Due: Tuesday, April 4, 2023, 11:59pm

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1 Good Augmenting Paths

Any algorithm for maximum flow that uses the Ford-Fulkerson method has to cope with the problem of picking a "good" augmenting path. If a "bad" augmenting path is chosen, the flow will increase, but only a little. This will lead to numerous iterations and a slow algorithm. What if there aren't any good augmenting paths? what if they just don't exist? The Max Flow Min Cut Theorem won't help us here because if there are still bad augmenting paths, then we have not achieved maximum flow yet. For this problem, you will show that there is always a sequence of at most E augmenting paths (starting from zero flow) that leads to the maximum flow.

Assignment:

1. You are given a flow network G. Suppose that you "cheat" and peek at a max flow f. (I.e., you look at the answer before doing the problem.) How would you pick a good augmenting path for the zero flow in G? (Remember that you are trying to minimize the number of iterations in the Ford-Fulkerson method.)
2. How would you update the max flow f so you can pick another good augmenting path? This augmenting path would augment the good path you picked above and still result in a valid flow for G. *Note:* Do this without constructing a residual graph.
3. Argue that you can keep picking good augmenting paths this way.
4. Argue that you will eventually reach max flow if you pick good augmenting paths this way.
5. Argue that you will pick at most E augmenting paths.

2 Petulant Baristas

A few years from now, after some soul-searching, you discover that your true passion is not computer science after all. It's managing a coffee shop! You cash in your life savings and open your own espresso bar. After a while you find out that attracting customers is the easiest part of your job. Your real problem is getting competent employees to show up for work.

Although your coffee shop employs k baristas, there are b bad weeks in the year when many of them refuse to work. After haggling for a while, you and the baristas finally agree on these rules:

- Each barista will provide a list of at least t days in bad weeks when they will work.
- You will assign a barista to work during a bad week only on the days on that barista's list.
- No barista will have to work for more than c days during bad weeks in the entire year.
- No barista will have to work for more than 3 days during the same bad week.

The baristas turn in their lists, but you struggle to follow the agreed upon rules and have coverage for the bad weeks, because you need at least one barista to work on every day of the bad weeks. Fortunately, you remember enough algorithms to know that this problem can be solved with a flow network.

1. Using a combination of a diagram, descriptive labels and English sentences, describe how you can transform the problem described above into a flow network.

2. Suppose that a solution does not exist — that is, there is no way to assign at least one barista to every bad day, given the constraints described. Show how a maximum flow in the network you constructed can be used to determine that no solution is possible.

3. Argue that your transformation works. That is, explain how a maximum flow of the flow network you constructed can be used to identify which baristas work on which bad days when a solution does exist.

3 The New Office

You are the chief operating officer of a company and you are considering moving your office to a new building across town. The new building has better facilities and lower heating and cooling costs. It is closer to most of your employees' homes, so a move would also reduce their commuting time. Happier employees have higher productivity, too.

Unfortunately, the CEO and founder of your company loves the view from his corner office in the old building and absolutely refuses to move. Since there is nothing more you can do to change the CEO's mind, you plan to move just a subset of your employees to the new building. Even with Zoom meetings, having your employees at two separate locations will incur some costs. So, you attempt to determine which employees you should move to the new building and which ones will stay behind with the CEO.

These are your parameters and constraints:

- Your company has n employees, including the CEO and yourself.
- For employee i, you have calculated the total benefit b_i gained by the company, if you move employee i to the new building. Assume that b_i is a non-negative integer dollar amount.
- Some employees work closely together and putting them in separate buildings would create a cost due to a reduction in productivity. For each pair of employees i and j, you have calculated a cost c_{ij} incurred by the company if you placed employees i and j in separate buildings. (The cost is the same whether you move employee i or employee j.) Assume that c_{ij} is a non-negative integer dollar amount.
- As stated previously, the CEO, employee #1, stays in the old building.

You *must* use network flow to solve this problem. The solution produced by your algorithm must maximize the sum of the total benefits minus the sum of the total costs.

Note: You have c_{ij} for every pair of employees. If two employees do not work together, the corresponding c_{ij} may be very small or zero. Also, your employees tend to work on multiple projects, so there are no convenient "clusters" for you to move around. Let max flow do the work for you, instead of trying to solve the problem directly.

1. Using a combination of a diagram, descriptive labels and English sentences, describe how you can transform the problem described above into a flow network.

2. Argue that your transformation works. That is, explain how a maximum flow of the flow network you constructed can be used to identify the set of employees to move.

3. Explain why the solution produced by your algorithm maximizes the sum of the total benefits minus the sum of the total costs.