MATH-GA 2840 HW#8

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1. (Haar wavelet) Define the discrete Haar wavelet $\mu_{2^s,p} \in \mathbb{R}^{2^n}$ at scale 2^s and position p by

$$\mu_{2^s,p}[j] := \begin{cases} -1/\sqrt{2^s} & \text{if } j \in \left\{p \cdot 2^s, p \cdot 2^s + 1, \ldots, p \cdot 2^s + 2^{s-1} - 1\right\} \\ 1/\sqrt{2^s} & \text{if } j \in \left\{p \cdot 2^s + 2^{s-1}, p \cdot 2^s + 2^{s-1} + 1, \ldots, (p+1) \cdot 2^s - 1\right\} \\ 0 & \text{otherwise} \end{cases}$$

where $0 < s \leq n$ and $0 \leq p \leq 2^{n-s} - 1$. Define the discrete Haar scaling function $\varphi_{2^s,p} \in \mathbb{R}^{2^n}$ at scale 2^s and position p by

$$\varphi_{2^s,p}[j] = \begin{cases} 1/\sqrt{2^s} & \text{if } j \in \{p \cdot 2^s, p \cdot 2^s + 1, \ldots, (p+1) \cdot 2^s - 1\} \\ 0 & \text{otherwise} \end{cases}$$

where $0 < s \leq n$ and $0 \leq p \leq 2^{n-s} - 1$.

(a) Define $V_0 := \mathbb{R}^{2^n}$. For $k > 0$, let $V_k \subset \mathbb{R}^{2^n}$ denote the subspace of all vectors that are constant on segments of size 2^k. That is

$$V_k := \left\{x \in \mathbb{R}^{2^n} : x[i] = x[j] \text{ if } \left\lfloor i/2^k \right\rfloor = \left\lfloor j/2^k \right\rfloor\right\}$$

Give an orthonormal basis for V_k (proving that it is indeed orthonormal and a basis). What is the dimension of V_k?

Since $V_k \subset \mathbb{R}^{2^n}$, and it contains the segments of size 2^k where $n > k$, then there will be another 2^{n-k} segments. Now we can define V_k with 2^{n-k} orthogonal vectors that are constant on 2^k but 0 otherwise. Then the dimension of V_k will be 2^{n-k}.

By the lecture notes, we can define the the basis of V_k is:

$$B := \left\{\varphi_{2^k,p} : 0 \leq p \leq 2^{n-k} - 1\right\}$$

Then

$$\begin{aligned} \left\langle \varphi_{2^k,p}, \varphi_{2^k,p} \right\rangle &= \sum_{j=0}^{2^n-1} \varphi_{2^k,p}[j]^2 = \sum_{j=2^k p}^{(p+1)2^k-1} 1/2^k = \frac{(p+1)2^k - 1 - 2^k p + 1}{2^k} \\ &= \frac{2^k}{2^k} \\ &= 1 \end{aligned}$$

Then the vectors $\varphi_{2^k,p}$ are unit length vectors.

Also, assume there are p and q: $0 \leq p \leq q \leq 2^{n-k} - 1$, then

$$\begin{aligned}\left\langle \varphi_{2^k,p}, \varphi_{2^k,q} \right\rangle &= \sum_{j=0}^{2^n-1} \varphi_{2^k,p}[j]\varphi_{2^k,q}[j] = \sum_{j=2^k p}^{(p+1)2^k-1} \varphi_{2^k,p}[j]\varphi_{2^k,q}[j] + \sum_{j=2^k q}^{(q+1)2^k-1} \varphi_{2^k,p}[j]\varphi_{2^k,q}[j] \\ &= 0 + 0 \\ &= 0\end{aligned}$$

Thus B forms an orthnormal basis of V_k.

(b) Show that one can project onto V_k by averaging (explain what needs to be averaged).

