmatrix} \quad b = \begin{bmatrix} x(0) \\ x(\frac{T}{N}) \\ ... \\ x(\frac{T(N-1)}{N}) \end{bmatrix}$$

The Nyquist-Shannon Sampling Theorem says that we need $N \geq 2k+1$ samples. If the Fourier Coefficient vector is sparse, then we can use compressed sensing to reconstruct the signal with fewer samples.

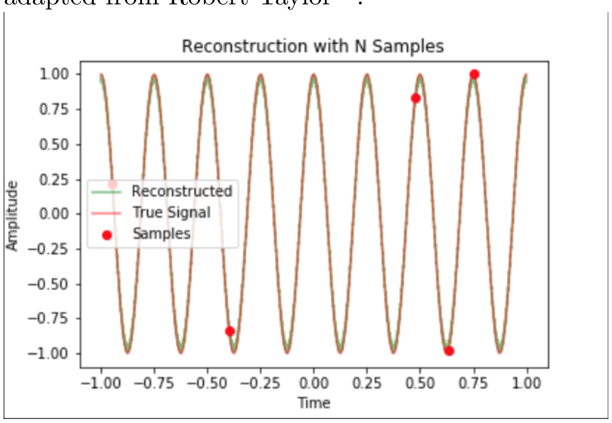
3 Graphs and Figures

3.1 Aliasing Experiment

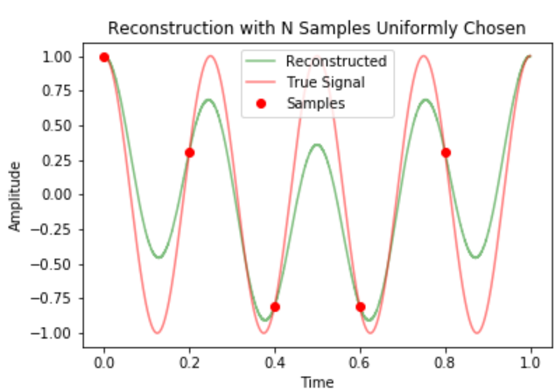
In class we have seen the issue of aliasing, where we may reconstruct the wrong signal if we have too few samples.



