Measurement in Non-Standard Bases
Change of Basis and Quantum Measurements

Quantum Computing Course

Contents

1 Motivation: Beyond the Computational Basis

1.1 Limitations of Computational Basis Measurements

In previous lectures, we studied quantum measurement exclusively in the **computational basis**:

$$\{|0\rangle, |1\rangle\}.$$

While this basis is natural for digital computation and hardware implementation, it represents only one possible way to extract information from a quantum system.

> **Key Question**
>
> If quantum states can exist in superpositions like $\alpha|0\rangle + \beta|1\rangle$, why should we only measure whether they're "0" or "1"? What about measuring whether they're "+" or "-"?

Quantum mechanics permits measurement in **any orthonormal basis**. This freedom is not just mathematical—it's essential for:

- Quantum algorithms that exploit interference patterns
- Testing quantum correlations (Bell inequalities)
- Quantum error correction syndromes
- Quantum state tomography

1.2 Physical Implementation Reality

All physical quantum hardware has a **native measurement basis**, typically the energy eigenbasis (which corresponds to $\{|0\rangle, |1\rangle\}$ for qubits). This is because:

- Measurement apparatus couples to specific physical observables (e.g., energy, charge, flux)
- The computational basis is often the easiest to distinguish physically
- Decoherence typically occurs toward this basis

> **Hardware Constraint**
>
> **Quantum computers measure only in their native computational basis.** All other measurements must be implemented via change of basis operations before measurement.

1.3 Applications Requiring Non-Standard Measurements

- **Deutsch-Jozsa algorithm:** Requires final measurement in $\{|+\rangle, |-\rangle\}$ basis
- **Bell inequality tests:** Measurement in rotated bases reveals quantum correlations
- **Quantum teleportation:** Bell basis measurements are crucial
- **Quantum key distribution:** Security relies on measurements in complementary bases

2 Mathematical Formalism of Measurement in Arbitrary Bases

2.1 Projective Measurements in Orthonormal Bases

Let $\{|\phi_0\rangle, |\phi_1\rangle\}$ be an orthonormal basis for a single-qubit Hilbert space. Orthonormality and completeness are expressed as

$$\langle\phi_i|\phi_j\rangle = \delta_{ij}, \qquad \sum_{i=0}^{1} |\phi_i\rangle\langle\phi_i| = I.$$

A **projective measurement** in this basis is described by the projection operators

$$P_0 = |\phi_0\rangle\langle\phi_0|, \qquad P_1 = |\phi_1\rangle\langle\phi_1|.$$

These projectors satisfy

$$P_i^2 = P_i, \qquad P_iP_j = 0 \ (i \neq j), \qquad P_0 + P_1 = I.$$

2.2 The Born Rule in an Arbitrary Basis

Let the quantum state of the qubit be $|\psi\rangle$, with $\langle\psi|\psi\rangle = 1$. The probability of obtaining outcome i, corresponding to the basis state $|\phi_i\rangle$, is given by the Born rule:

$$p(i) = \langle\psi| P_i |\psi\rangle = |\langle\phi_i|\psi\rangle|^2.$$

This expression holds for measurements in *any* orthonormal basis.

2.3 Post-Measurement State Update

If outcome i is obtained, the post-measurement state is

$$|\psi_i\rangle = \frac{P_i|\psi\rangle}{\sqrt{p(i)}} = |\phi_i\rangle,$$

up to an unobservable global phase.

Thus, a projective measurement in the basis $\{|\phi_0\rangle, |\phi_1\rangle\}$ prepares the system in the corresponding basis state, independent of the initial state.

Important Distinction

Computational-basis measurement: Projects onto $|0\rangle$ or $|1\rangle$.

Arbitrary-basis measurement: Projects onto $|\phi_0\rangle$ or $|\phi_1\rangle$.

