Due: Tuesday, May 16, 2023, 11:59pm

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1 Hoeffding's Inequality

Hoeffding's Inequality is a generalization of Chernoff Bounds and applies to independent bounded random variables, regardless of their distribution.

> **Hoeffding's Inequality:** Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $a \leq X_i \leq b$ for all i. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathrm{E}[X]$. Then, for all $\delta > 0$:

$$\mathrm{Prob}\left[\, X \geq (1+\delta)\mu \,\right] \leq e^{-\frac{2\delta^2\mu^2}{n(b-a)^2}}$$

Suppose we roll a 6-sided die where the faces are labeled 1, 2, 3, 4, 5 and 6. The expected value of the single die roll is 3.5. The probability that a single die roll is ≥ 4 is 0.5. If we roll the die 100 times, the expected average value is still 3.5. However, the probability that the average value is ≥ 4 should drop exponentially. Use Hoeffding's Inequality to bound the probability that the average value from rolling a 6-sided die 100 times is ≥ 4.

Questions:

1. What are the values of a and b?
2. How should you define the random variables X_i?
3. What are the values of μ and δ?
4. What bound do you get from Hoeffding's Inequality for the probability that the average of 100 die rolls is ≥ 4?

2 Approximate graph coloring?

[Adapted from Kleinberg & Tardos]

Graph coloring is a decision problem. An undirected graph is either 3-colorable or not 3-colorable. Often there is more than one way to convert a decision problem into an optimization problem. For graph coloring, the usual way to express graph coloring as an optimization problem is to ask for a coloring of the given graph that uses as few colors as you can.

In this problem, we take an alternative approach — we keep the number of colors constant, say three. That is, we are always considering an assignment of three colors to the vertices of the graph. We say that an edge is *satisfied* if its endpoints are assigned different colors. If every edge is satisfied, then we have a 3-coloring of the graph. The optimization problem is to assign one of three colors to each vertex of the graph that maximizes the number of edges that are satisfied. Let's call this problem MaxColor.

Side Note: MaxColor is definitely the "cheating" version of graph coloring as an optimization problem. There are no good approximation algorithms for the normal version of graph coloring.

For example, there are rather sophisticated algorithms that can color a graph using about $O(n^{1/5})$ colors under the assumption that the graph is 3-colorable. (That is, even knowing the graph is 3-colorable does not help you color the graph with few colors.)

Assignment:

1. Devise and describe a randomized approximation algorithm for MaxColor that achieves an expected approximation ratio of 1.5.
2. Provide an analysis of the expected number of edges that are satisfied by your algorithm.
3. State and briefly justify the running time of your algorithm.