Due: Tuesday, April 18, 2023, 11:59pm

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1 Finding a three coloring

This problem explores the difference between decision problems (where we only provide yes/no answers) and search problems (where we actually find a witness to some property).

In computational complexity theory, we often work with algorithms that have access to an *oracle*. The oracle is a set and the algorithm is allowed to make membership queries to the oracle. That is, the algorithm can ask an oracle A whether a string x a member of the set A. The algorithm receives the answer from the oracle in a single step (i.e., the complexity of determining whether $x \in A$ is not included the running time of the algorithm). The oracle only provides a yes/no answer.

Consider the usual three coloring decision problem defined below.

Three Coloring (3COL)

INPUT: an undirected graph $G = (V, E)$

QUESTION: Does there exist an assignment of colors to each vertex $v \in V$ such that no more than 3 colors are used and for every edge $(u, v) \in E$, u and v are assigned different colors.

Assignment:

1. Suppose we are given 3COL as an oracle. Devise and describe a polynomial-time algorithm that finds a three-coloring of an undirected graph G, or states that no three-coloring of G is possible.

   When a three-coloring is possible, your algorithm must assign a color to each vertex of G and guarantee that adjacent vertices are assigned different colors.

   Recall that the oracle will only return a yes/no answer and will not provide you any evidence as to why the graph is or is not three colorable.

   Hint: you will need to repeatedly use the oracle on graphs that you construct.

2. Explain why your algorithm works correctly.

3. Briefly state and justify the running time of your algorithm. (Calls to the oracle return in $\Theta(1)$ time.)

4. Argue that if 3COL can be decided in polynomial time, then your algorithm can be used to find a three-coloring in polynomial time.

2 All But Five 3-Colorable

Consider the Three Coloring problem again:

Three Coloring (3COL)

INPUT: an undirected graph $G = (V, E)$

QUESTION: Does there exist an assignment of colors to each vertex $v \in V$ such that no more than 3 colors are used and for every edge $(u, v) \in E$, u and v are assigned different colors.

All But Five Three-Colorable (AB53C)

INPUT: a connected undirected graph $G = (V, E)$.

QUESTION: Does there exist $V' \subseteq V$ and an assignment of one of three colors to each vertex in V' such that no two adjacent vertices are assigned the same color and $|V'| \geq |V| - 5$? In other words, is G 3-colorable if we are allowed to not color up to 5 vertices?

Show that AB53C is NP-complete by constructing a $\leq_{\mathrm{m}}^{\mathrm{P}}$-reduction f from 3COL to AB53C.

1. Describe a polynomial-time function f that given input G_1 outputs G_2 where G_1 and G_2 are undirected graphs. Note that G_2 must be a connected graph.

   N.B.: the reduction function f has G_1 as its **ONLY** input. It has no other information. In particular, it does not know whether G_1 has a 3-coloring.

2. Briefly argue that f runs in time polynomial in the size of G.

3. Prove that if $G_1 \in$ 3COL then $G_2 \in$ AB53C where $G_2 = f(G_1)$ for the function f you described in part 1.

4. Prove that if $G_2 \in$ AB53C then $G_1 \in$ 3COL where $G_2 = f(G_1)$ for the function f you described in part 1.

N.B.: the proofs in parts 3 and 4 must be separate proofs. Also, these proofs must hold for all undirected graphs, not just conveniently chosen examples. (That's why a proof is required.)

3 Quarter Clique.

Consider the following two problems:

CLIQUE

INPUT: an undirected graph $G = (V, E)$ and a natural number k, where $1 \le k \le |V|$.

QUESTION: Does G have a k-clique? I.e, does there exist $V' \subseteq V$ such that $|V'| = k$ and every pair of vertices in V' is connected by an edge in E?

QUARTERCLIQUE (QC)

INPUT: an undirected graph $G = (V, E)$.

QUESTION: Does G have a clique with $|V|/4$ vertices? I.e, does there exist $V' \subseteq V$ such that $|V'| = |V|/4$ and every pair of vertices in V' is connected by an edge in E?

Prove that QUARTER-CLIQUE is NP-complete by constructing a $\le_m^P$-reduction from k-CLIQUE:

1. Describe a $\le_m^P$-reduction f from CLIQUE to QC.

   You should describe how an instance of CLIQUE is transformed into an instance of QC. Make sure that your reduction is in the correct direction for showing that QC is NP-complete. Pictures are great, but do include enough English to make your description unambiguous.

   N.B.: in the definition of CLIQUE, the value k can range from 1 to $|V|$.

2. Prove that if $(G_1, k) \in$ CLIQUE then $G_2 \in$ QC where $G_2 = f(G_1, k)$ for the function f you described in part 1.

3. Prove that if $G_2 \in$ QC then $(G_1, k) \in$ CLIQUE where $G_2 = f(G_1, k)$ for the function f you described in part 1.

N.B.: the proofs in parts 2 and 3 must be separate proofs. Also, these proofs must hold for all undirected graphs G_1 and all values of k, not just conveniently chosen examples. (That's why a proof is required.)