Finite Element Discretization of the Static Linear PDE

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We derive the finite element (FE) discretization of the following equations:

$$\left\{ \begin{array}{rcl} -\mathrm{div}(k(x)\nabla u(x)) + f(x) & = & 0 \ \text{ in } \Omega, \\ u(x) & = & g(x) \text{ on } \ \Gamma_D \quad \text{(Dirichet B.C.)}, \\ k(x)\frac{\partial u}{\partial n}(x) & = & h(x) \text{ on } \ \Gamma_N \quad \text{(Neumann B.C.)}, \\ u(x) & = & c + p(x) \ \text{ on } \Gamma_F \quad \text{(Floating B.C.)}, \end{array} \right. \tag{1}$$

where $u(x), c$ are unknown; $f(x), g(x), h(x), p(x)$ are given known functions and the boundary of Ω is $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_F$.
The floating boundary condition implies

$$\int_{\Gamma_F} \sigma \cdot n = \int_{\Gamma_F} k\frac{\partial u}{\partial n} = 0. \tag{2}$$

We write the weak form of (1) by multiplying the first equation with a test function ϕ.

$$\begin{aligned} & -\int_\Omega \phi \mathrm{div}(k\nabla u) + \int_\Omega f\phi = 0, \\ & \int_\Omega k\nabla\phi\nabla u + \int_{\partial\Omega} k\frac{\partial u}{\partial n}\phi + \int_\Omega f\phi = 0, \\ & \int_\Omega k\nabla\phi\nabla u + \int_{\Gamma_D} k\frac{\partial u}{\partial n}\phi + \int_{\Gamma_N} k\frac{\partial u}{\partial n}\phi + \int_{\Gamma_F} k\frac{\partial u}{\partial n}\phi + \int_\Omega f\phi = 0, \\ & \int_\Omega k\nabla\phi\nabla u + \int_{\Gamma_D} k\frac{\partial u}{\partial n}\phi + \int_{\Gamma_F} k\frac{\partial u}{\partial n}\phi + \int_\Omega f\phi + \int_{\Gamma_N} h\phi = 0. \end{aligned}$$

We replace u with a function v satisfying $u = v + u_0$, where u_0 is a function such that $u_0(x) = g(x)$ on Γ_D and $u_0(x) = p(x)$ on Γ_F. We obtain the following equation

$$\int_\Omega k\nabla\phi\nabla v + \int_{\Gamma_D} k\frac{\partial u}{\partial n}\phi + \int_{\Gamma_F} k\frac{\partial u}{\partial n}\phi + \int_\Omega f\phi + \int_{\Gamma_N} h\phi + \int_\Omega k\nabla\phi\nabla u_0 = 0,$$

where the primary unknown is v satisfying $v(x) = 0$ on Γ_D and $v(x) = c$ on Γ_F.

To define a well-posed variational problem (and to formulate the FE discretization based on that), we choose two versions φ and ψ (both in $H^1(\Omega)$) of the test functions ϕ. The functions φ satisfy $\varphi = 0$ on $\Gamma_D \cup \Gamma_F$. The functions

ψ satisfy $\psi = 1$ on Γ_F, $\psi = 0$ on Γ_D. This results in two equations:

$$\int_\Omega k\nabla\varphi\nabla v + \underbrace{\int_{\Gamma_D} k\frac{\partial u}{\partial n}\varphi}_{\substack{=0\\(\varphi=0 \text{ on } \Gamma_D)}} + \underbrace{\int_{\Gamma_F} k\frac{\partial u}{\partial n}\varphi}_{\substack{=0\\(\varphi=0 \text{ on } \Gamma_F)}} + \int_\Omega f\phi + \int_{\Gamma_N} h\varphi + \int_\Omega k\nabla\varphi\nabla u_0 = 0,$$

$$\int_\Omega k\nabla\psi\nabla v + \underbrace{\int_{\Gamma_D} k\frac{\partial u}{\partial n}\psi}_{\substack{=0\\(\psi=0 \text{ on } \Gamma_D)}} + \underbrace{\int_{\Gamma_F} k\frac{\partial u}{\partial n}\psi}_{\substack{=\int_{\Gamma_F} k\frac{\partial u}{\partial n}=0\\(\text{by eqn (2)})}} + \int_\Omega f\psi + \int_{\Gamma_N} h\psi + \int_\Omega k\nabla\psi\nabla u_0 = 0,$$

or

$$\begin{aligned} \int_\Omega k\nabla\varphi\nabla v + \int_\Omega f\varphi + \int_{\Gamma_N} h\varphi + \int_\Omega k\nabla\varphi\nabla u_0 &= 0,\\ \int_\Omega k\nabla\psi\nabla v + \int_\Omega f\psi + \int_{\Gamma_N} h\psi + \int_\Omega k\nabla\psi\nabla u_0 &= 0. \end{aligned} \tag{3}$$

