Fixed Income Derivatives

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Abstract

This document explores interest rate futures contracts, focusing on U.S. Treasury Bond futures and their valuation. It introduces key concepts, trading mechanics, and valuation techniques, including a binomial interest rate tree model for bond option pricing. The analysis concludes with insights into the role of interest rate derivatives in managing market risks and volatility.

Fixed Income

Overview

Fixed-income securities, such as bonds, are financial instruments that promise a fixed return over time. They differ from equity, which represents ownership in a company, as bonds are debt instruments in which the issuer borrows money and agrees to pay back the principal and interest.

They represent lending money to a company or government in exchange for periodic interest payments and principal repayment at maturity. Unlike equity, which typically has one class, bonds can have multiple classes with varying terms.

Bonds: Price and Valuation

Definition: A bond is a debt instrument where the issuer (borrower) pays periodic interest (coupons) and repays the face value (FV) at maturity.

Formula: The price of a bond is calculated as:

$$P = \sum_{i=1}^{t} \frac{C}{(1+r)^i} + \frac{FV}{(1+r)^t}$$

The bond price (P) is calculated as the sum of the present value of coupon payments (C) discounted by the rate (r) over time (t), plus the present value of the face value (FV) at maturity.

Explanation: The bond price depends on coupon payments, which provide regular income; (FV) repaid at maturity; and(r), reflecting the market's required return. These factors collectively determine the bond's present value.

Example: Consider a bond that has a face value (FV) of $1,000, an annual coupon rate of 5%, and a maturity of 10 years. The discount rate (r) used to calculate the present value of the bond is 4%.

$$P = \sum_{t=1}^{10} \frac{50}{(1+0.04)^t} + \frac{1,000}{(1+0.04)^{10}} \approx 1,081.11$$

The bond price would be higher than its face value since the coupon rate exceeds the discount rate.

Risk and Maturity

Bond prices are inversely related to risk and directly influenced by maturity. Higher risk decreases the value of the bond, while longer maturities (time t) increase the sensitivity to interest rate changes(discount rate r).

Fixed vs. Floating Rates

Fixed Rate Bonds: Fixed coupon rate ($c = k$).
Floating Rate Bonds: Coupon adjusts based on inflation or other benchmarks ($c = inflation + x$).
Reverse Floaters: Coupon decreases as a variable benchmark increases ($c = x - variable$).

Example: A 10-year bond with a face value of $1,000 offers a 5% coupon rate while market rates are at 4%. The bond price will exceed $1,000 due to its higher coupon payments relative to market rates.

Latest News in Fixed Income Markets: As of late 2024, rising inflation concerns have led central banks globally to adjust their monetary policies, impacting yield curves and increasing demand for inflation-protected securities like floating-rate bonds.

Yield to Maturity (YTM)

Yield to Maturity (YTM) is the total return an investor can expect to earn if the bond is held until maturity, assuming all coupon payments are reinvested at the same rate.

$$P = \sum_{t=1}^{N} \frac{C}{(1+YTM)^t} + \frac{FV}{(1+YTM)^N}$$

The YTM reflects the internal rate of return (IRR) for a bond's cash flows, combining coupon income and capital gain or loss. It serves as a benchmark for risk-free rates.

Duration

Duration measures a bond's sensitivity to interest rate changes and is the weighted average time until cash flows are received.

$$D = \sum_{t=1}^{N} \frac{t \cdot PV(C_t)}{P}$$

It quantifies how much a bond's price is expected to change in response to a 1% change in YTM. It is the first derivative of the Yield curve

Convexity

Convexity measures the curvature of the price-yield relationship of a bond, providing a more accurate estimate of price changes due to interest rate movements.

$$\text{Convexity} = \frac{\sum_{t=1}^{N} t(t+1) \cdot PV(C_t)}{P \cdot (1+r)^2}$$

Convexity accounts for non-linear price changes, complementing duration in interest rate risk analysis acting as the first derivative of Duration and second of Yield curve

Interconnectivity Between Bond Price, YTM, Duration, and Convexity

Bonds with higher convexity trade at a premium, as they are less sensitive to interest rate changes, offering more excellent price stability. Low-convexity bonds may be preferred in stable interest-rate environments due to lower premiums. Convexity, derived as the second derivative of the price in relation to yield, implies higher bond prices, lower yields, and increased price stability during rate fluctuations. Duration, on the other hand, measures a bond's sensitivity to yield changes. High-yield bonds generally exhibit lower sensitivity.