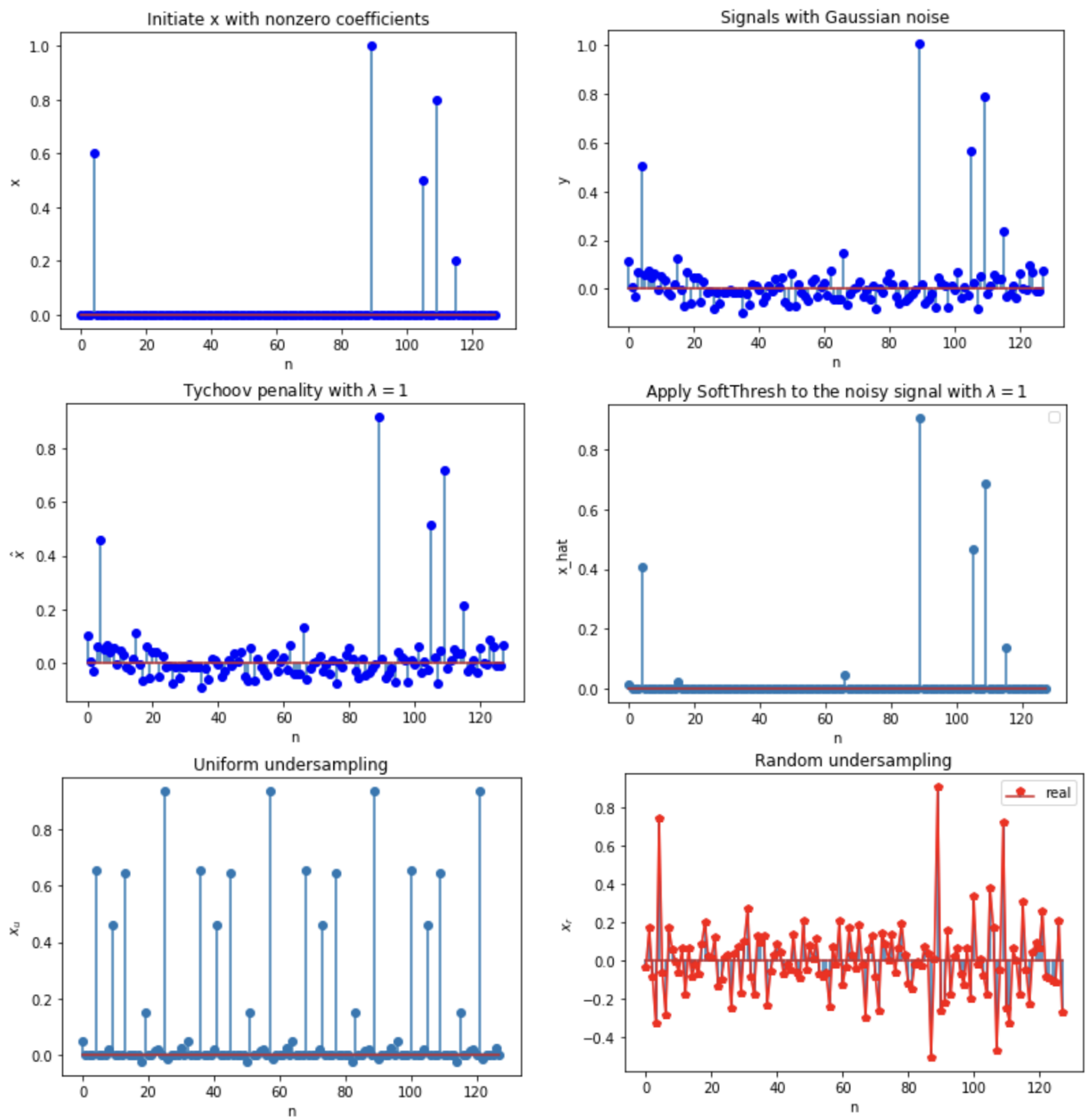
Because the signal $x(t) = .5exp(-i2\pi 4t) + .5exp(i2\pi 4t)$ has a sparse Fourier Coefficient Vector, we can apply compressed sensing. The following code is adapted from Robert Taylor[3]:



It is important to note that for this reconstruction, we sample points randomly instead of uniformly. If we were to sample uniformly as in the Aliasing example, our reconstructed signal looks as follows:



3.2 Second Experiment



Another example showing how L1 Regularization can recover sparse solutions and the importance of random sampling.

4 Citations

1. Irena Orović, Vladan Papić, Cornel Ioana, Xiumei Li, Srdjan Stanković, ”Compressive Sensing in Signal Processing: Algorithms and Transform Domain Formulations”, Mathematical Problems in Engineering, vol. 2016, Article ID 7616393, 16 pages, 2016. https://doi.org/10.1155/2016/7616393

2. Emmanuel Candès ”Compressive Sensing – A 25 Minute Tour” First EU-US Frontiers of Engineering Symposium, Cambridge, September 2010 https://www.raeng.org.uk/publications/other/candes-presentation-frontiers-of-engineering

3. Robert Taylor ”Compressed Sensing in Python” July 20, 2016 http://www.pyrunner.com/weblog/2016/05/26/compressed-sensing-python/

4. E. J. Candes and T. Tao, ”Near-Optimal Signal Recovery From Random Projections: Universal Encoding Strategies?,” in IEEE Transactions on Information Theory, vol. 52, no. 12, pp. 5406-5425, Dec. 2006, doi: 10.1109/TIT.2006.885507.

5. Terrence Tao, ”NTNU's Onsager Lecture, Compressed Sensing” https://www.youtube.com/watch?v=i2aY7tZ5S7Ulist=PLC94A02A1218B24DFindex=2