3 The Hadamard Basis: $\{|+\rangle, |-\rangle\}$

3.1 Definition and Orthonormality

The most important non-standard basis for qubits is the **Hadamard basis** (also called X-basis or diagonal basis):

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

These states form an orthonormal basis:

$$\langle +|+|+|+\rangle = 1, \quad \langle -|-|-|-\rangle = 1, \quad \langle +|-|+|-\rangle = 0.$$

3.2 Geometric Interpretation on the Bloch Sphere

On the Bloch sphere representation:

- $|0\rangle$ points to the **north pole** (+Z direction)
- $|1\rangle$ points to the **south pole** (-Z direction)
- $|+\rangle$ points to the **positive X-axis** (equator, $\phi = 0$)
- $|-\rangle$ points to the **negative X-axis** (equator, $\phi = \pi$)

Thus, measuring in $\{|+\rangle, |-\rangle\}$ corresponds to measuring the **Pauli X observable**:

$$X = |+\rangle\langle +| - |-\rangle\langle -|.$$

3.3 Measurement Probabilities in the Hadamard Basis

For a general state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:

$$\langle +|\psi|+|\psi\rangle = \frac{\alpha+\beta}{\sqrt{2}}, \qquad \langle -|\psi|-|\psi\rangle = \frac{\alpha-\beta}{\sqrt{2}}.$$

The measurement probabilities are:

$$p(+) = \frac{|\alpha+\beta|^2}{2}, \qquad p(-) = \frac{|\alpha-\beta|^2}{2}.$$

Example 1: Equal Superposition

For $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$:

$$p(+) = 1, \quad p(-) = 0.$$

Measuring $|+\rangle$ in $\{|+\rangle, |-\rangle\}$ basis always yields ”+”.

Example 2: Computational Basis State

For $|\psi\rangle = |0\rangle$:

$$p(+) = \frac{|1+0|^2}{2} = \frac{1}{2}, \quad p(-) = \frac{|1-0|^2}{2} = \frac{1}{2}.$$

Equal probabilities—this makes sense since $|0\rangle$ is halfway between $|+\rangle$ and $|-\rangle$ on the Bloch sphere.

4 Change of Basis as a Unitary Operation

4.1 The Basis-Change Theorem

Let $\{|\phi_0\rangle, |\phi_1\rangle\}$ be an orthonormal basis. There exists a unitary operator U such that:

$$U|0\rangle = |\phi_0\rangle, \qquad U|1\rangle = |\phi_1\rangle.$$

Equivalently, U maps the computational basis to the new basis.

Construction of U

Given $|\phi_0\rangle = a|0\rangle + b|1\rangle$ and $|\phi_1\rangle = c|0\rangle + d|1\rangle$ with $\langle\phi_0|\phi_1|\phi_0|\phi_1\rangle = 0$:

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

More abstractly: $U = |\phi_0\rangle\langle 0| + |\phi_1\rangle\langle 1|$.

4.2 Equivalence of Measurements: Formal Proof

Theorem: Measurement in basis $\{|\phi_0\rangle, |\phi_1\rangle\}$ is equivalent to:

1. Apply $U^\dagger$ to the state
2. Measure in computational basis $\{|0\rangle, |1\rangle\}$

Proof: The probability of outcome $|\phi_i\rangle$ in the original measurement is:

$$p(i) = |\langle\phi_i|\psi|\phi_i|\psi\rangle|^2.$$

After applying $U^\dagger$, the state becomes $U^\dagger|\psi\rangle$. The probability of measuring $|i\rangle$ in computational basis is:

$$|\left\langle i|U^\dagger|\psi\middle|i|U^\dagger|\psi\right\rangle|^2 = |\langle\phi_i|\psi|\phi_i|\psi\rangle|^2 = p(i).$$

Thus, the probabilities are identical. The post-measurement states are also equivalent (up to the basis change).

4.3 General Recipe for Arbitrary Basis Measurement

To measure state $|\psi\rangle$ in basis $\{|\phi_0\rangle, |\phi_1\rangle\}$:

1. **Construct** U such that $U|0\rangle = |\phi_0\rangle$, $U|1\rangle = |\phi_1\rangle$
2. **Apply** $U^\dagger$ to $|\psi\rangle$
3. **Measure** in computational basis $\{|0\rangle, |1\rangle\}$
4. **Interpret:** Outcome 0 corresponds to $|\phi_0\rangle$, outcome 1 to $|\phi_1\rangle$

5 The Hadamard Gate as a Basis-Change Operator

5.1 Properties of H: Self-Inverse and Real

The Hadamard gate:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

has key properties:

- **Self-inverse:** $H^\dagger = H$, $H^2 = I$
- **Basis change:** $H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$
- **Real and symmetric:** $H = H^\top = H^\dagger$

5.2 Circuit Implementation for $\{|+\rangle, |-\rangle\}$ Measurement

To measure in $\{|+\rangle, |-\rangle\}$ basis:

$$\boxed{|\psi\rangle \longrightarrow H \longrightarrow M \longrightarrow \text{classical bit}}$$