Let $proj_{V_k} u$ be the projection of u onto V_k, then:

$$proj_{V_k} u = \sum_{p=0}^{2^{n-k}-1} \left\langle u, \varphi_{2^k,p} \right\rangle \varphi_{2^k,p}$$

And since basis of V_k only contains non-overlapping support, then

$$proj_{V_k} u[j] = \left\langle u, \varphi_{2^k,q} \right\rangle \varphi_{2^k,q}[j]$$

Then based on the question condition,

$$= 1/\sqrt{2^k} \sum_{j=2^k q}^{(q+1)2^k-1} u[j](1/\sqrt{2^k}) = (1/2^k) \sum_{j=2^k q}^{(q+1)2^k-1} u[j]$$

Thus we now can say $proj_{V_k} u$ is the average of the elements $u[j]$ for $2^k q \leq j \leq (q+1)2^k - 1$, which means that elements of $proj_{V_k} u[j]$ are the average value of u on the same 2^k length segment.

(c) Show that $V_{k+1} \subset V_k$

If we want to show V_{k+1} is the subset of V_k then we can try to prove the basis of V_{k+1} is also nested inside the basis of V_k.

We know:

$$\begin{aligned}\frac{1}{\sqrt{2}} \varphi_{2^k,2p}[j] &= \begin{cases} 1/\sqrt{2^{k+1}} & \text{if } j \in \left[2p2^k, (2p+1)2^k - 1\right\} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1/\sqrt{2^{k+1}} & \text{if } j \in \left[p2^{k+1}, (2p+1)2^k - 1\right\}, \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

and A

$$\begin{aligned}\frac{1}{\sqrt{2}} \varphi_{2^k,2p+1}[j] &= \begin{cases} 1/\sqrt{2^{k+1}} & \text{if } j \in \left[(2p+1)2^k, (2p+2)2^k - 1\right\}, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1/\sqrt{2^{k+1}} & \text{if } j \in \left[(2p+1)2^k, (p+1)2^{k+1} - 1\right\}, \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Then,

$$\frac{1}{\sqrt{2}}\left((\varphi_{2^k,2p}) + (\varphi_{2^k,2p+1})\right)[j] = \begin{cases} 1/\sqrt{2^{k+1}} & \text{if } j \in \left[p2^{k+1}, (p+1)2^{k+1} - 1\right\}, \\ 0 & \text{otherwise} \end{cases}$$

Which is exactly the function definiton of $\varphi_{2^{k+1},p}[j]$. Also $|B_{V_{k+1}}| = 2^{n-k-1} < |B_{V_k}|$. Then, $V_{k+1} \subset V_k$.

(d) Fix $0 \leq k < n$. Consider the set

$$W_{k+1} = \{x \in V_k : \langle x, y\rangle = 0 \text{ for all } y \in V_{k+1}\}$$

the orthogonal complement of V_{k+1} in V_k, so that $V_k = V_{k+1} \oplus W_{k+1}$. Give an orthonormal basis for W_{k+1} (proving that it is indeed orthonormal and a basis).

Since V_k and V_{k+1} have the dimensions 2^{n-k} and 2^{n-k-1}, and we need to locate the condition of W_{k+1} that $2^{n-k} - 2^{n-k-1} = 2^{n-k-1}$. Then let us assume:

$$B_W := \left\{\psi_{2^{k+1},p} : 0 \leq p \leq 2^{n-k-1}\right\}$$

Then

$$\begin{aligned}\left\langle \varphi_{2^{k+1},p}, \psi_{2^{k+1},p}\right\rangle &= \sum_{j=0}^{2^n-1} \psi_{2^{k+1},p}[j]\varphi_{2^{k+1},p}[j] \\ &= -\sum_{j=2^{k+1}p}^{p2^{k+1}+2^k-1} \frac{1}{\sqrt{2^{k+1}}} + \sum_{j=2^{k+1}p+2^k}^{(p+1)2^{k+1}-1} \frac{1}{\sqrt{2^{k+1}}} \\ &= \frac{2^k}{\sqrt{2^{k+1}}}(-1+1) \\ &= 0\end{aligned}$$

$$\begin{aligned}\left\langle \varphi_{2^{k+1},p}, \psi_{2^{k+1},q}\right\rangle &= \sum_{j=0}^{2^n-1} \psi_{2^{k+1},p}[j]\varphi_{2^{k+1},q}[j] \\ &= 0\end{aligned}$$

So both are results of non-overlapping support.