Example:

FV = 1000, C = 60, N = 5, YTM = 6%.

$$Price = \sum_{t=1}^{5} \frac{60}{(1+0.06)^t} + \frac{1000}{(1+0.06)^5} \approx 1000$$

$$Duration = \frac{\sum_{t=1}^{5} t \cdot \frac{60}{(1+0.06)^t} + 5 \cdot \frac{1000}{(1+0.06)^5}}{1000} \approx 4.29\,\text{years}$$

$$\text{Convexity} = \frac{\sum_{t=1}^{5} t(t+1) \cdot \frac{60}{(1+0.06)^t} + 5(5+1) \cdot \frac{1000}{(1+0.06)^5}}{1000 \cdot 1.06^2} \approx 19.45$$

Financial Derivatives

Financial derivatives are contracts whose value depends on an underlying asset's price, used for hedging.

Futures contract

A **Futures contract** obligates the buyer and seller to exchange an asset at a predetermined price at a future date. The futures price is calculated by the formula:

$$F = S_0 \times (1 + r)^t$$

Where F is the futures price, S_0 is the spot price, r is the interest rate, and t is the time to maturity.
Example:
Given: $S_0 = 100$, $r = 0.05$, $t = 1$ year.
The futures price is calculated as:

$$F = S_0 \times (1 + r)^t = 100 \times (1 + 0.05)^1 = 100 \times 1.05 = 105$$

Thus, the futures price is \$105.

Hedge Risk with Futures:

Hedging involves locking in a price to reduce risk. For example, a seller of oil profits when P_T (price at maturity) is locked using a short forward to protect from price volatility. Futures are traded in units that match the underlying asset but vary in size.
Example: The spot price of gold is 1400, and the 1-year futures price is also 1400. The US interest rate is 5%. To check for an arbitrage opportunity, we calculate the futures price using the formula $F = S_0 \times (1 + r)^T$. Substituting the given values, we get:

$$F = 1400 \times 1.05 = 1470.$$

Since the calculated futures price (1470) is higher than the spot price (1400), an arbitrage opportunity exists by selling the futures contract and buying the underlying asset.
Long Future Hedge: Used when you know you'll purchase an asset in the future and want to lock in the price.
Cost of asset at time of purchase = S_2.
Gain on futures= $F_2 - F_1$.
Net paid = $S_2 - (F_2 - F_1) = F_1 + b$, where b is the basis at purchase. Basis risk arises due to uncertainty about the basis when the hedge is closed out.
Short Futures Hedge: Used when you know you'll sell an asset in the future and want to lock in the price.
Price of asset at time of sale = S_2.

Gain on futures = $F_1 - F_2$.
Net received = $S_2 + (F_1 - F_2) = F_1 + b$.
Optimal Hedge Ratio: the Hedge ratio is the ratio of the size of the position in a derivative (such as a futures contract) to the size of the underlying asset position, aimed at minimizing risk. It is used to determine how much of a derivative is needed to offset the price movements of the underlying asset, thus reducing potential losses.

$$h_z = \frac{\sigma_s}{\sigma_f}$$

Where σ_s and σ_f are the standard deviations of the spot and futures prices
Why we should Hedge Equity Returns? Hedging reduces the costs of selling and repurchasing portfolios and ensures that returns earned are risk-free returns plus excess portfolio returns over market returns.

Forward Contracts:

A bilateral agreement between a buyer and seller to trade an underlying asset at a future date.
Long Forward: Profit increases as the price of the underlying asset rises at maturity.
Short Forward: Profit increases as the price of the underlying asset falls at maturity.
Arbitrage: Risk-free profit opportunity from price discrepancies in different markets.
Example: Spot price $F = 1400$, interest rate $r = 0.05$. $F_{\text{forward}} = F \times (1 + r)$

$$F_{\text{forward}} = 1400 \times (1 + 0.05) = 1400 \times 1.05 = 1470$$

Arbitrage Opportunity: Buy at 1400, sell at 1470, profit $= 1470 - 1400 = 70$. Short when forward price is high, long when it's low. You buy at the lower price ($1400) and sell at the higher price ($1470), ensuring a risk-free profit of $70.

Options

Option: An option is a financial instrument that grants the holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price (the strike price) within a specified period or at a specified expiration date.

Call Option: A call option gives the holder the right (but not the obligation) to buy an underlying asset at a specified strike price on or before the expiration date. The profit from a call option arises if the price of the underlying asset exceeds the strike price at maturity.