- **Hardware executes:** Apply H, then measure in $\{|0\rangle, |1\rangle\}$
- **Interpretation:** Outcome 0 means $|+\rangle$, outcome 1 means $|-\rangle$

5.3 Interpretation: X-basis Measurement via HZH

An alternative perspective: Measuring X (Pauli operator) on state $|\psi\rangle$:

$$\langle\psi| X |\psi\rangle = \langle\psi| HZH |\psi\rangle .$$

This shows that measuring the expectation value of X is equivalent to measuring Z on $H|\psi\rangle$.

6 General Basis Change: Constructing U for Arbitrary Bases

6.1 Single-Qubit Case: Relation to Rotation Operators

Any single-qubit basis $\{|\phi_0\rangle, |\phi_1\rangle\}$ can be obtained from $\{|0\rangle, |1\rangle\}$ by a rotation on the Bloch sphere. The unitary U can be expressed as:

$$U = e^{-i\theta/2} R_{\hat{n}}(\phi)$$

for some axis $\hat{n}$ and angles θ, ϕ.

Example: Y-basis

The Y-basis is $\{|+i\rangle, |-i\rangle\}$ where:

$$|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \quad |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle).$$

The basis change unitary is $U = HS^\dagger$ where $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$.

6.2 Multi-Qubit Bases and Tensor Product Structures

For multi-qubit systems, we can measure each qubit in a different basis. The overall unitary is a tensor product:

$$U_{\text{total}} = U_1 \otimes U_2 \otimes \cdots \otimes U_n.$$

Example: Bell Basis Measurement

The Bell basis for two qubits is:

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}$$

Measurement in this basis requires H on first qubit followed by CNOT.

6.3 Non-Standard Bases in Higher Dimensions

For a d-dimensional system, measurement in basis $\{|\phi_i\rangle\}_{i=0}^{d-1}$ uses unitary U with columns $|\phi_i\rangle$:

$$U = \sum_{i=0}^{d-1} |\phi_i\rangle \langle i|.$$

7 Partial Measurement in Non-Standard Bases

7.1 Procedure for Single-Qubit Basis Change in Multi-Qubit Systems

Consider a two-qubit state $|\Psi\rangle_{AB}$. To measure qubit A in basis $\{|\phi_0\rangle, |\phi_1\rangle\}$:

1. Apply $U_A^\dagger$ to qubit A only (where $U_A|0\rangle = |\phi_0\rangle$, $U_A|1\rangle = |\phi_1\rangle$)
2. Measure qubit A in computational basis
3. The post-measurement state of qubit B depends on the outcome

7.2 Conditional States and Correlations

Example: Bell State with X-basis Measurement

Start with Bell state: $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Measure qubit 1 in $\{|+\rangle, |-\rangle\}$ basis:

1. Apply H to qubit 1: $H_1|\Phi^+\rangle = \frac{1}{2}(|0+\rangle + |1-\rangle)$
2. Measure qubit 1 in computational basis:
   - Outcome 0: $|0\rangle_1$, remaining state: $|+\rangle_2$
   - Outcome 1: $|1\rangle_1$, remaining state: $|-\rangle_2$

Thus, measuring one qubit of a Bell state in X-basis leaves the other in an X-basis eigenstate.

7.3 Circuit Representation with Controlled Basis Changes

For adaptive measurements where basis choice depends on previous outcomes:

$$\boxed{|\psi\rangle_A \longrightarrow \textcircled{M} \longrightarrow \text{classical control} \longrightarrow \boxed{U(\text{outcome})}_B \longrightarrow \textcircled{M}}$$

8 Practical Considerations and Common Pitfalls

8.1 Hardware Limitations: Only Computational Basis is Physical

- **Gate errors:** $U^\dagger$ is imperfect (finite gate fidelity)
- **Timing:** Basis change adds circuit depth and decoherence time
- **Calibration:** Different bases may have different measurement fidelities

8.2 Error Propagation through Basis Changes

If U has error ϵ, the measurement probabilities acquire error:

$$p_{\text{actual}}(i) = |\langle i| (U^\dagger + \delta U) |\psi\rangle|^2 \approx p_{\text{ideal}}(i) + 2\,\text{Re}[\langle\psi|U|i|\psi|U|i\rangle \langle i|\delta U|\psi|].$$

8.3 Classical Post-Processing Interpretation

An alternative viewpoint: Instead of physically changing basis, measure in computational basis and classically compute:

$$p(i) = |\langle\phi_i|\psi|\phi_i|\psi\rangle|^2 = |\sum_j U_{ij}^* \langle j|\psi|j|\psi\rangle|^2.$$

This requires knowing the full quantum state, which defeats the purpose of quantum computation.

