Catalan Numbers

Problem Description

Catalan is a name given to numbers that form a particular series and that arise when we count, for example:

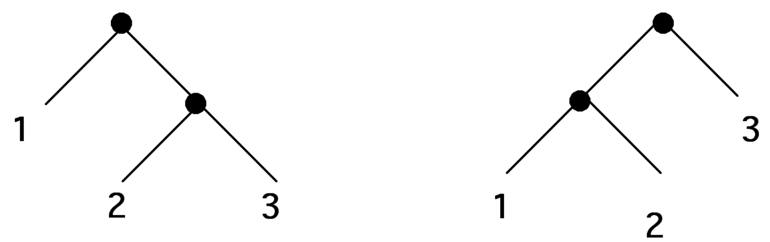
- The number of triangulations of a polygon.
- The number of ways n items may be paired by brackets without changing their order (for example to indicate the order in which we want to multiply the numbers)
- The number of ordered binary trees we can create out of n items without changing their order

For example, if we have a list of ordered items there are the following increasing numbers of different ways of bracketing as the number of items increases:

1. (1 2)
2. (1 (2 3)) ((1 2) 3)
3. (1 (2 (3 4))) (1 ((2 3) 4)) ((1 2) (3 4)) ((1 (2 3)) 4) (((1 2) 3) 4)

etc.

If we express the second line in the example above, $C(2)$, as a binary tree we get the following two possible trees:



Briefly, we can write the number of possibilities as (by definition we set $C(0) = 1$):

$$C(1) = 1 \qquad C(2) = 2 \qquad C(3) = 5$$

Task

1. By hand draw the binary tree equivalences for line 3 of our example, that is to say, for $C(3)$.
2. By hand, work out $C(4)$, i.e. given the five numbers 1 2 3 4 and 5, how many different ways can we place brackets to pair off the numbers.
3. Write a program that uses 32-bit integers to calculate $C(n)$ for $n = 1 \ldots 50$.

Some help

Look closely at $C(3)$ and note the ways the numbers are paired in brackets. You will observe that it includes the solutions for the previous two, $C(1)$ and $C(2)$. $C(3)$ can be generated from solutions for $C(0)$, $C(1)$, and $C(2)$. Similarly, you should find how $C(4)$ can be generated from $C(0) \ldots C(3)$. It turns out that the number of possibilities, $C(n)$, can be expressed in the following formula:

$$\sum_{i=0}^{n-1} C(i) \times C(n-1-i)$$

Relates to Objectives

1.1 1.2 1.3 1.4 2.1 2.2 2.4 2.7 2.8 2.9 2.10 3.1 3.4 3.5 4.1 4.2 4.3

(Group)