- **Profit at Maturity**: Profit occurs when the share price at maturity exceeds the strike price.

- **Out-of-the-Money**: If the share price is below the strike price, the option is out-of-the-money, meaning the option holder will not exercise the option.
- **Black-Scholes Formula for Call Option**: The Black-Scholes formula for a European call option is given by:

$$C_0 = S \cdot N(d_1) - E \cdot e^{-rT} \cdot N(d_2)$$

Where:

- C_0 = The price of the call option
- S = Current price of the underlying asset
- E = Strike price of the option
- r = Risk-free interest rate
- T = Time to expiration (in years)
- $N(d_1)$ = Cumulative normal distribution function for d_1
- $N(d_2)$ = Cumulative normal distribution function for d_2

The values of d_1 and d_2 are calculated as follows:

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Where:

- σ = Volatility of the underlying asset (standard deviation of asset return)

Call Option Intrinsic Value example:

$$C_0 = 100 \cdot N(0.432) - 95 \cdot e^{-0.05 \cdot 0.5} \cdot N(0.331) \approx 10.27$$

Put Option: A put option gives the holder the right (but not the obligation) to sell an underlying asset at a specified strike price on or before the expiration date. The profit from a put option arises if the price of the underlying asset falls below the strike price at maturity.

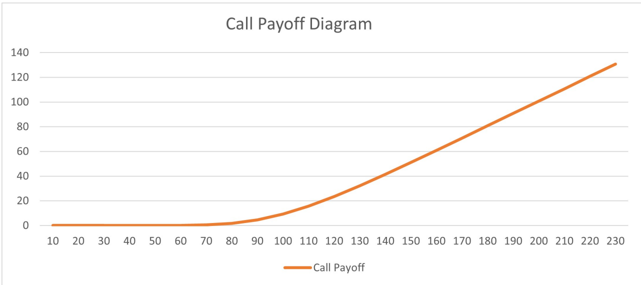
- **Profit at Maturity**: Profit occurs when the strike price exceeds the price of the underlying asset at maturity.
- **Out-of-the-Money**: If the strike price is below the price of the underlying asset, the option is out-of-the-money, meaning the option holder will not exercise the option.

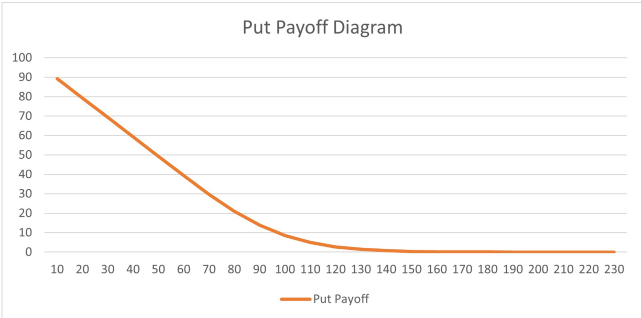
- **Black-Scholes Formula for Put Option**: The Black-Scholes formula for a European put option is given by:

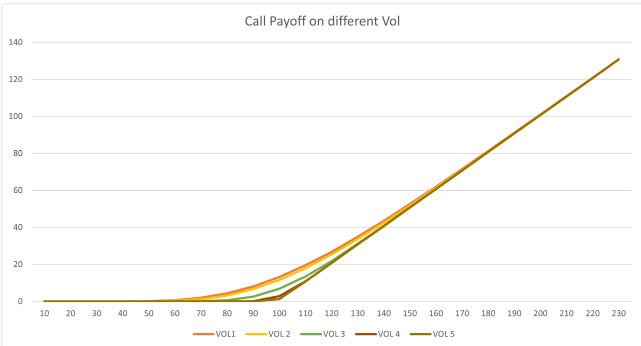
$$P_0 = E \cdot e^{-rT} \cdot N(-d_2) - S \cdot N(-d_1)$$

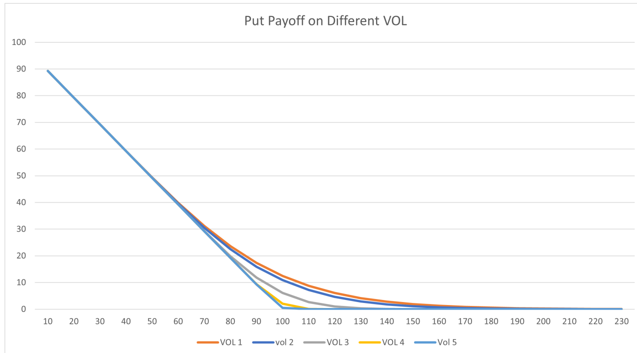
Put Option Intrinsic Value Example:

$$P_0 = 95 \cdot e^{-0.05 \cdot 0.5} \cdot N(-0.331) - 100 \cdot N(-0.432) \approx 4.82$$









The Greeks in Options Pricing

1. Delta (Δ): Delta measures the change in the price of an option for a small change in the price of the underlying asset.

$$\Delta_{\text{call}} = N(d_1), \quad \Delta_{\text{put}} = N(d_1) - 1$$

2. Gamma (Γ): Gamma measures the rate of change of Delta with respect to changes in the price of the underlying asset.

$$\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T}}$$

3. Vega (ν): Vega measures the sensitivity of the option's price to changes in the volatility of the underlying asset.

$$\nu = S\sqrt{T}N'(d_1)$$

4. Theta (Θ): Theta measures the rate of time decay of an option.

$$\Theta_{\text{call}} = -\frac{SN'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2), \quad \Theta_{\text{put}} = -\frac{SN'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

5. Rho (ρ): Rho measures the sensitivity of the option's price to changes in the risk-free interest rate.

$$\rho_{\text{call}} = KTe^{-rT}N(d_2), \quad \rho_{\text{put}} = -KTe^{-rT}N(-d_2)$$

Interconnection of the Greeks: The Greeks are interconnected because they represent different risk factors affecting options. Delta and Gamma are closely linked, with Gamma representing the rate of change in Delta. Vega shows how volatility impacts the option price, while Theta measures time decay. Rho reflects the effect of interest rates. Together, they help traders manage risk and adjust their strategies for optimal hedging.

Interest rate futures

Interest rate futures contracts are powerful financial instruments used in portfolio management and risk hedging. These contracts are settled in cash and allow investors to speculate on movements in interest rates.

Treasury Bond Futures Structure

The structure of U.S. Treasury Bond futures is unique:

- The underlying asset is a hypothetical 20-year bond with a 6% coupon and a par value of $100,000.
- Sellers have the flexibility to deliver any qualifying Treasury bond, often selecting the **cheapest-to-deliver (CTD)** bond.
- The minimum tick size is $\frac{1}{32}$, and futures prices are expressed as the quoted price minus an optional value component.

Trading Mechanics and Applications

Futures contracts serve several strategic purposes:

- Adjusting portfolio duration or creating duration-zero portfolios (DZD).
- Engaging in **cross-hedging** by purchasing futures in related commodities.
- Futures require **margin maintenance**, with clearinghouses enforcing minimum margin requirements.
- Traders can open or liquidate positions based on market conditions.

Eurodollar Futures

Eurodollar futures cater specifically to interest rate speculation:

- Investors anticipating declining interest rates take long positions in these futures.
- The minimum tick size is $\frac{1}{100}$, allowing for precise price adjustments.

Valuation and Delivery Considerations

Valuation involves several steps:

1. Calculate conversion values for potential deliverable bonds.
2. Analyze the characteristics of the CTD bond.
3. Estimate the futures contract value.

4. Evaluate the embedded call option on bonds.

Bond prices are calculated using:

- **Clean Price:** Excludes accrued interest.
- **Dirty Price:** Includes accrued interest.

The conversion rate increases with higher coupon rates but cannot exceed 1. Interest rate fluctuations may change the CTD bond over the contract's duration.

Futures Contract Pricing

The relationship between futures and spot prices is given by:

$$F = (S - I)e^{rt}$$

where:

- F: Futures price
- S: Spot price
- I: Accrued interest
- r: Interest rate
- t: Time to maturity

Solved Example

Problem Statement

Suppose the spot price of a Treasury bond is \$100, accrued interest is \$2, the annual interest rate is 5% ($r = 0.05$), and the time to maturity is 6 months ($t = 0.5$). Calculate the futures price.

Solution

Using the formula:

$$F = (S - I)e^{rt}$$

Substitute the values:

$$F = (100 - 2)e^{0.05 \times 0.5}$$

$$F = 98 \cdot e^{0.025}$$

$$F \approx 98 \cdot 1.0253$$

$$F \approx 100.48$$

Thus, the futures price is approximately \$100.48.

Conversion Value and Related Concepts

Conversion Value

The conversion value is used to adjust the quoted price of a bond to account for differences in coupon rates and maturities. It is calculated as:

$$\text{Conversion Value} = \frac{\text{Quoted Price of Bond}}{\text{Conversion Factor}}$$

where:

- **Quoted Price of Bond:** The market price of the bond.
- **Conversion Factor:** A coefficient provided by the exchange to standardize eligible bonds.

Cheapest-to-Deliver (CTD) Bond

The CTD bond minimizes the cost of delivery for the seller of the futures contract. The delivery cost is:

$$\text{Delivery Cost} = \text{Quoted Bond Price} - (\text{Futures Price} \times \text{Conversion Factor})$$

Conversion Factor

Conversion factors are provided by the exchange to account for differences in coupon rates and maturities. Bonds with higher coupon rates have higher conversion factors, but the maximum value cannot exceed 1.

Clean Price and Dirty Price

- **Clean Price:** The bond price excluding accrued interest.
- **Dirty Price:** The bond price including accrued interest, calculated as:

$$\text{Dirty Price} = \text{Clean Price} + \text{Accrued Interest}$$

Accrued Interest

Accrued interest represents the interest earned by the bondholder since the last coupon payment.

Switching of CTD Bond

The CTD bond may change over the contract's life due to interest rate fluctuations, adding complexity to delivery strategies.

Example: Conversion Value and Delivery Cost

Problem: A trader delivers a bond for a Treasury Bond futures contract with:

- Quoted price: \$105.50
- Conversion factor: 1.1
- Futures price: \$102.75

Find the conversion value and delivery cost.

Solution:

1. Conversion Value:

$$\text{Conversion Value} = \frac{\text{Quoted Price}}{\text{Conversion Factor}} = \frac{105.50}{1.1} \approx 95.91$$

2. Delivery Cost:

$$\text{Delivery Cost} = \text{Quoted Price} - (\text{Futures Price} \times \text{Conversion Factor})$$

$$\text{Delivery Cost} = 105.50 - (102.75 \times 1.1) \approx -7.525$$

Conclusion: The conversion value is \$95.91, and the delivery cost of $-\$7.525$ indicates this bond is not ideal for delivery.

Options on Bonds via Binomial Model

Overview

Options on bonds provide the holder the right to buy (call) or sell (put) a bond at a strike price K. The binomial model evaluates these options by modeling bond price dynamics, incorporating factors such as volatility, time to maturity, and interest rates.

Key Factors in the Binomial Model

1. Price Movement. The bond price evolves over discrete time steps, influenced by upward (u) and downward (d) factors:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the bond's price volatility, and Δt is the time step length.

2. Risk-Neutral Probability. The probability p of an upward movement reflects a risk-neutral world:

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

where r is the risk-free interest rate.

3. Terminal Payoffs. At maturity:

$$\text{Call: } \max(S_{n,j} - K, 0), \quad \text{Put: } \max(K - S_{n,j}, 0)$$

4. Backward Induction. Starting from the terminal payoffs, the option value at earlier steps is computed as:

$$V_{i,j} = e^{-r\Delta t}\big(pV_{i+1,j+1} + (1-p)V_{i+1,j}\big)$$

Example: Pricing a Call Option

Given: $S_0 = 100$, $\sigma = 0.2$, $r = 0.05$, $K = 105$, $T = 1$, $\Delta t = 0.5$.

Step 1: Price Movements.

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\sqrt{0.5}} \approx 1.151, \quad d = e^{-\sigma\sqrt{\Delta t}} \approx 0.869$$

Step 2: Risk-Neutral Probability.

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05\cdot 0.5} - 0.869}{1.151 - 0.869} \approx 0.578$$

Step 3: Terminal Bond Prices.

$$S_u = S_0 \cdot u = 100 \cdot 1.151 = 115.1, \quad S_d = S_0 \cdot d = 100 \cdot 0.869 = 86.9$$

Step 4: Terminal Payoffs.

$$\text{Call Payoff: } \max(115.1 - 105, 0) = 10.1, \quad \max(86.9 - 105, 0) = 0$$

Step 5: Option Value. At $t = 0$:

$$V_0 = e^{-r\Delta t} \cdot (p \cdot 10.1 + (1-p) \cdot 0) = e^{-0.05\cdot 0.5} \cdot (0.578 \cdot 10.1) \approx 5.58$$

Role of Volatility

Volatility σ directly impacts u, d, and p, influencing the breadth of price movements and option value. Higher volatility increases the probability of extreme price changes, thus raising option premiums due to greater potential payoffs.

The binomial model incorporates bond price volatility, risk-neutral probabilities, and interest rates to accurately price options on bonds. This discrete framework enables flexibility in evaluating European and American options, considering early exercise rights and changing market conditions.

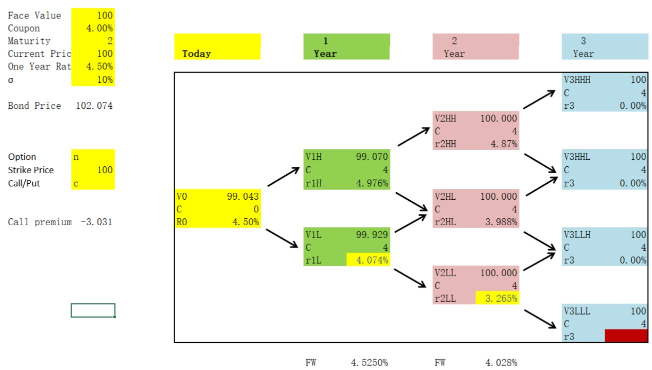


Figure 1: Binomial Model

Valuation of Bond Options Using Binomial Interest Rate Trees

The binomial tree model evaluates bond options by calculating bond prices across different nodes, accounting for interest rate volatility, and determining option premiums via backward induction.

Key Inputs:

- **Bond Characteristics:**
  - Face Value (FV): 100
  - Coupon (C): 4%
  - Maturity: 2 years
  - Current Price: 100
  - One-Year Rate (r_0): 4.5%
  - Volatility (σ): 10%
- **Option Details:**
  - Strike Price (K): 100
  - Type: Call

Steps in the Binomial Tree Model:

1. Interest Rate Tree Construction

The tree models interest rate evolution:

- Upward movement: $r_{\text{up}} = r_0 \cdot e^{\sigma}$

- Downward movement: $r_{\text{down}} = r_0 \cdot e^{-\sigma}$

For Year 1:

$$r_{1H} = 4.976\%, \quad r_{1L} = 4.074\%$$

For Year 2:

$$r_{2HH} = 4.87\%, \quad r_{2HL} = 3.988\%, \quad r_{2LL} = 3.265\%$$

2. Bond Price Calculation at Each Node

Bond price (P) at node i, j uses discounted cash flows:

$$P = \sum_{t=1}^{T} \frac{C}{(1 + r_t)^t} + \frac{FV}{(1 + r_t)^T}$$

Example: At V_{1H}:

$$P = \frac{4}{1.04976} + \frac{104}{(1.04976)^2} = 99.070$$

Similarly, at V_{1L}:

$$P = \frac{4}{1.04074} + \frac{104}{(1.04074)^2} = 99.929$$

3. Option Valuation via Backward Induction

At maturity (Year 2):

$$\text{If } P > K, \quad C = P - K; \quad \text{otherwise } C = 0$$

Example: V_{2HH}: $C = 0$, V_{2LL}: $C = 0$.

At earlier nodes (e.g., Year 1):

$$C_{\text{current}} = e^{-r_t \Delta t} \left(p_{\text{up}} C_{\text{up}} + p_{\text{down}} C_{\text{down}} \right)$$

4. Final Option Premium

At V_0, the call premium is:

$$C_0 = e^{-r_0 \Delta t} \left(p_{\text{up}} C_{\text{up}} + p_{\text{down}} C_{\text{down}} \right) = -3.031$$

Current News Section

Recent developments in the financial markets highlight the critical role of interest rate futures and options:

Federal Reserve Rate Cut Expectations

The Federal Reserve is expected to lower its benchmark interest rate by 0.25% on December 18, 2024. Current trading in Fed funds futures suggests a 95% probability of this cut. However, persistent inflation concerns may temper expectations for additional rate cuts in 2025. The yield on 10-year U.S. Treasury notes recently exceeded three-month bill yields for the first time since 2022, reflecting market anticipation of monetary policy adjustments.

Market Performance

The Dow Jones Industrial Average has experienced a historic 10-day losing streak as investors await clarity from the Federal Reserve's policy meeting. Despite short-term declines, all sectors remain positive for the year, with communication services leading at a 44.08% increase.

Treasury Futures Activity

U.S. Treasury Bond futures continue to serve as a vital hedging tool amidst market uncertainty. These contracts have demonstrated strong liquidity and efficiency in managing interest rate risk, as evidenced by their role in price discovery and volatility spillovers.