Key Insight

The power of quantum measurement comes from **physically** changing basis before measurement, not classical post-processing.

9 Comparison and Summary

9.1 Side-by-Side: Standard vs. Non-Standard Measurements

Aspect	**Standard Basis**	**Arbitrary Basis**
Basis vectors	$|0\rangle, |1\rangle$	$|\phi_0\rangle, |\phi_1\rangle$
Projectors	$|0\rangle\langle 0|, |1\rangle\langle 1|$	$|\phi_0\rangle\langle\phi_0|, |\phi_1\rangle\langle\phi_1|$
Implementation	Direct measurement	Apply $U^\dagger$, then measure
Hardware	Native operation	Requires gates
Information	Z-component	Arbitrary observable

9.2 Key Conceptual Takeaways

1. **Generalized Born Rule:** $p(i) = |\langle\phi_i|\psi|\phi_i|\psi\rangle|^2$ for any orthonormal basis $\{|\phi_i\rangle\}$
2. **Basis-Change Equivalence:** Measurement in $\{|\phi_i\rangle\} = U^\dagger$ + measurement in $\{|i\rangle\}$
3. **Hardware Reality:** All quantum hardware measures only in computational basis
4. **Hadamard Basis:** $\{|+\rangle, |-\rangle\}$ with H as basis-change unitary
5. **Partial Measurements:** Can measure subsystems in different bases

9.3 Applications in Upcoming Topics

- **Next lecture:** Deutsch-Jozsa algorithm uses H before final measurement
- **Quantum algorithms:** Grover's algorithm, quantum Fourier transform
- **Quantum information:** Bell tests, quantum key distribution
- **Quantum error correction:** Syndrome measurements in different bases

10 Exercises and Further Exploration

10.1 Problem Set: Basis Change Calculations

1. For state $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$, calculate probabilities for measurement in:
   (a) Computational basis
   (b) $\{|+\rangle, |-\rangle\}$ basis
   (c) $\{|+i\rangle, |-i\rangle\}$ basis
2. Construct the unitary U that maps $\{|0\rangle, |1\rangle\}$ to basis $\{\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\}$.
3. Show that measuring $|+\rangle$ in $\{|+\rangle, |-\rangle\}$ basis always gives "+", while measuring $|0\rangle$ gives 50/50 outcomes.
4. For the Bell state $|\Phi^+\rangle$, calculate the post-measurement state of qubit 2 when qubit 1 is measured in $\{|+\rangle, |-\rangle\}$ basis and outcome is "-".
5. Prove that for any unitary U, measurement in basis $\{U|0\rangle, U|1\rangle\}$ is equivalent to applying $U^\dagger$ then measuring in computational basis.

10.2 Exploration: Other Important Bases

- **Y-basis:** $\{|+i\rangle, |-i\rangle\}$ with $U = HS^\dagger$
- **Breidbart basis:** $\{\cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle, \sin(\pi/8)|0\rangle - \cos(\pi/8)|1\rangle\}$ Used in optimal quantum cloning and cryptography.
- **Mutually unbiased bases (MUBs):** Bases where $|\langle\phi_i|\psi_j|\phi_i|\psi_j\rangle|^2 = 1/d$ for all i, j. For qubits: X, Y, Z bases are mutually unbiased.

10.3 Looking Ahead: POVMs and Generalized Measurements

Advanced Preview

Beyond projective measurements lie **POVMs (Positive Operator-Valued Measures)**:

$$E_i \geq 0, \quad \sum_i E_i = I, \quad p(i) = \langle\psi| E_i |\psi\rangle .$$

POVMs can distinguish non-orthogonal states and are more general than projective measurements. We'll cover these in advanced topics.

Final Thought

The ability to measure in arbitrary bases is not just a mathematical curiosity—it's what enables quantum algorithms to outperform classical ones. By changing basis before measurement, we extract different types of information from quantum states, information that would be exponentially hard to compute classically.