To obtain the FE discretization, we choose a basis set $\{\varphi_i\}_{i=1}^n \cup \{\Psi\}$ such that $\varphi = 0$ on $\Gamma_D \cup \Gamma_F$ and $\Psi = 0$ on Γ_D, $\Psi = 1$ on Γ_F. The basis function Ψ can be defined by $\Psi = \sum_{l=1}^{n_F} \psi_l$ where $\{\psi_l\}_{l=1}^{n_F}$ are the nodal basis functions on Γ_F. The FE solution for v is given by

$$v(x) \simeq v_j\varphi_j(x) + c\Psi(x).$$

We use this and the interpolant $u_0 \simeq g_j\varphi_j(x) + p_l\psi_l(x)$ to write the discretized version of the integral equations (3).

$$v_j\int_\Omega k\nabla\varphi_i\nabla\varphi_j + c\int_\Omega k\nabla\varphi_i\nabla\Psi + \int_\Omega f\varphi_i + \int_{\Gamma_N} h\varphi_i + g_j\int_\Omega k\nabla\varphi_i\nabla\phi_j + p_l\int_\Omega k\nabla\varphi_i\nabla\psi_l = 0,$$

$$v_j\int_\Omega k\nabla\Psi\nabla\varphi_j + c\int_\Omega k\nabla\Psi\nabla\Psi + \int_\Omega f\Psi + \int_{\Gamma_N} h\Psi + g_j\int_\Omega k\nabla\Psi\nabla\phi_j + p_l\int_\Omega k\nabla\Psi\nabla\psi_l = 0.$$

To write the corresponding linear system, we define the following matrices and vectors

$$K_{ij} := \int_\Omega k\nabla\varphi_i\nabla\varphi_j, \qquad K_{il}^F := \int_\Omega k\nabla\varphi_i\nabla\psi_l,$$

$$\widetilde{K}_i := \int_\Omega k\nabla\varphi_i\nabla\Psi = \sum_{l=1}^{n_F} K_{il}, \qquad \widetilde{K}_l^F := \int_\Omega k\nabla\Psi\nabla\psi_l = \sum_{l=1}^{n_F} K_{il}^F,$$

$$\widetilde{\widetilde{K}} := \int_\Omega k\nabla\Psi\nabla\Psi = \sum_{l_1=1}^{n_F}\sum_{l_2=1}^{n_F}\int_\Omega k\nabla\psi_{l_1}\nabla\psi_{l_2},$$

$$f_i := \int_\Omega f\varphi_i, \quad h_i := \int_{\Gamma_N} h\varphi_i, \quad \widetilde{f} := \int_\Omega f\Psi, \quad \widetilde{h} := \int_{\Gamma_N} h\Psi.$$

Then the linear system is

$$\begin{aligned} &K_{ij}v_j + \widetilde{K}_i c + f_i + h_i + K_{ij}g_j + K_{il}^F p_l = 0,\\ &\widetilde{K}_j v_j + \widetilde{\widetilde{K}} c + \widetilde{f} + \widetilde{h} + \widetilde{K}_j g_j + \widetilde{K}_l^F p_l = 0, \end{aligned}$$

or

$$\begin{pmatrix} \mathbf{K} & \widetilde{\mathbf{K}} \\ \mathbf{K^T} & \widetilde{\widetilde{\mathbf{K}}} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ c \end{pmatrix} + \begin{pmatrix} \mathbf{f} \\ \widetilde{\mathbf{f}} \end{pmatrix} + \begin{pmatrix} \mathbf{h} \\ \widetilde{\mathbf{h}} \end{pmatrix} + \begin{pmatrix} \mathbf{K^F} \\ (\widetilde{\mathbf{K}}^{\mathbf{F}})^T \end{pmatrix} (\mathbf{p}) + \begin{pmatrix} \mathbf{K} \\ \widetilde{\mathbf{K}}^T \end{pmatrix} (\mathbf{g}) = 0,$$

or

$$\begin{pmatrix} \mathbf{K} & \widetilde{\mathbf{K}} \\ \mathbf{K^T} & \widetilde{\widetilde{\mathbf{K}}} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ c \end{pmatrix} + \begin{pmatrix} \mathbf{f} \\ \widetilde{\mathbf{f}} \end{pmatrix} + \begin{pmatrix} \mathbf{h} \\ \widetilde{\mathbf{h}} \end{pmatrix} + \begin{pmatrix} \mathbf{K} & \mathbf{K^F} \\ \widetilde{\mathbf{K}}^T & (\widetilde{\mathbf{K}}^{\mathbf{F}})^T \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{g} \end{pmatrix} = 0.$$