Then, for $\psi_{2^{k+1},p}$ in B_W

$$\begin{aligned}\left\langle \psi_{2^{k+1},p}, \psi_{2^{k+1},p}\right\rangle &= \sum_{j=0}^{2^n-1} \psi_{2^{k+1},p}[j]^2 \\ &= \sum_{j=p^{k+1}}^{p2^{k+1}+2^k-1} \psi_{2^{k+1},p}[j]^2 + \sum_{j=p2^{k+1}+2^k}^{(p+1)2^{k+1}-1} \psi_{2^{k+1},p}[j]^2 \\ &= \sum_{j=p2^{k+1}}^{p2^{k+1}+2^k-1} \frac{1}{2^{k+1}} + \sum_{j=p2^{k+1}+2^k}^{(p+1)2^{k+1}-1} \frac{1}{2^{k+1}} \\ &= \frac{(p+1)2^{k+1} - 1 - p2^{k+1} + 1}{2^{k+1}} \\ &= 1\end{aligned}$$

And, for $\psi_{2^{k+1},p}, \psi_{2^{k+1},q}$ in B_W

$$\begin{aligned}\left\langle \psi_{2^{k+1},p}, \psi 2^{k+1}, q \right\rangle &= \sum_{j=0}^{2^n-1} \psi_{2^{k+1},p}[j]\psi_{2^{k+1},q}[j] \\ &= \sum_{j=p2^{k+1}}^{(p+1)2^{k+1}-1} \psi_{2^{k+1},p}[j]\psi_{2^{k+1},q}[j] + \sum_{j=q2^{k+1}}^{(q+1)2^{k+1}-1} \psi_{2^{k+1},p}[j]\psi 2^{k+1}, q[j] \\ &= 0\end{aligned}$$

Therefore B_W is an orthnormal basis.

(e) For $1 \leq k \leq n$ give an orthonormal basis for the set

$$W_{\leq k} = \left\{ x \in \mathbb{R}^{2^n} : \langle x, y \rangle = 0 \text{ for all } y \in V_k \right\}$$

the orthogonal complement of V_k in $\mathbb{R}^{2^n}$, so that $\mathbb{R}^{2^n} = V_k \oplus W_{\leq k}$. Give an orthonormal basis for $W_{\leq k}$ (proving that it is indeed orthonormal and a basis).

We can define $W_{\leq k}$ to be $\oplus_{j=1}^{k} W_j = W_k \oplus W_{k-1} \oplus W_{k-2} \oplus W_1$, then:

Basis of $B_{W_{\leq k}}$ will be: $\bigcup_{j=1}^{k} B_{W_j}$,

and B_W from d part we know it has 2^{n-j} orthonormal vectors. Then we can assume that there are $u \in V_m$ and $v \in V_n$. If m=n-1, then from part d

$$u \in V_n \text{and } \langle u, v \rangle = 0$$

Else based on part c, $V_{n-1} \subset V_m$ and $\langle u, v \rangle = 0$ again.

Thus, $B_{W_{\leq k}}$ has all orthonormal vectors.

And for the dimension of $B_{W_{\leq k}}$:

$$|B_{W_{\leq k}}| = \sum_{j=1}^{k} |B_{W_j}| = \sum_{j=1}^{k} 2^{n-j} = 2^n - 2^{n-k}$$

Thus,

$$B_{W_{\leq k}}$$

has $2^n - 2^{n-k}$ orthonormal vectors.

In [ ]: