Measurement in Non-Standard Bases
Change of Basis and Quantum Measurements

Quantum Computing Course

1 Motivation

In previous lectures, we studied quantum measurement in the *computational basis*

$$\{|0\rangle, |1\rangle\}.$$

However, quantum mechanics allows measurement in *any orthonormal basis*. In this lecture, we extend the notion of measurement to non-standard bases and show how such measurements are implemented using change of basis.

Throughout this lecture, we restrict ourselves to:

- Projective measurements
- Finite-dimensional Hilbert spaces
- Standard basis measurement as the physical primitive

2 Measurement in an Arbitrary Orthonormal Basis

Let $\{|\phi_0\rangle, |\phi_1\rangle\}$ be an orthonormal basis of a single-qubit Hilbert space.

A measurement in this basis is described by the projectors:

$$P_0 = |\phi_0\rangle\langle\phi_0|, \qquad P_1 = |\phi_1\rangle\langle\phi_1|.$$

For a quantum state $|\psi\rangle$, the probability of outcome i is

$$p(i) = \langle\psi| P_i |\psi\rangle = |\langle\phi_i|\psi|\phi_i|\psi\rangle|^2,$$

and the post-measurement state is

$$|\psi_i\rangle = \frac{P_i |\psi\rangle}{\sqrt{p(i)}}.$$

3 The $|+\rangle, |-\rangle$ Basis

The most important non-standard basis for a qubit is the *Hadamard basis*:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

These states satisfy:

$$\langle +|+|+|+\rangle = 1, \quad \langle -|-|-|-\rangle = 1, \quad \langle +|-|+|-\rangle = 0.$$

4 Measurement in the $|+\rangle, |-\rangle$ Basis

Let

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Then:

$$\langle +|\psi|+|\psi\rangle = \frac{\alpha + \beta}{\sqrt{2}}, \qquad \langle -|\psi|-|\psi\rangle = \frac{\alpha - \beta}{\sqrt{2}}.$$

Thus, the measurement probabilities are:

$$p(+) = \frac{|\alpha + \beta|^2}{2}, \qquad p(-) = \frac{|\alpha - \beta|^2}{2}.$$

5 Change of Basis as a Unitary Transformation

Let U be a unitary operator such that:

$$U|0\rangle = |\phi_0\rangle, \qquad U|1\rangle = |\phi_1\rangle.$$

Then measurement in the basis $\{|\phi_0\rangle, |\phi_1\rangle\}$ can be implemented as:

1. Apply $U^\dagger$
2. Measure in the computational basis

Justification

$$\langle\phi_i|\psi|\phi_i|\psi\rangle = \left\langle i|U^\dagger|\psi\middle|i|U^\dagger|\psi\right\rangle,$$

so the measurement probabilities are preserved.

6 Hadamard Gate as a Basis-Change Operator

The Hadamard gate is defined as:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It satisfies:

$$H|0\rangle = |+\rangle, \qquad H|1\rangle = |-\rangle,$$

and

$$H^\dagger = H.$$

Thus, measuring in the $|+\rangle, |-\rangle$ basis is equivalent to:

1. Apply H
2. Measure in the computational basis

7 Circuit Representation of Basis-Change Measurement

$$|\psi\rangle \xrightarrow{H} H|\psi\rangle \xrightarrow{\text{measurement}} \text{classical outcome}$$

Quantum hardware measures only in the computational basis; all other measurements are implemented via unitary change of basis.

8 Partial Measurement in a Non-Standard Basis

Consider a two-qubit state $|\Psi\rangle_{AB}$.

To measure qubit A in the $|+\rangle, |-\rangle$ basis:

1. Apply H to qubit A
2. Measure qubit A in the computational basis

The post-measurement state of qubit B depends on the measurement outcome.

9 Comparison: Standard vs Non-Standard Basis Measurement

Standard Basis	Non-Standard Basis
Measure $|0\rangle, |1\rangle$	Measure $|\phi_0\rangle, |\phi_1\rangle$
No preprocessing	Apply unitary $U^\dagger$ first
Direct probabilities	Probabilities via overlaps

10 Key Takeaways

- Measurement can be performed in any orthonormal basis
- Measurement probabilities depend on state overlaps
- All non-standard measurements reduce to standard-basis measurement
- Change of basis is implemented using unitary operators
- The Hadamard gate enables $|+\rangle, |-\rangle$ basis measurement

11 Looking Ahead

Non-standard basis measurements are essential for:

- Quantum algorithms (Deutsch–Jozsa, Simon, Grover)
- Quantum teleportation
- Bell inequality tests
- Quantum error correction