Recitation 5

DS-GA 1013 Mathematical Tools for Data Science

1. Which of the following cosine functions all have a period of 2π?

   A. $cos(t), cos(t/2), cost(t/3), \ldots$

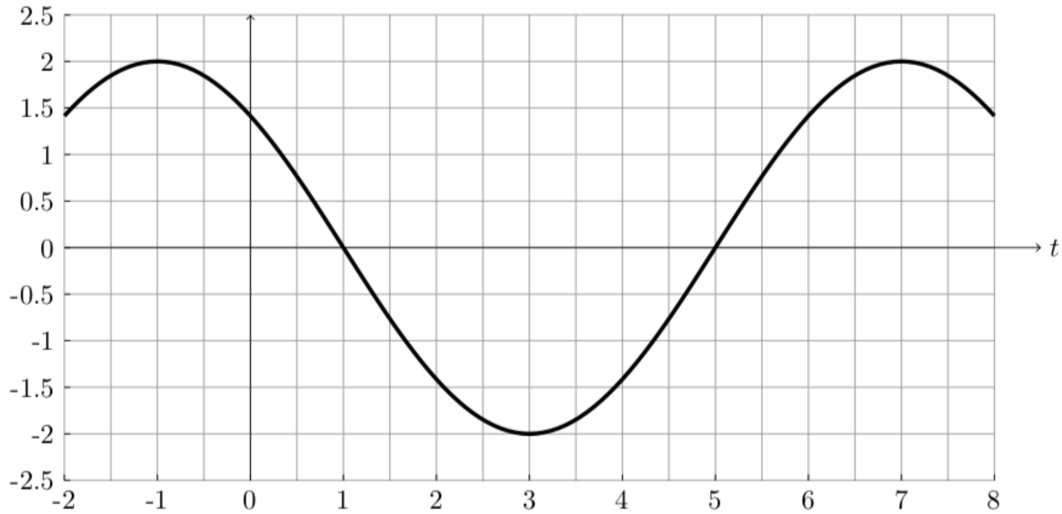
   B. $cos(\pi t), cos(2\pi t), cos(3\pi t), \ldots$

   C. $cos(t), cost(2t), cos(3t) \ldots$

2. What is the fundamental period of

   1. $sin(\pi t/3)$
   2. $|sin(t)|$
   3. $cos^2(3t)$
   4. $f(t) = cos(t) + cos(2t) + cos(3t)$?

3. Express the following sinusoidal function in the form $Acos(\omega t - \varphi)$ where $A \in \mathbb{R}^+ \cup \{0\}$ and $\omega, \varphi \in \mathbb{R}$



4. Now express the sinusoid above as $\sum_j r_j e^{i\varphi_j} e^{i\omega_j t}$ where $r_j \in \mathbb{R}^+ \cup \{0\}$ and $\omega_j, \varphi_j \in \mathbb{R}$

5. What's the fundamental period of $e^{j\omega t}$? What is the projection of $e^{j\omega t}$ to both the axes on complex plane? Animation. Negative frequency.

   Here we list some useful facts about complex numbers. Below $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

   - $z = a + bi = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$
   - $(a + bi)(c + di) = ac - bd + (ad + bc)i$
   - $|a + bi|^2 = a^2 + b^2 = (a + bi)(a - bi) = (a + bi)\overline{(a + bi)}$
   - $|zw| = |z||w|$, $|z + w| \leq |z| + |w|$
   - $e^{a+bi} = e^a(cos(b) + i\sin(b))$, $e^z e^w = e^{z+w}$

- $|e^{a+bi}| = e^a$
- $z = \overline{z}$ if and only if $z \in \mathbb{R}$
- $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $z - \overline{z} = 2i\,\mathrm{Im}(z)$
- $\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$
- $\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$ and $\langle \vec{x}, c\vec{y} \rangle = \overline{c}\langle \vec{x}, \vec{y} \rangle$
- $\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle$
- For $\vec{x} \in \mathbb{C}^n$, $\vec{x}^* := \overline{(\vec{x})}^T$
- For $A \in \mathbb{C}^{m\times n}$, $A^* = \overline{A}^T$
- For $\vec{x}, \vec{y} \in \mathbb{C}^n$, $\langle \vec{x}, \vec{y} \rangle = \vec{y}^*\vec{x} = \sum_{i=1}^n \overline{\vec{y}[i]}\vec{x}[i]$

6. Compute $z = 1 + e^{2\pi i/n} + e^{4\pi i/n} + \cdots + e^{2(n-1)\pi i/n}$ where $n \geq 1$. Explain your answer geometrically.

7. For $z, z_1, z_2 \in \mathbb{C}$, if
$$\left| z - \left(\frac{z_1 + z_2}{2}\right)\right| = \frac{|z_1 - z_2|}{2}$$
then show that
$$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

8. Show that $|r_1 e^{it} - r_2 e^{is}| \geq |r_1 - r_2|$ for all $r_1, r_2 > 0$ and $t, s \in \mathbb{R}$.

9. Prove that Cauchy-Schwarz holds in a complex inner product space:
$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|.$$

10. (Sines and cosines) Let $x : [-1/2, 1/2) \to \mathbb{R}$ be a real-valued square-integrable function defined on the interval $[-1/2, 1/2)$, i.e. $x \in L_2[-1/2, 1/2)$. The Fourier series coefficients of x, are given by
$$\hat{x}[k] := \langle x, \varphi_k \rangle = \int_{-1/2}^{1/2} x(t) \exp\left(-i2\pi kt\right) \,\mathrm{d}t, \quad k \in \mathbb{Z}, \tag{1}$$
and the corresponding Fourier series of order k_c equals
$$\mathcal{F}_{k_c}\{x\}(t) = \sum_{k=-k_c}^{k_c} \hat{x}[k] \exp\left(i2\pi kt\right). \tag{2}$$
As we will discuss in class, this is a representation of x in a basis of complex exponentials. In this problem we show that for real signals the Fourier series is equivalent to a representation in terms of cosine and sine functions.

    1. Prove that $\hat{x}[k] = \overline{\hat{x}[-k]}$ for all $k \in \mathbb{Z}$. [Hint: What is $\overline{e^{it}}$?]
    2. Show that the Fourier series of x of order k_c can be written as
    $$\mathcal{F}_{k_c}\{x\}(t) = a_0 + \sum_{k=1}^{k_c} a_k \cos(2\pi kt) + b_k \sin(2\pi kt),$$
    for some $a_0, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{R}$. [Hint: Group terms in $\mathcal{F}_{k_c}\{x\}(t)$ corresponding to $\pm k$ and use previous part. What is the real part of zw for $z, w \in \mathbb{C}$?]
    3. Give expressions for the coefficients a_k, b_k for $k \geq 1$ from the previous part as real integrals. Interpret them in terms of inner products.
    4. Suppose $x(t) = \cos(2\pi(t + \varphi))$ for some fixed $\varphi \in \mathbb{R}$. What are the Fourier coefficients of x?
    5. Suppose that f is also even (i.e., $x(-t) = x(t)$). Prove that the Fourier coefficients are all real (i.e., that $\hat{x}[k] \in \mathbb{R}$ for all $k \in \mathbb{Z}$).